NONLINEAR LOAD-DISPLACEMENT ANALYSIS OF STRUCTURES HAVING MEMBERS WITH NON-WARPING CROSS-SECTIONS

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ABSTRACT

Solid or hollow rectangular and circular cross-section members are most extensively used in medium and small size steel structures. Such cross-sections are non-warping cross-section. They are extensively used because of their strength and the ease to assemble. The governing differential equations and the corresponding stiffness equations for such members are derived. The load-displacement and stability characteristics of such members are investigated. The structure deformations are also indicated. Such knowledge will be very helpful for engineers who are involved in the construction of such structures.

INTRODUCTION

Most steel structures constructed in developing countries are small or medium size buildings or frames. Most members in such constructions have hollow or solid cross-section rectangular or circular cross-sections. Following the linearized finite displacement theory and the finite element technique, the governing differential equations and the stiffness equations have been derived. Two classes of structures with non-warping cross-sections are identified. The first class of members with non-warping cross-sections are members of frames which act as beams or beam-columns, the internal bending moment and internal shear force are the important internal actions. The second class of non-warping cross-section members are members of trusses; where the only internal action in such members is the internal axial force. With proper transformation and updating procedures the load-displacement curves of structures containing such members have been plotted.

THEORETICAL FORMULATION

Non-Warping Beam-Columns

Solid or hollow rectangular and circular cross-section members are common examples of members with non-warping cross-sections. The displacement field u, v, and w of an arbitrary point on the cross-section may be given as

$$u = u_c - v_s'(y.\cos\phi) - w_s'(z.\cos\phi + y.\sin\phi) \qquad \text{(1a)}$$

$$v = v_s(y - y_s)(1 - \cos\phi) - (z - z_s)\sin\phi \qquad \text{(1b)}$$

$$w = w_s - (y - y_s).\sin\phi - (z - z_s)(1 - \cos\phi) \qquad \text{(1c)}$$

The following virtual work principle will be used for the required derivations;

$$\begin{aligned}&\int_v(\sigma_{ij}^o.\delta e_{ij}^{NL} + \sigma_{ij}.\delta e_{ij}^L)dV\\ &-\int_L(p_i^o.\delta u_i^{NL} + p_i.\delta u_i^L)dx\\ &-\left[F_{ij}^o.\delta u_{ik}^{NL} + F_{ik}.\delta u_{ik}^L\right]_{k=ij} = 0\end{aligned} \qquad (2)$$

The non-vanishing strain components are;

$$\begin{aligned}e_{xx} &= u_c' - v_s''(y - z\phi) - w_s''(z + y\phi)\\ &+ \frac{1}{2}\left[(v_s')^2(w_s')^2\right] + (z_s v_s' - y_s w_s')\phi'\\ &+ \left[(y - y_s)^2 + (z - z_s)^2\right](\phi')^2\end{aligned} \qquad \text{(3a)}$$

$$e_{xs} = \frac{1}{2}\Theta\,\phi' \qquad \text{(3b)}$$

For a thin-walled non-warping beam, the normal stress σ_{xx}^o at the reference state is expressed in terms of the stress resultants for axial force and bending moments, as follows;

$$\sigma_{xx}^o = \left(\frac{N^o}{A}\right) + \left(\frac{M_y^o}{I_{yy}}\right)y + \left(\frac{M_z^o}{I_{zz}}\right)z \qquad (4)$$

Substituting the relevant values, which are given in the preceding expressions, into the virtual work equation given in Eq.(2), the following is obtained:

$$\begin{aligned}&\int_v\left\{\frac{N^o}{A} + \left(\frac{M_y^o}{I_{yy}}\right)y + \left(\frac{M_z^o}{I_{zz}}\right)z\right\}\delta\Big[z.v_c''\phi - y.w_s''\phi\\ &+ \frac{1}{2}\left\{(v_s')^2 + (w_s')^2\right\} + (z_s v_s' - y_s w_s')\phi'\\ &\frac{1}{2}\left\{(y - y_s)^2 + (z - z_s)^2\right\}(\phi')^2\\ &+ E(u_c' = y.v_s'' - z.w_s'').\delta(u_c' - y.v_s'' - z.w_s'')\\ &+ G\Theta\,\phi'.\delta(\Theta\,\phi'')\Big]dV - F^T\delta d = 0\end{aligned} \qquad (5)$$

Through variational treatment of Eq.(5), one obtains the following governing differential (equilibrium) equations;

$$EAu_c'' = 0 \tag{6a}$$

$$EI_{yy}v_s^{(4)} - N^o v_s'' + (-z_s N^o + M_z^o)\phi'' = 0 \tag{6b}$$

$$EI_{zz}.w_s^{(4)} - N^o.w_s'' + (y_s N^o - M_y^o)\phi'' = 0 \tag{6c}$$

$$GJ\phi'' + (z_s N^o - M_z^o)v_s'' - (y_s N^o - M_y^o)w_s'' + \left(r_s^2 N^o + \beta_y M_y^o + \beta_z M_z^o\right)\phi'' = 0 \tag{6d}$$

The associated boundary conditions being

$$u_c = y_{ck} \quad \text{or } n_x EAu'_{c=F_{xk}} \tag{7a}$$

$$v_s = v_{sk} \quad \text{or } n_x \begin{bmatrix} -EI_{yy}v_s''' + N^o v_s' \\ +(z_s N^o - M_z^o)\phi' \end{bmatrix} = F_{yk} \tag{7b}$$

$$-v_s'' = -v_{sk}' \quad or \left[-EI_{yy}v_s'' - M_z^o\phi\right] = d_{yk} \tag{7c}$$

$$w_s = w_{sk} \quad \text{or } n_x \begin{bmatrix} -EI_{zz}w_s''' + N^o w_s' \\ +(-y_s N^o + M_y^o)\phi' \end{bmatrix} = f_{zk} \tag{7d}$$

$$-w_s' = -w_{sk}' \quad \text{or } n_x\left[-EI_{zz}w_s'' + M_y^o\phi\right] = D_{zk} \tag{7e}$$

$$\phi = \phi_k \quad or\ n_x \left[GJ\phi' + z_s N^o v_s' - y_s N^o w_s' + (r_x^2 N^o + \beta_y M_y^o + \beta_z M_z^o)\phi' \right] = D_{xk} \tag{7f}$$

where, $n_x = -1$ at $x = 0$ and $n_x = 1$ at $x = L$.

The above differential equations are important from a theoretical point of view only. For practical application, a relevant stiffness equation must be developed. For this purpose, the forces and displacements must be expressed in a discrete format. Thus;

$$F = \left\langle F_x^T, F_y^T, F_z^T, T_x^T \right\rangle^T \tag{8a}$$

$$d = \left\langle U^T, V^T, W^T, \Phi^T \right\rangle^T \tag{8b}$$

in which, the components of the force vector are;

$$F_x = \left\langle F_{xi}, F_{xj} \right\rangle^T \tag{9a}$$

$$F_y = \left\langle F_{yi}, D_{yi}, F_{yj}, D_{yj} \right\rangle^T \tag{9b}$$

$$F_z = \left\langle F_{zi}, D_{zi}, F_{zj}, D_{zj} \right\rangle^T \tag{9c}$$

$$T_x = \left\langle C_{xi}, C_{xj} \right\rangle^T \tag{9d}$$

and, those of the displacement vector are,

$$U = \left\langle u_{ci}, u_{cj} \right\rangle^T \tag{10a}$$

$$V = \left\langle v_{si} - v_{si}', v_{si}, v_{sj}' \right\rangle^T \tag{10b}$$

$$W = \left\langle w_{si}, -w_{si}', w_{sj}, -w_{sj}' \right\rangle^T \tag{10c}$$

$$\Phi_s = \left\langle \phi_i, \phi_j \right\rangle^T \tag{10d}$$

At this stage, the following interpolation functions, which are the well known Hermite interpolating polynomial, are introduced;

$$N_1 = 1 - \left(\frac{x}{L}\right) \tag{11a}$$

$$N_2 = \left(\frac{x}{L}\right) \tag{11b}$$

$$N_3 = 1 - 3\left(\frac{x^2}{L^2}\right) + 2\left(\frac{x^3}{L^3}\right) \tag{11c}$$

$$N_4 = -x + 2\left(\frac{x^2}{L}\right) - \left(\frac{x^3}{L^2}\right) \tag{11d}$$

$$N_5 = 3\left(\frac{x^2}{L^2}\right) - 2\left(\frac{x^3}{L^3}\right) \tag{11e}$$

$$N_6 = \left(\frac{x^2}{L}\right) - \left(\frac{x^3}{L^2}\right) \tag{11f}$$

The above interpolating polynomials may be arranged as follows;

$$A = \left\langle N_1, N_2 \right\rangle^T \tag{12a}$$

$$B = \left\langle N_3, N_4, N_5, N_6 \right\rangle^T \tag{12b}$$

Using the above interpolating polynomials, the displacement components at an arbitrary cross-section in the region ($0<x<L$) can be written in terms of the displacements at the ends of the element, in the following approximate form;

$$u_c = A^T U \tag{13a}$$

$$v_s = B^T V \tag{13b}$$

$$w_s = B^T W \tag{13c}$$

$$\phi = B^T \Phi_s \tag{13d}$$

The displacements in the above equation Eqs. (13) are introduced into the virtual work equation of Eq. (5), giving finally the following stiffness equation;

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \\ T_x \end{Bmatrix} = \begin{bmatrix} K_{11} & & \text{Symm.} & \\ 0 & K_{22} & & \\ 0 & 0 & K_{33} & \\ 0 & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \\ \Phi_s \end{Bmatrix} \tag{14}$$

where

$$K_{11} = \left(\frac{EA}{L}\right)K_0 \tag{15a}$$

$$K_{22} = \left(\frac{EI_{yy}}{L^3}\right)K_1 + \left(\frac{N^o}{L}\right)K_2 \tag{15b}$$

$$K_{33} = \left(\frac{EI_{zz}}{L^3}\right)K_1 + \left(\frac{N^o}{L}\right)K_2 \tag{15c}$$

$$K_{42} = \left(\frac{N^o z_s}{L}\right)K_4 + \left(\frac{M_z^o}{L}\right)K_5 \tag{15d}$$

$$K_{43} = -\left\{\left(\frac{N^o y_s}{L}\right)K_4 + \left(\frac{M_y^o}{L}\right)K_5\right\} \tag{15e}$$

$$K_{44} = \left(\frac{GJ}{L}\right)K_o + \left[\left(\frac{N^o}{L}\right)r_o^2 + \left(\frac{M_y^o}{L}\right)\beta_y + \left(\frac{M_z^o}{L}\right)\beta_z\right]K_o \quad (15f)$$

in which,

$$K_4 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \quad (16a)$$

$$K_5 = \begin{bmatrix} -1 & L & 1 & 0 \\ 1 & 0 & -1 & -L \end{bmatrix} \quad (16b)$$

The other block matrices are as given in Ref. [10]

General Stiffness for Truss Members

In truss structures, it is generally assumed that loads are applied at the joints. Members are joined together using frictionless pins. Therefore, all members of such trusses are two-force members, and they are either in tension or compression. The whole member is taken as one element, and no bending moments arise anywhere in the truss.

Consider now a truss element of length L and a cross-sectional area of A. The stress in an element at an arbitrary reference state is,

$$\sigma_{xx}^o = \frac{N^o}{A} \quad (17a)$$

The Green's strain-displacement relations for the axially loaded truss ember is given by;

$$\varepsilon_{xx} = u' + \frac{1}{2}\left[(v')^2 + (w')^2\right] \quad (17b)$$

The virtual work equation to be used here is;

$$\int_v \left\{\left[\frac{N^o}{A}\right]\delta\left\{\frac{1}{2}[(v')^2 + (w')^2\right\} + Eu'\delta u'\right\}dV - F^T\delta d = 0 \quad (18)$$

Substituting the relevant terms into Eq. (18), and through variational treatment, one obtains the governing (equilibrium) equations as follows;

$$\begin{aligned} EAu'' &= 0 \\ N^o v'' &= 0 \\ N^0 w'' &= 0 \end{aligned} \quad (19a\text{-}c)$$

with the corresponding boundary conditions being;

$$u = u_k \quad \text{or} \quad n_x EAu' = F_{xk} \quad (20a)$$

$$u = u_k \quad \text{or} \quad n_x N^o v' = F_{yk} \quad (20b)$$

$$u = u_k \quad \text{or} \quad n_x N^o w' = F_{zk} \quad (20c)$$

Next, to develop the incremental stiffness, one needs to redefine the force and displacement vector as in the following. First, regarding the force vector;

$$F = \left\langle F_x^T, F_y^T, F_z^T\right\rangle^T \quad (21)$$

in which,

$$\begin{aligned} F_x &= \left\langle F_{xi}, F_{xj}\right\rangle^T \\ F_y &= \left\langle F_{yi}, F_{yj}\right\rangle^T \\ F_z &= \left\langle F_{zi}, F_{zj}\right\rangle^T \end{aligned} \quad (22a\text{-}c)$$

Second, regarding the displacement vector;

$$d = \left\langle U^T, V^T, W^T\right\rangle^T \quad (23)$$

in which

$$U = \left\langle u_i, u_j\right\rangle^T \quad (24a)$$

$$V = \left\langle v_i, v_j\right\rangle^T \quad (24b)$$

$$W = \left\langle w_i, w_j\right\rangle^T \quad (24c)$$

Making use of linear interpolating functions, the incremental stiffness equation for the truss element can be derived from Eq. (18) in the form

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{bmatrix} K_{11} & & \text{Symm.} \\ 0 & K_{22} & \\ 0 & 0 & K_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} \quad (25)$$

in which,

$$K_{11} = \left(\frac{EA}{L}\right)K_o \quad (26a)$$

$$K_{22} = \left(\frac{N^o}{L}\right)K_o \quad (26b)$$

$$K_{33} = \left(\frac{N^o}{L}\right)K_o \quad (26c)$$

NUMERICAL EXAMPLES

Plane Structures

Cantilever Beam

In the first example, the in-plane finite displacement behavior of a circular cantilever beam which is subjected an axial force at the free end is investigated. An initial disturbing moment is used. The horizontal displacement '*u*' and the vertical displacement '*w*' are plotted against the applied load, and are shown in Fig.1. The computed results are compared with the analytical solutions obtained from the elliptic integration approach, showing excellent agreement, as shown in the same figure. The deformed shape of the cantilever beam for level 3 is shown on the right-hand side of Fig. 1. The deformations have been exaggerated and it is to be noted that the applied load is still horizontal up to the end of the loading history.

Portal Frame One

In the second example, the displacement behavior of a fixed base portal frame is investigated. The members have hollow square cross-section, and a small horizontal disturbing force is applied. The load-displacement curve is shown in Fig. 2. Again, at the right-hand side in the figure, the deformed shape of the frame at level 3 is shown. It is observed that the frame buckles unsymmetrically.

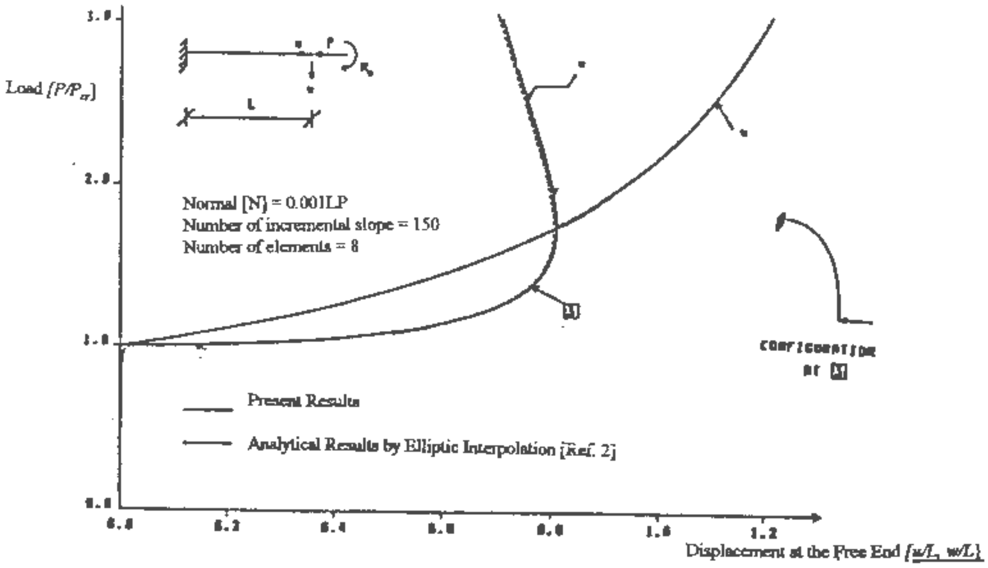


Figure 1 Plane load-displacement behavior of cantilever beam

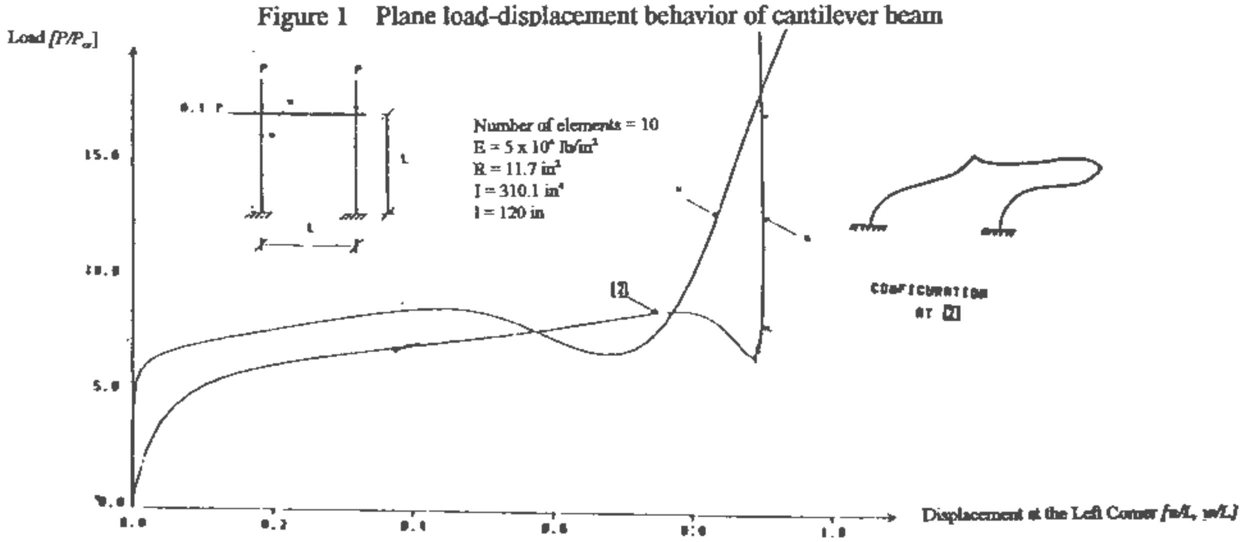


Figure 2 Plane load-displacement behavior of a fixed portal frame (unsymmetrical mode)

Portal Frame Two

The same portal frame as the one used in the preceding example is treated herein, the only difference now being that the disturbing actions being symmetrically placed couples at the loaded nodes. The resulting load-displacement curves are shown in Fig. 3. The deformed shape of this portal frame at level 2 is shown at the right-hand end of the figure. It is seen that this portal frame buckles symmetrically.

Symmetrically Loaded Circular Arch

Circular and parabolic arches are usually used in bridges, building roofs, and other structures. Thus, the load-displacement behaviors of such structures is of great importance for structural design. In the present case, a circular arch subjected to a downward vertical load at the apex is investigated. The load-displacement diagram for this case is shown in Fig.4. On the right-hand side of the figure, the deformed shape of the structure at level 3 is shown. It is clear that this arch buckles symmetrically. No horizontal or '*u*' displacements occurs.

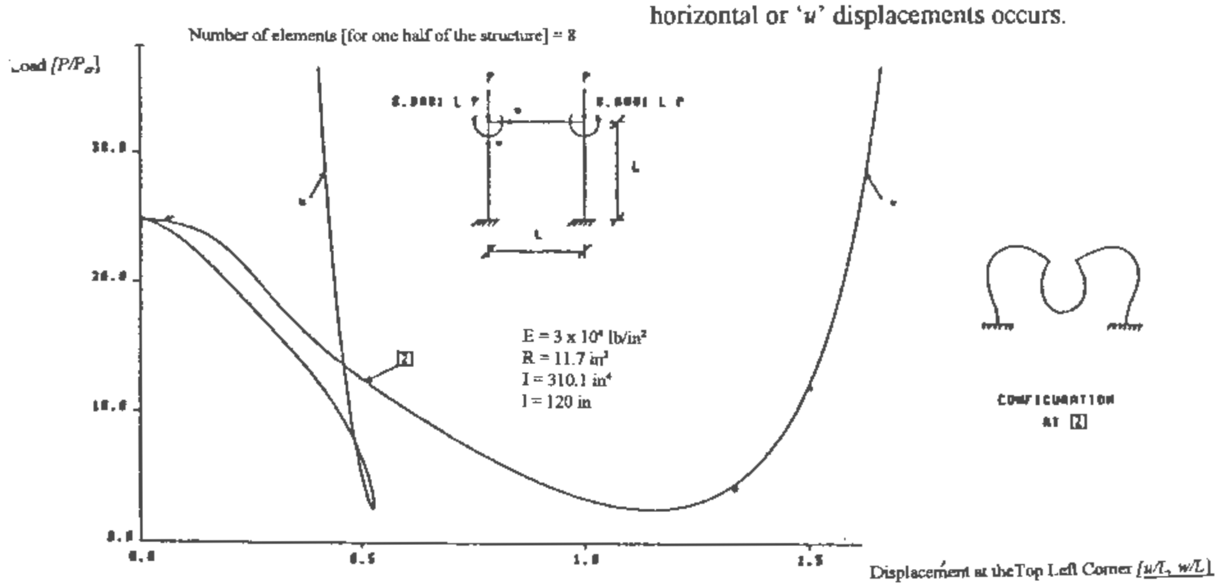


Figure 3 Plane load-displacement behavior of a fixed portal frame (symmetric mode)

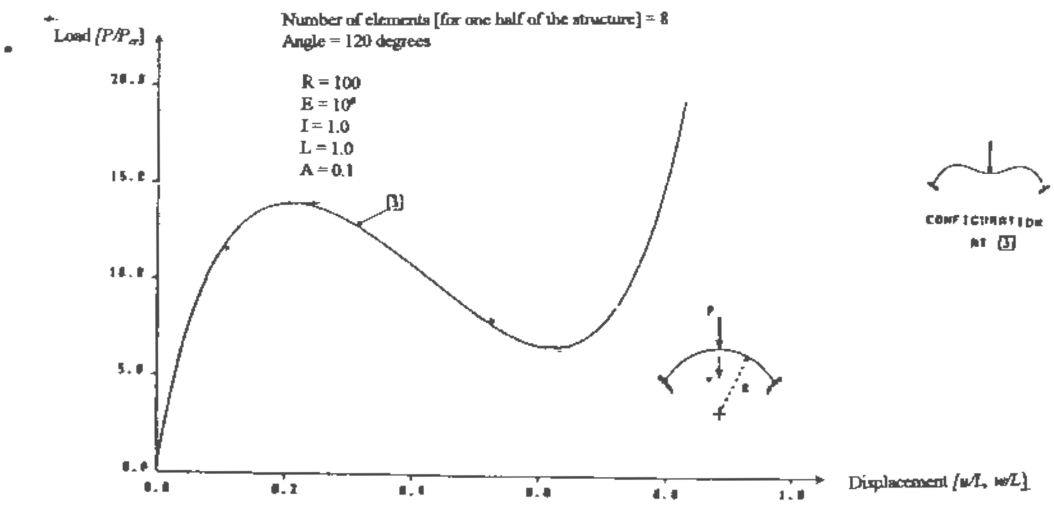


Figure 4 Plane load-displacement behavior of a fixed circular arch (symmetric mode)

Unsymmetrically Loaded Circular Arch

In this final example for planar structures, the arch that was treated in the preceding example is used, except being unsymmetrically loaded in the present case. In this case, both the '*u*' and '*w*' displacements occur, as it can be observed from the load-displacement diagram, shown in Fig 5. At the right-hand side of the load-displacement diagram, the deformed shape of the arch for position 2 is shown. It is clearly seen that the arch buckles unsymmetrically.

Spatial Structures: A Twelve Member Hexagonal Structure

The twelve member hexagonal structure under a vertical load applied at the crest is considered next. All members are of uniform square cross-section which does not warp. All supports of the structure are hinges. The crest of the structure, which is subjected to a vertical load, deflects vertically only. The structure could be studied using two idealizations; the first one using the assumption where all joints between members uses perfectly smooth pins, and the second one using the assumption which states that all joints between members are rigid joints. If the first assumption is used, the structure becomes a truss, and the only internal action in all the members is the axial force. With the second assumption, the structure is becomes a frame, and the important internal actions are bending moment, shear force, and axial force. The load-displacement diagram shown in Fig. 6 shows results following the two assumptions raised.

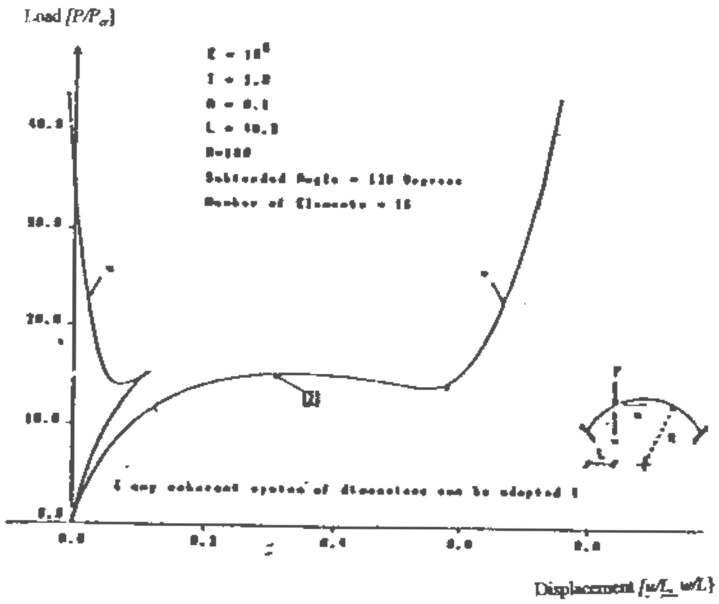
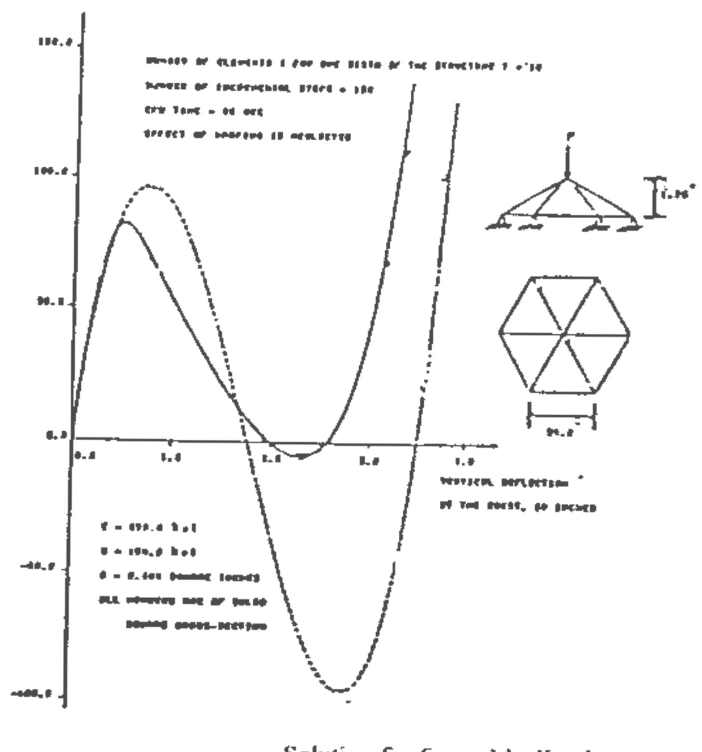


Figure 5 Plane load-displacement behavior of a fixed circular arch (unsymmetric mode)



______ Solution for frame Idealization.

.... Solution for truss Idealization

Figure 6 Spatial load-displacement behavior of a hexagonal frame with hinged supports

SUMMARY AND CONCLUSIONS

In medium or small size steel structures, the type of members used have rectangular or circular hollow cross-sections, which are non-warping. Such members are light and strong. The members are not in danger of lateral-torsional instability, which is a common deficiency in larger hot-rolled I-section members. Solid rectangular or circular cross-sections members, which are heavier, are used only when stronger members are required. The assembly of the hollow rectangular and circular cross-section members is relatively easy.

The load-displacement behavior of such members is investigated using the linearized finite displacement approach also employing the finite element method. The method employs an efficient transformation and updating strategy. Numerical results were produced for a number of planar structures and a spatial structure. It has been found possible to investigate the non-linear finite displacement with sufficient accuracy, by the direct solution of the tangent stiffness equation helped by an accurate updating procedure, thus making it unnecessary to perform iterations. Imposing relatively small increments, the accumulation of error can be reduced to an acceptable level. Neither iteration nor convergences checks are required for the present scheme. Under these circumstances, it can be concluded that the present scheme is quite suitable for practical use.

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