NON-STATIONARY VIBRATIONS OF MECHANICAL SYSTEMS WITH SLOWLY VARYING NATURAL FREQUENCIES DURING ACCELERATION THROUGH RESONANCE
(APPROXIMATE ANALYTICAL SOLUTION)

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ABSTRACT

The paper presents an approximate analytical solution for investigation of vibration responses in linear Single-Degree-of-Freedom-Systems (SDOF) with slowly varying natural frequency subjected to a transient excitation force with constant amplitude. The solution employs the WBKJ-approximation method, the method of Variation of the constants and in the final analysis using error functions with complex variables. This has been used to study systems with an ideal energy source, the excitation frequency changes lineorly as function of time, the natural frequency changes slowly and linearly with time and the damping remains constant. The influence of the changing parameters on the non-stationary resonance response with special emphasis on the maximum amplitude and the speed are presented.

INTRODUCTION

Rotating machines like motors, turbines, compressors etc. are generally subjected to periodic forces and the system parameters remain more or less constant. However, during transition processes the system parameters change and, consequently, the natural frequencies too, due to reasons of changing gyroscopic moments, centrifugal forces, bearing characteristics, mass exchange etc.. The time dependent excitation force associated with the transition process together with the natural frequency changes results in non-stationary vibrations which could be detrimental if the machine passes through critical speeds before reaching the operating speed.

Indeed, these machines are continuous systems, however, making simplifying assumptions they can be modeled mathematically by N linear, coupled-differential-equations as for multi-degree-of-freedom-system. Using modal transformation, they can be reduced to N decoupled equations from which the dynamic behaviour of each mode can be studied. Thus, the influence of the time dependent excitation force and other parameter changes can be investigated based on a linear SDOF system.

Non-stationary vibrations that occur in mechanical systems during run-up and shut-down and passage through resonance, modeled as SDOF systems with a linearly changing speed, constant natural frequency and damping have been investigated by a number of authors using exact or numerical methods [2], [1], [7] and others. In case of simultaneously changing natural frequency and speed, the governing equation is a second order differential equation with time dependent coefficients. This problem has been investigated either using direct numerical integration [8] or reducing the differential equation to a first-order one employing the asymptotic method followed by numerical integration [6]. In this paper, the problem is investigated using approximate analytical solution derived as an extension to the solution given by Markert [7] and Goloskokow/Fillipow [4] for the case of constant parameters, which is based on complex error functions.

Results of the numerical computations obtained from the derived solution are presented as an example and the influences of the simultaneously changing excitation and natural frequencies are summarized.

MATHEMATICAL MODEL

With the assumption that the system mass and the damping remains constant and the stiffness changes with time, the equation of motion of a linear SDOF system subjected to transient excitation is described by

$$m\ddot{q} + c\dot{q} + k(t)q = f(t) \qquad (1)$$

Where the parameters m, c, and $k(t)$ are the mass, damping and the time dependent stiffness of the system. Here, the run-up and shut-down process is modeled through a quasi-harmonic excitation force as,

$$f(t) = \hat{f} \cos(\Theta_e(t)) \qquad (2)$$

Assuming a constant acceleration [2], [1], [7], the excitation frequency $\Omega(t)$ changes linearly with time

$$\Omega(t) = \Omega_0 + \dot{\Omega}t \qquad (3)$$

The phase, representing the transition process, changes with time as

$$\Theta_e(t) = \int \Omega(t)dt = \frac{1}{2}\dot{\Omega}t^2 + \Omega_0 t + \Theta_0 \qquad (4)$$

Further, the change in natural frequency that occurs in rotating machinery following the system stiffness change $k(t)$ during this transition process is in most practical cases a function both of the standstill natural frequency and the changing excitation frequency. However, assuming the natural frequency changes slowly, its change with time is approximated through a linear function

$$\omega(t) = \omega_0 + \dot{\omega}t \qquad (5)$$

Figure 1 shows the changes of the excitation and the natural frequencies with time.

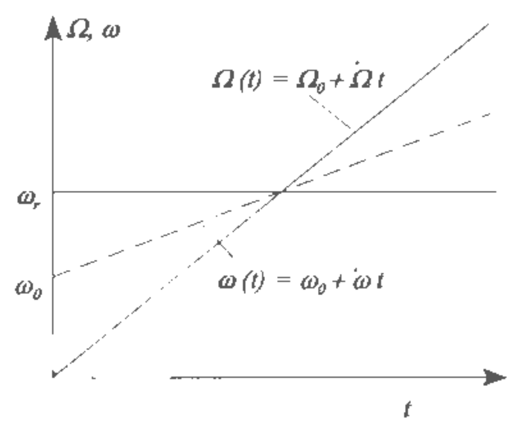

Figure 1 Changes of the excitation and natural frequencies with time

Normalizing the time in non-dimensional form with reference to the resonance frequency ω_r (intersection of the excitation and the natural frequencies)

$$\tau = \omega_r t \qquad (6)$$

the motion and the excitation with reference to a characteristic amplitude

$$Q = \frac{q}{\hat{q}}; \qquad Q_e = \frac{f(t)}{m}\frac{1}{\omega_r^2 \hat{q}} \qquad (7)$$

the equation of motion becomes

$$Q'' + 2\zeta_0 Q' + \rho(\tau)^2 Q = Q_e(\tau) \qquad (8)$$

The coefficients thus become

$$\rho(\tau) = \frac{\omega(\tau)}{\omega_r}; \qquad \zeta_0 = \frac{c}{2m\omega_r} \qquad (9)$$

The time dependent excitation and natural frequencies are

$$\eta(\tau) = \alpha\tau + \eta_0; \qquad \rho(\tau) = \rho_0 + \beta\tau \qquad (10)$$

respectively where the parameters are η_0, α, ρ_0 and β are given by,

$$\eta_0 = \frac{\Omega_0}{\omega_r}; \quad \alpha = \frac{\dot{\Omega}}{\omega_r^2}; \quad \rho_0 = \frac{\omega_0}{\omega_r}; \quad \beta = \frac{\dot{\omega}}{\omega_r^2} \qquad (11)$$

The normalized transient excitation function is thus

$$Q_e(\tau) = \hat{Q}_e \cos\left[\frac{1}{2}\alpha\tau^2 + \eta_0\tau + \Theta_0\right] \qquad (12)$$

GENERAL SOLUTION

The equation of motion given in equation (12) is a differential equation with variable coefficients which are linear functions of time τ. Because of the variable coefficients, it is evident that the analytical solution for the normalized equation can only be approximate.

The approximate homogeneous solution using WBKJ-method is obtained by eliminating the first derivative term in the equation of motion through transformation with

$$Q = Y e^{-\int \zeta_0 d\tau} \qquad (13)$$

With the transformation the derivatives of the original coordinates will be

$$Q'=-Y\,\zeta_0 e^{-\int\zeta_0 d\tau}+Y'\,e^{-\int\zeta_0 d\tau}$$
$$Q''=\zeta_0^2 Y\,e^{-\int\zeta_0 d\tau}+Y''\,e^{-\int\zeta_0 d\tau}-Y'\,2\zeta_0\,e^{-\int\zeta_0 d\tau} \quad (14)$$

Substituting equation (13 & 14) reduces the normalized equation (8) to

$$Y''+G^2(\tau)Y=Q_e\,e^{\int\zeta_0 d\tau} \quad (15)$$

where,

$$G^2(\tau)=[(\rho_0+\beta\tau)^2\quad\zeta_0^2] \quad (16)$$

Here, since $\zeta_0 \ll (\rho_0+\beta\tau)$ and also $\zeta_0 \ll 1$, the following approximation

$$G(\tau)\approx\sqrt{1-\zeta_0^2}\,[\rho_0+\beta\tau] \quad (17)$$

is ntroduced. Provided the condition

$$G^2 \gg \left|\frac{G''(\tau)}{2G(\tau)}\quad\frac{3}{4}\left(\frac{G'(\tau)}{G(\tau)}\right)\right| \quad (18)$$

is fulfilled, which is proved in [9]. The two homogeneous solutions back transformed are

$$Q_{h1}(\tau)=\frac{1}{\sqrt{G(\tau)}}e^{-\int\zeta_0 d\tau}\sin\Theta_h(\tau)$$
$$Q_{h2}(\tau)=\frac{1}{\sqrt{G(\tau)}}e^{-\int\zeta_0 d\tau}\cos\Theta_h(\tau) \quad (19)$$

whereby the phase using the above approximation for $G(\tau)$ is

$$\Theta_h=\int_0^\tau G(\tau)d\tau=\sqrt{1-\zeta_0^2}(\rho_0\tau+\frac{1}{2}\beta\tau^2) \quad (20)$$

The complete solution of the inhomogeneous normalized equation of motion employing the method of *Variation of the Constants* gives

$$Q(\tau)=C_1Q_{h1}(\tau)+C_2Q_{h2}(\tau)+$$
$$+\frac{1}{\sqrt{G(\tau)}}e^{-\int\zeta_0 d\tau}\int\frac{1}{\sqrt{G(s)}}\,Q_e(s)e^{\int\zeta_0 ds} \quad (21)$$
$$.\sin[\Theta_h(\tau)-\Theta_h(s)]$$

Since G is a function of time, it causes difficulty integrating the solution given above. Therefore, an approximation for the term $\sqrt{G}$ is further introduced by [9].

$$\sqrt{G}=(\sqrt{1-\rho_0^2})^{\frac{1}{2}}, \quad (22)$$

replacing the time function $\rho(\tau)=\rho_0+\beta\tau$ by the mean constant value of one which is the value at resonance.

With the above two approximations, the solution of the normalized equation of motion simplifies in a form

$$Q(\tau)=Q_h+\int_0^\tau Q_e(s)Q_i(\tau,s)ds \quad (23)$$

where

$$Q_i(\tau,s)=\frac{1}{\sqrt{1-\zeta_0^2}}e^{-\zeta_0(\tau-s)}.$$
$$.\sin\left\{\sqrt{1-\zeta_0^2}\left[\rho_0(\tau-s)+\frac{1}{2}\beta(\tau^2-s^2)\right]\right\} \quad (24)$$

For the case where the excitation $Q_e(s)$ is a quasi-harmonic function with a constant amplitude

$$Q_e(s)=\hat{Q}_e\cos\left[\frac{1}{2}\alpha s^2+\eta_0 s+\Theta_0\right] \quad (25)$$

where the phase of the cosine function represents the run-up or shut-down process, the integral in the above solution cannot be given in closed form through simple functions. Thus, the times τ and s are further replaced by complex time functions $v(\tau)$, $u(\tau)$ and $v^*(s)$, $u^*(s)$ respectively, which have linear relationship with τ. The new time variables are

$$v(\tau)=\frac{(1+i)}{2\sqrt{\alpha-\beta^*}}\left[(\alpha\quad\beta^*)\tau+\eta_0+i\lambda_1\right] \quad (26)$$

$$u(\tau)=\frac{(1+i)}{2\sqrt{\alpha+\beta^*}}\left[(\alpha+\beta^*)\tau+\eta_0+i\lambda_2\right] \quad (27)$$

where

$$\lambda_1=-\zeta_0+i\sqrt{1-\zeta_0^2}\rho_0;\ \lambda_2=-\zeta_0-i\sqrt{1-\zeta_0^2}\rho_0; \quad (28)$$

$$\beta^* = \sqrt{1-\zeta_0^2}\,\beta \tag{29}$$

The substitutions for τ, s with v, u and v^*, u^* and also their differentials ds by

$$ds = \frac{(1-i)}{\sqrt{\alpha-\beta^*}}dv^*;\quad ds = \frac{(1-i)}{\sqrt{\alpha-\beta^*}}du^* \tag{30}$$

respectively, yield the homogeneous solution to be

$$\bar{Q}_h = \left[\bar{C}_1 e^{-(v^2-v_0^2)} + \bar{C}_2 e^{-(u^2-u_0^2)}\right] e^{i(\Theta_e(\tau)-\Theta_0)} \tag{31}$$

and the particular integral

$$\bar{Q}_p = \frac{(1+i)}{2\sqrt{1-\zeta_0^2}}\left(\frac{1}{\sqrt{\alpha+\beta^*}}e^{-u^2}\int_{u_0}^{u} e^{u^{*2}}du^* - \frac{1}{\sqrt{\alpha-\beta^*}}e^{-v^2}\int_{u_0}^{v} e^{v^{*2}}dv^*\right)e^{i\Theta_e(\tau)} \tag{32}$$

Finally, replacing the integrals with the known complex error functions $w(v)$, $w(u)$ [5]

$$e^{-v^2}\int_0^v e^{v^{*2}}dv^* = \frac{\sqrt{\pi}}{2i}\left[w(v) - e^{-v^2}\right] \tag{33}$$

and obtaining the values of the constants $\bar{C}_1$ and $\bar{C}_2$

$$\bar{C}_1 = -\frac{i}{2\rho_0\sqrt{1-\zeta_0^2}}[\bar{Q}_0 - \lambda_2\bar{Q}_0];\qquad \bar{C}_2 = \frac{i}{2\rho_0\sqrt{1-\zeta_0^2}}[\bar{Q}_0 - \lambda_1\bar{Q}_0] \tag{34}$$

from the initial conditions of the displacement $\bar{Q}_0$ and velocity $\bar{Q}_0'$, the general solution becomes

$$\bar{Q}(\tau) = \frac{i}{2\sqrt{1-\zeta_0^2}}\left(\bar{B}_1 w(v) - \bar{B}_2 w(u) + \bar{C}_1^* e^{v_0^2-v^2}\ \bar{C}_2^* e^{u_0^2-u^2}\right)e^{i\Theta_e(\tau)} \tag{35}$$

whereby the constants are

$$\bar{B}_1 = \frac{(1+i)}{2}\frac{\sqrt{\pi}}{\sqrt{\alpha-\beta^*}}$$
$$\bar{B}_2 = \frac{(1+i)}{2}\frac{\sqrt{\pi}}{\sqrt{\alpha+\beta^*}}$$
$$\bar{C}_1^* = -\bar{B}_1 w(v_0) - \left[\frac{\bar{Q}_0' - \lambda_2\bar{Q}_0}{\rho_0}\right]e^{-i\Theta_0} \tag{36}$$
$$\bar{C}_2^* = -\bar{B}_1 w(u_0) - \left[\frac{\bar{Q}_0' - \lambda_1\bar{Q}_0}{\rho_0}\right]e^{-i\Theta_0}$$

This solution describes the transient vibration response of the system that occurs during run-up and shut-down processes when the natural frequency of thé system changes simultaneously with time. Thus, the amplitude during the transition is given by

$$\hat{Q}(\tau) = \frac{i}{2\sqrt{1-\zeta_0^2}}\left(\bar{B}_1 w(v) - \bar{B}_2 w(u) + \bar{C}_1^* e^{v_0^2-v^2} - \bar{C}_2^* e^{u_0^2-u^2}\right) \tag{37}$$

which is a complex time dependent function.

NUMERICAL RESULTS

In the previous section an approximate analytical solution to compute the non-stationary vibration that takes place during run-up or shut-down processes in mechanical systems, modeled as a linear SDOF with a linearly changing excitation frequency and simultaneously linearly changing natural frequency, is presented. The verification of the above solution was

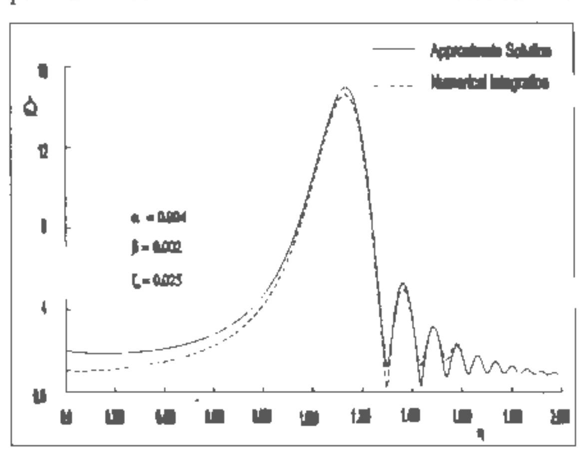

Figure 2 Comparison of the approximate analytical and numerical solutions

done comparing it with the solution obtained from numerical integration method (*Runge-Kutta 4th Order*). A numerical example of the transient vibration envelops is presented in Fig. 2 showing the very small differences between the two solutions.

Using the solution derived above, the influence of a changing natural frequency on the transient vibration during run-up and shut-down and passage through resonance is investigated. As an example, the effects of linearly increasing (β>0) and decreasing (β<0) natural frequencies both on the resonance amplitude and its occurrence with reference to a constant frequency ($\beta=0$), for one value of acceleration (run-up) $\alpha=0.004$ and constant damping $\zeta_0=0.025$ is shown in Fig. 3.

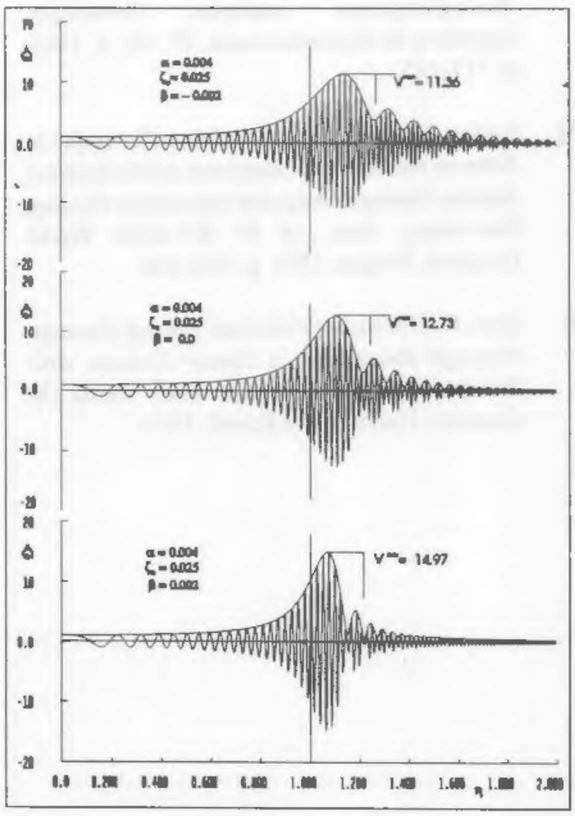

Figure 3 Non-stationary vibration during passage through resonance

From Fig. 3, the observed phenomena as influence of changing natural frequency here represented by β for a given value of acceleration α are:

The transient resonance vibration for the case with changing natural frequency is similar to that with constant natural frequency $\beta=0$, provided the relative slope between the excitation and the eigenfrequency $\alpha^{eff}=\alpha-\beta$ remains the same, as α itself.

The resonance amplitude depends both on the acceleration α and the rate of change of the natural frequency β. The maximum amplitude becomes larger when the $\alpha^{eff}<\alpha$ and is smaller otherwise.

The resonance frequency also depends on α and β. The shifting of the resonance will be smaller to that for the case with $\beta=0$ and approaches the stationary resonance frequency $\eta_r \approx 1$ provided the relative slope $(\alpha-\beta)$ is smaller than α, otherwise it is the reverse.

APPROXIMATE FORMULAS

For the purpose of practical application, many authors [2], [7] etc., suggested approximation formulas for the estimation of the maximum amplitude and the frequency of occurrence. However, these formulas are either limited to undamped case or they are not sufficiently accurate for a wide range of α and ζ_0 values. Furthermore, they are all limited for the case of constant natural frequency ($\beta=0$).

The maximum transient magnification factor and its frequency ratio that occurs during passage through resonance both for the case of constant and varying natural frequency ($\beta\neq0$) can be estimated accurately using the formulas suggested in [8] and [9]. The maximum magnification factor for the run-up (α>0) case can be approximated by

$$V^{max}=\frac{1}{2\zeta_0+0.75\sqrt{\alpha^{eff}}-0.44\sqrt{\zeta_0}\sqrt[4]{\alpha^{eff}}} \tag{38}$$

for shut-down (α<0) case by

$$V^{max}=\frac{1}{2\zeta_0+0.77\sqrt{\alpha^{eff}}-0.58\sqrt{\zeta_0}\cdot\sqrt[4]{\alpha^{eff}}} \tag{39}$$

The frequency ratio η_r at which the maximum response occurs can be approximated by

$$\eta_r=1+\frac{\alpha^{eff}}{0.45\sqrt{|\alpha^{eff}|}+0.25\zeta_0} \tag{40}$$

which is valid both for run-up and shut-down cases.

CONCLUSION

Approximate analytical solution to investigate the non-stationary response of mechanical systems with slowly varying natural frequency was presented. The derived solution was verified comparing it with the results obtained through numerical solution. The resonance amplitudes computed with this method indeed is slightly greater, due to the approximations introduced. However, it is sufficiently accurate for the purpose of investigating the vibration phenomena in linear mechanical systems that occurs during passage through resonances and having parameters varying with time.

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