VIBRATION OF A HORIZONTAL SHAFT CONTAINING A TRANSVERSE CRACK

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ABSTRACT

In this paper, the influence of transverse crack of a simple rotor system on the non-resonant and resonant response are discussed. The existence of cracks on the rotor changes the stiffness and damping properties, as a result the characteristic feature of vibration. For theoretical investigation, the stiffness of the cracked shaft is modeled approximately with a harmonically varying function to simulate the opening and closing of the crack with the rotation. On the basis of this idealized model, the influence of a crack and its depth both on the stationary and non-stationary vibration behavior is investigated and numerical results presented.

INTRODUCTION

Because of the extent of damage that can occur as the result of failure, lots of theoretical and experimental work on the problem of cracked rotor, based on simplified model or real rotor has been published. Most of the studies [1, 2, 3, 4, 6, 7, 8] are based on simplified rotor, Jeffcott rotor considering different crack models. In [7], the theoretical investigations has been verified experimentally.

The theoretical studies in [2, 7], presented the effect of crack on the natural frequency making different assumptions and formulating different crack models. The spring characteristic of the crack were expressed by Ishada et. al. [] with a power series while Gasch and Henry with a piece wise linear stiffness. Hamidi and others [2], also studied the influence of cracks on the modal characteristics of rotors, but the study was limited on the natural frequency. Liao et. al. [7], investigated the steady state response of a cracked Jeffcot rotor, modeling the crack by a hinge model and expressing the changes in stiffness due to a crack by a Fourier series and for a zero unbalance. The change of the natural frequency due to the crack was verified experimentally.

The works done so far on the problem of cracked rotors concentrated mainly on the natural frequency changes and the steady state behavior as the crack developed. The influence of the crack on the transient behavior, the situation during the start up and shut down, is not investigated. Therefore, the purpose of this work is to study the effect of a crack on shafts both on the steady and the non-stationary behavior, based on Mayes stiffness model [8] so that its information can be utilized in the early detection of cracks in rotating machinery diagnosis.

EQUATION OF MOTION OF ROTOR SYSTEM

To study the vibration phenomena of a cracked shaft, consider a simple Jeffcot rotor shown in Fig. (1), with a transverse crack located in the middle of the span with the section of the shaft at the crack location shown Fig. 1(b). The position of points on the section may be described with respect to the fixed coordinate system (XYZ) whose Z axis coincides with the bearing center line and also with the coordinate $x'y'$ rotating with the crack section.

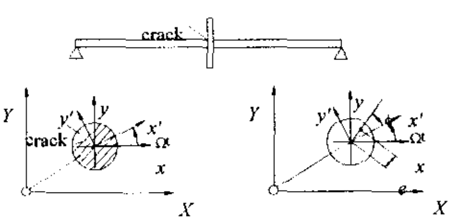

Figure 1 Jeffcot rotor

The equation of motion of the horizontal rotor without a crack and excited with unbalanced rotating mass is of general form

$$\mathbf{M}\ddot{\vec{x}} + \mathbf{C}\dot{\vec{x}} + \mathbf{K}\vec{x} = \vec{f}_{wt} + \vec{f}_{un} \qquad (1)$$

$\vec{x}, \dot{\vec{x}}$ and $\ddot{\vec{x}}$ are the displacement, velocity and acceleration vectors in the inertial frame of reference. $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are the time independent mass, damping and stiffness matrices of the un-cracked shaft. $\vec{f}_{wt}$ and $\vec{f}_{un}$ Are the weight and the unbalance mass excitation forces.

Crack Model

The mathematical model of the stiffness and the damping parameters in the Jeffcot rotor assumes symmetric and equal both in the x- and the y-directions of deflections. With the development of the crack on the cross-section of the shaft however, these will not be the case. The existence of the transverse crack alters the stiffness. The assumed symmetry of the stiffness in the x- and in the y-directions ceases and the magnitude will have two components, a constant and time dependent components. Similarly the damping parameter changes.

With the development of a crack, the stiffness matrix given in Eq. (1), becomes time dependent $\mathbf{K} = \mathbf{K}(t)$. The stiffness parameter can be modeled, considering the crack as a segment of a circle that is infinitely thin and with a depth of a as shown in Fig. 2. Initially, without the crack, the stiffness of the shaft both in the x- and in the y-directions is determined from

$$k_x = k_y = \frac{48EI}{l^3} \qquad (2)$$

With the crack initiated, it opens when the load on the cracked side is tensile and closes when it is under compression. When it opens, the section properties I_x and I_y reduces as a result the bending

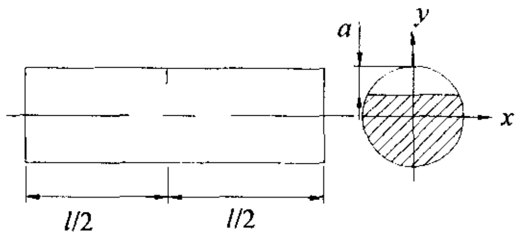

Figure 2 Development of a crack

stiffness k_x and k_y obtained from Eq. (2) will have a minimum value respectively. On the other hand, when it closes the shaft behaves as if there is no crack on it and the stiffness becomes $k_x = k_y = k_0$.

For the breathing crack, the bending stiffness both in the x'- and y'-directions, thus varies between a maximum value k_0 when the crack closes and minimum k_{x-m}, k_{y-m} respectively. A good approximation of the cyclic stiffness variation will be a harmonic variation, within a period of $(2\pi / \Omega)$, where Ω is the shaft angular speed of rotation. For a constant speed of rotation, the variation of the stiffness in the x'- and y'-directions can thus be modeled after [7] by

$$k_x(t) = \frac{k_{xm}}{2} + \frac{\Delta k_x}{2}\cos\theta(t)$$
$$k_y(t) = \frac{k_{ym}}{2} + \frac{\Delta k_y}{2}\cos\theta(t) \qquad (3)$$

where

k_0 Shaft stiffness with no crack

Δk_x $= k_0 - k_{x-m}$ stiffness reduction in the x-direction with the crack open

Δk_y $= k_0 - k_{y-m}$ stiffness reduction in the y-direction with the crack open

$k_{x,y}(t)$ stiffness with respect to the rotating coordinate x', y'

k_{xm} $= k_0 + k_{x-m}$ the mean stiffness about x' axis

k_{ym} $= k_0 + k_{y-m}$ the mean stiffness about y' axis

k_{x-m} the min. stiffness in the x' direction for a given crack depth a

k_{y-m} the min. stiffness in the y' direction for a given crack depth a

$\theta(t)$ The shaft angular rotation with respect to x-y coordinate

Transforming the time dependent stiffness given by Eq. (3) to the space fixed coordinates x and y with

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} x \\ y \end{Bmatrix} \quad (4)$$

yields the stiffness in the inertial frame XYZ as

$$\mathbf{K}(t) = \mathbf{T}^T \mathbf{k} \mathbf{T} \quad (5)$$

with

$$\mathbf{k} = \begin{bmatrix} k_x(t) & 0 \\ 0 & k_y(t) \end{bmatrix} \quad (6)$$

the stiffness matrix in the rotating coordinate $x'y'$, with the elements form Eq. (3). The transformed stiffness matrix finally, can be written as a superposition of constant and time dependent components.

$$\mathbf{K}(t) = \mathbf{K}_c + \Delta \mathbf{K}(t) \quad (7)$$

Equation of Motion with Crack

The equation of motion of the rotor shown in Fig. 1, with a crack will be of the same form as that of the un-cracked one, except the changes in the stiffness matrix term. Employing the transformed crack model given by Eq. (7), the equation of motion in the fixed coordinate will be of the form

$$\mathbf{M}\ddot{\vec{x}} + \mathbf{C}\dot{\vec{x}} + [\mathbf{K}_c + \Delta\mathbf{K}(t)]\vec{x} = \vec{f}_{wt} + \vec{f}_{un} \quad (8)$$

where $\mathbf{K}_c$ is the constant stiffness matrix of the un-cracked shaft

$$\mathbf{K}_c = k_0 \begin{bmatrix} (2+\kappa_x+\kappa_y)/4 & 0 \\ 0 & (2+\kappa_x+\kappa_y)/4 \end{bmatrix} \quad (9)$$

andis the time dependent component of the stiffness matrix due to the crack. From Eqs. (3), (5) and (6)

$$\Delta\mathbf{K}(t) = \frac{1}{2}\Delta k_a \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \quad (10)$$

The parameters

$$\begin{aligned} f_{11} &= C\cos(\Omega t) + 2\cos(2\Omega t) - \cos(3\Omega t) \\ f_{12} &= -\sin(\Omega t) + 2\sin(2\Omega t) - \sin(3\Omega t) \\ f_{22} &= (2+C)\cos(\Omega t) - 2\cos(2\Omega t) + \cos(3\Omega t) \\ C &= \frac{4-3\kappa_x-\kappa_y}{\kappa_x-\kappa_y}; \quad \Delta k = k_0\frac{\kappa_x-\kappa_y}{4}; \\ \kappa_x &= k_{x-\min}/k_0; \quad \kappa_y = k_{y-\min}/k_0; f_{12} = f_{21} \end{aligned} \quad (11)$$

is the phase that depends on the mode of operation and generally, it is described by the integral of the speed of rotation and its variation is

$$\theta(t) = \int \Omega(t)\,dt \quad (12)$$

In a non-dimensional form Eq. (8) can be rewritten as

$$\vec{X}'' + 2\zeta\vec{X}' + \left[1 + \Delta\mathbf{K}(\tau)/k_c\right]\vec{X} = \left(\vec{f}_{wt} + \vec{f}_{un}\right)/k_c \tag{13}$$

assuming the damping ratios both in the x and y directions are the same, with

$$\omega_0 t = \tau;\quad \omega_0 = \sqrt{k_c/m},\ c/m = 2\zeta\omega_0;$$
$$x/e = X \quad or\ x/\Delta_{st} = X,\ k_c = k_0\frac{2+\kappa_1+\kappa_2}{4} \tag{14}$$

Equation (13) describes the behavior of the cracked rotor, both during the steady and transient operation.

Steady State Response

The equation of motion of the cracked shaft Eq. (14) is a differential equation with a harmonically varying coefficients. Thus, difficult to obtain the complete exact analytical solution. The complete solution however, is given as a superposition of the homogeneous and particular solution,

$$\vec{X}(\tau) = \vec{X}_{\text{hom}} + \vec{X}_{st} \tag{15}$$

The homogeneous part decays away if the system is stable. The steady solution vector of the equation of motion is obtained as the sum of three vectors. X_{st} - the response due to the unbalanced excitation, X_{wt} - the gravitational force and X_d - the non-linear force due to the crack.

In the equation of motion, the time dependent stiffness component $\Delta\mathbf{K}(\tau)$ is small compared to the constant stiffness term $\mathbf{K}_c$ for all, except for large cracks. Thus, an approximate solution of the equation of motion that gives the steady response can be determined by perturbation method.

The first approximation of the steady state solution is obtained neglecting the time dependent term and reducing it as two decoupled differential equations

$$\mathbf{X}_c'' + 2\zeta\mathbf{X}_c' + \mathbf{X}_c = \vec{f}(\tau) \tag{16}$$

where the right hand side is

$$\vec{f}(\tau) = \eta^2\begin{Bmatrix}\cos\theta(\tau)\\ \sin\theta(\tau)\end{Bmatrix} + \begin{Bmatrix}0\\ -\Delta_{st}/e\end{Bmatrix} \tag{17}$$

In this case, the angular rotation of the rotor is constant and the normalized phase $\theta(\tau)$ is the described by

$$\theta(\tau) = \theta_0 + \eta_0\tau \tag{18}$$

where θ_0 , $\eta_0 = \Omega/\omega_0$ are the non-dimensional initial phase speed ratio respectively. Part of the steady state solution due to the unbalanced excitation will be

$$X = \frac{\eta_0^2}{\sqrt{(1-\eta_0^2)^2 + (2\zeta\eta_0)^2}}\cos\left[\theta(\tau) - \phi\right]$$
$$Y = \frac{\eta_0^2}{\sqrt{(1-\eta_0^2)^2 + (2\zeta\eta_0)^2}}\sin\left[\theta(\tau) - \phi\right] \tag{19}$$

Substitute this solution in Eq. (13), as a first approximation to $\Delta K(t)\cdot X_c$. Hence, we obtain the equation in matrix form as

$$\vec{X}_\tau'' + 2\zeta\vec{X}_\tau' + \vec{X}_\tau = -\Delta\mathbf{K}(\tau)\vec{X}_c \tag{20}$$

The solution of Eqs. (16) and (20) yields the response as a superposition

$$\vec{X}(\tau) = \vec{X}_c(\tau) + \vec{X}_\tau(\tau) \tag{21}$$

When there is no unbalance, the excitation on the rotor will be due to gravitational force only

$$\vec{f}(\tau) = \left\{ \begin{matrix} 0 \\ -\Delta_{st} \end{matrix} \right\} \tag{22}$$

For this case, the total solution will remain of the same form Eq. (21)

$$\vec{X}_g(\tau) = \vec{X}_{gc}(\tau) + \vec{X}_{gr}(\tau) \tag{23}$$

Transient Response

The rotating speed of rotating machines are usually much higher than the main critical speeds. Therefore, they have to pass through the critical speeds during starting and shut down. So it is essential to analyze the behavior of the rotor when translating through the critical speeds before reaching the operating speeds also to observe the influence of developed cracks.

The transition of the rotor through the critical speed, can satisfactorily be simulated with linearly varying excitation speed assuming sufficient power supply. Thus, the excitation speed can be described by

$$\Omega(t) = \Omega_0 + \dot{\Omega} t \tag{24}$$

and the phase

$$\theta(t) = \int \Omega(t) dt = \theta_0 + \Omega_0 t + \frac{1}{2}\dot{\Omega} t^2 \tag{25}$$

When normalized with the Eq.(14), the phase describing the transition through resonance becomes

$$\theta(\tau) = \theta_0 + \eta_0 \tau + \frac{1}{2}\alpha \tau^2 \tag{26}$$

where $\alpha = \dot{\Omega} / \omega_0^2$ is the rate of change of speed. The equation of motion Eq. (8) with the phase Eq. (26), describes the non-stationary operation of a cracked rotor. The equation of motion in this case does not have a closed form solution and can be solved numerically.

Numerical Results

The numerical investigation presented are for the model shown in Fig. (1) and the vibration response follow Eq. (8) and phase Eq. (26) for steady state ($\alpha = 0$).

Figures 3 and 4 show the dynamic displacement of the shaft with cracked-forced excitation in the horizontal and vertical directions respectively due to the gravity load only for $\zeta = 0.0075$. With the assumptions used and with the existence of transverse cracks, the displacements consist of the three harmonics. Resonance therefore, occurs at n_0 for the n harmonics, at $n_0 / 2$ for the $2 \times n$ and $n_0 / 3$ for the $3 \times n$ harmonics, where n_0 is the natural frequency of the shaft with no crack. The higher harmonic component resonances are non-existent in un-cracked rotor.

For a constant damping and excitation due to the gravitational load only, the maximum peak resonance displacements in the horizontal direction occur at $n = n_0 / 2$ while in the vertical direction it occurs at $n = n_0$.

The peak amplitude in the vertical direction due to the gravitational load only occurs at $n = n_0$. These two figures show that the peak value at resonance is strongly influenced by the gravitational load and the degree of crack depth. Furthermore, with the increase in crack depth, the contribution of the higher harmonic components becomes more significant.

Figures 5 and 6 shows the dynamic displacement of a cracked and un-cracked rotor subjected to the combination of the gravity and unbalance, for the eccentricity to static deflection ratio of 0.035. With no crack, the resonance peak amplitude, both in the horizontal and vertical directions can be approximated by $1/(2\zeta)$ and occurs at $n = n_0$. However, with the existence of a crack, the resonance peaks at n_0 and also at the higher harmonics are strongly influenced and they either increase or decrease the amplitude depending on the position of the crack with respect to the unbalance mass. The increase is when the unbalance mass is in the same side of the crack and the vice versa when it is on the opposite side of the crack.. As shown in figure (5&6), the resonance peak increases from with increase in crack depth when the position of unbalance is in the opposite side of the crack. For other unbalance positions, the peak will be smaller.

Figures 7 and 8 show the peak amplitude in the horizontal and vertical directions with the increase in crack depth and different damping. From the figures, for low damping, the resonance amplitudes both in the horizontal and vertical directions for all harmonics may increase with crack depth when the unbalance is in the side of the crack This marked increase however disappears for higher values of damping the effect.

For the non-stationary case, the equation of motion Eq. (8) with the phase form Eq. (26) are numerically computed to obtain the displacement in the horizontal and vertical directions during passage of the rotational speeds through the critical speeds.

Figure 9 shows the displacement in the horizontal and vertical direction of the cracked rotor for constant damping (ζ = 0.075), when passing through the main critical speed, with an acceleration of α = 0.0075. The response indicates, the peak amplitudes at the higher harmonics disappear as there is no time for the build up. On the other hand, the peak amplitude at n_0 increases strongly compared to the amplitude increase of un-cracked rotor with the same acceleration and the frequency at which it occurs is shifted to the right.

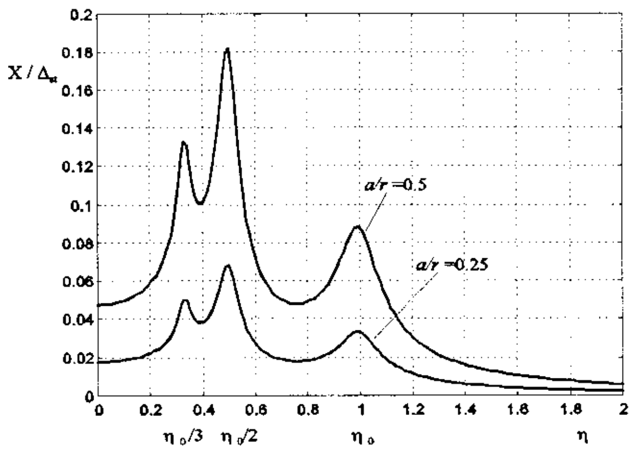

Figure 3 Dynamic displacement due to gravity excitation with damping ζ = 0.075

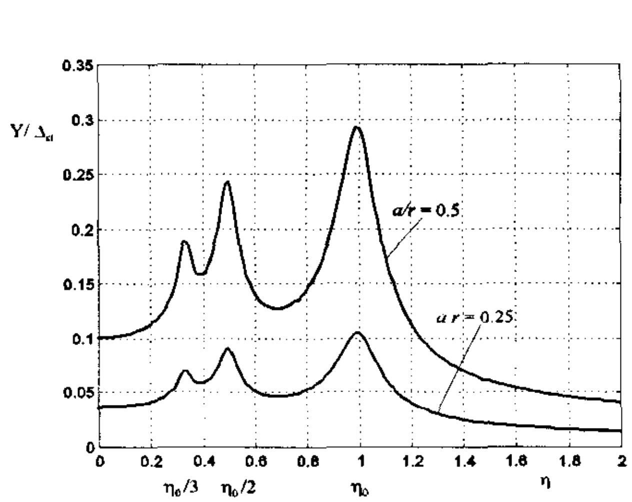

Figure 4 Dynamic displacement in the vertical direction due to gravitational load with a damping $\zeta = 0.075$

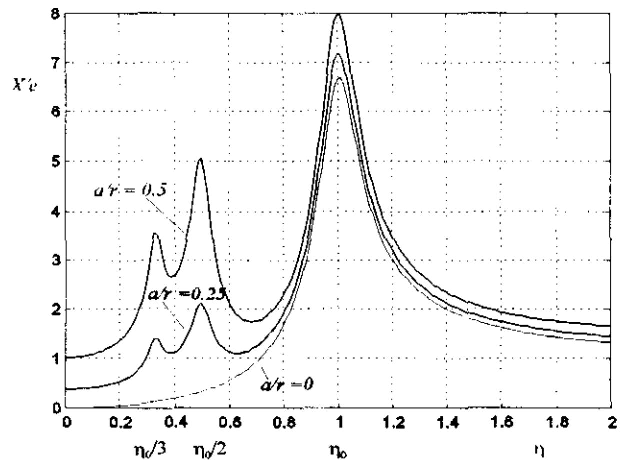

Figure 5 Dynamic displacement in the horizontal direction due to gravity and unbalance excitation with damping $\zeta = 0.075$, $e = 0.035\Delta_{st}$

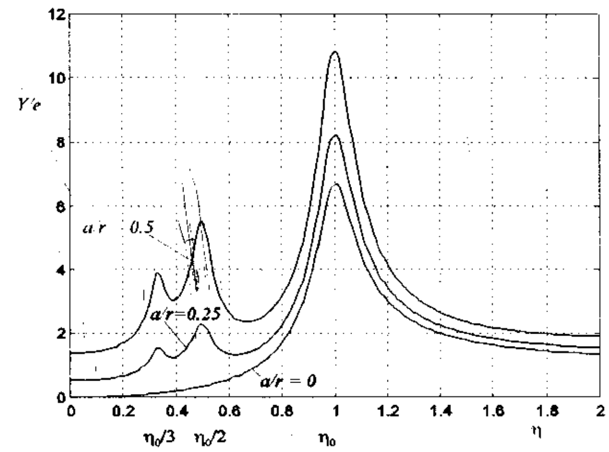

Figure 6 Dynamic displacement in the vertical direction due to gravity and unbalance excitation with damping ζ = 0.075, e = 0.035Δ_{st}

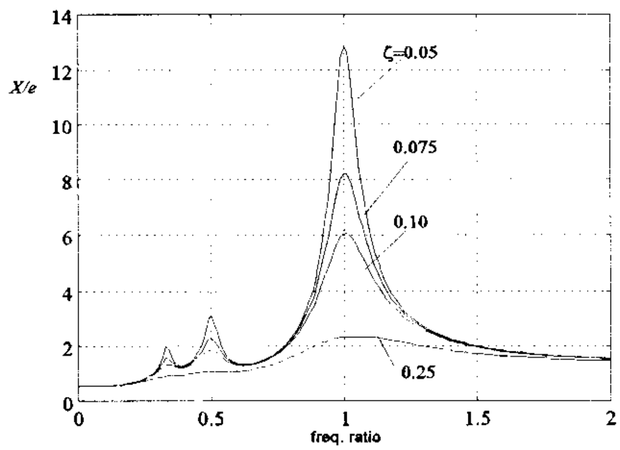

Figure 7 Dynamic displacement variation in the horizontal direction for different values of damping and a/r = 0.075, e = 0.035Δ_{st}

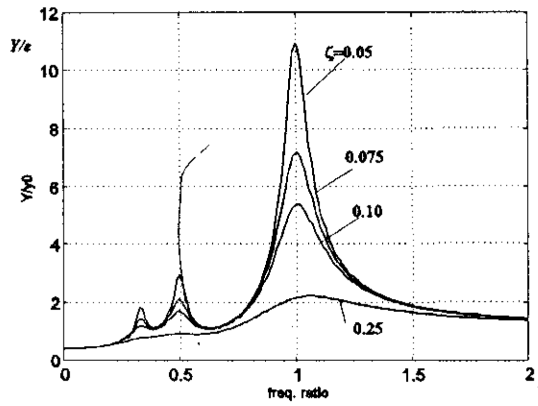

Figure 8 Dynamic displacement variation in the vertical direction for different values of damping and $a / r = 0.25, e = 0.035\Delta_{st}$

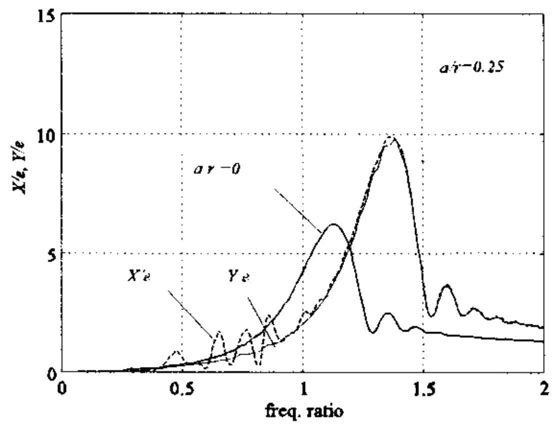

Figure 9 Transient vibration amplitude when accelerating through the resonance $\alpha = 0.0075, \zeta = 0.075$

CONCLUSION

According to the numerical results obtained in the previous section, the following conclusions can be drawn:

- besides the first harmonic, there exists higher harmonics in stationary response and among the harmonics, the second harmonic is most obvious.
- because of the existing higher harmonics, fractional critical speeds such as ½ and 1/3 speed will occur. Thus when the speed approaches the fractional critical speeds, the corresponding resonance will occur.
- the magnitude of peak amplitudes at resonance may increase or decrease with the existence of crack. It depends on the orientation of the unbalanced mass relative to the crack. For all

damping, it increases when the unbalance is on the same side of the crack and decreases when it is opposite to it.

- for higher damping values, the characteristic observed due to the crack, appearance of resonances at the frequencies n_0 / 2 and n_0 / 3 can not be observed.

- the transient response indicates, that the amplitude at the main resonance increases and at the same time is shifted to a higher frequency.

The response changes due to the existence of cracks are significant changes and can be observed on vibration measurement during machinery health monitoring. These changes both from the steady and transient responses are indicators of the initiation of cracks and can be utilized in the diagnosis of machinery health in practice.

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