A FEM ANALYSIS OF A FLEXIBLE MANIPULATOR (ARM)

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ABSTRACT

This paper presents dynamic behaviour analysis and time history of responses of a flexible arm. Lagrange equations are employed to develop the governing equations of motion which are discretized by using the finite element method. An illustrative example of a pin-pin beam-like flexible arm is treated and results obtained are presented.

INTRODUCTION

In most engineering applications, analysis of multi-body systems or mechanisms is done under the assumption that links are rigid. A link is assumed to be rigid if any pair of its material points do not allow relative displacement. In practice, however small the deflection may be, any loaded link is subjected to deformations which, in most cases, may be negligible. But in special applications like spatial structures, robot arms and manipulators, high speed machine elements, etc. where light weight is of great importance, these deformations play an important role in the dynamic analysis. Imposed weight limits on light structural elements result in highly flexible systems. Thus, dynamic analysis of flexible manipulators, such as robot arms, which takes into account the non-rigidity of the elements is essential.

Dynamics of flexible bodies is an area of on-going research and many researchers have concentrated their effort in the analysis of such systems. The methods for the analysis of flexible multi-body systems are divided into three groups:

(a) simplified method based on elasto-dynamics,

(b) methods based on defining deformations with respect to moving reference frames, and

(c) methods that define the overall motion plus deformation with respect to an inertial frame [1].

Sunada and Dubowsky [7] used the simplified elasto-dynamic method in which the deformation is uncoupled from rigid-body motion. This method does not account for coupling terms which may influence the results. For general purpose applications where coupling terms strongly influence the solution, the dynamic analysis is based on the second method. Serena and Bayo [4], Song and Haug [5] applied the second method in which rigid-body variables and deformation variables are used to express the deformation with respect to moving frames. Moreover, they applied finite element discretization of the flexible body, where deformation variables were considered by nodal variables resulting from the finite element discretization. This method has the advantage in that it makes use of known linear finite element theory. The third method uses large deformation theory which develops nonlinear finite elements in the formulation using variables that define rotation, translation and deformation of the body at the same time.

In this paper we will discuss the dynamic formulation of the problem of flexible manipulators based on the method of defining the deformation with respect to a moving frame of reference. Following Serna and Bayo, FEM descretization of the flexible body is used to determine deformation variables from nodal variables. The method presented is then demonstrated by applying the theory to a pin-pin flexible arm. The FEM solution presented herein can easily be extended to include various cross-sectional behaviours and boundary conditions of the flexible arm.

KINEMATICS OF THE PIN-PIN BEAM-LIKE ARM WITH CONSTANT CROSS-SECTION

Consider the beam-like arm OA fixed at O that represents a flexible body which vibrates under pin-pin conditions. The reference system (x,y) is a moving reference system that moves along with the arm. The reference φ characterizes the rigid body motion of the arm. The elastic deformation of the arm is measured relative to the reference φ and the slope of the elastic deformation is measured by the angle θ.

Figure 1 Flexible arm motion behaviour

The velocity of a point P on the arm is obtained by considering both rigid-body motion and elastic deformation of the arm. In component form, V_P is given by

$$\begin{aligned}(V_P)_y &= \dot{v} + \dot{\varphi}(x + u)\\ (V_P)_x &= \dot{u} - \dot{\varphi}v \qquad (1)\\ \dot{\phi} &= \dot{\varphi} + \dot{\theta}\end{aligned}$$

where v and u are local displacements of point P; x and y define the position of P relative to the x-y frame of reference; φ is the rigid-body displacement; θ is the elastic displacement and, $\dot{\varphi}$ and $\dot{\theta}$ are the respective time derivatives. The elastic deformations of the arm are very small in comparison to the overall dimensions of the arm. Hence u is negligible in comparison to x, and neglecting $\dot{\varphi}v$ for simplicity, the velocity components are obtained to be

$$\begin{aligned}V_P = (V_P)_y &= \dot{v} + \dot{\varphi}x \qquad (2)\\ \dot{\phi} &= \dot{\varphi} + \dot{\theta}\end{aligned}$$

KINETIC AND POTENTIAL ENERGY OF THE ARM

Kinetic energy of the flexible arm is obtained from

$$T = \frac{1}{2}\int_m dm\cdot(V_P)^2 + \frac{1}{2}\int_m dm\cdot(y\dot{\phi})^2 \qquad (3)$$

Substituting for the mass dm and V_P, we obtain the kinetic energy equation as

$$T = \frac{1}{2}\int_0^L [\rho A(\dot{v} + \dot{\varphi}\,x)^2 + \rho I(\dot{\varphi} + \dot{\theta})^2]\,dx \qquad (4)$$

where

I is the area moment of inertia, and
$dm = \rho A dx$
A is the cross-sectional area.

Simplifying the right hand side yields

$$T = \frac{1}{2}\Big[\,\dot{\varphi}^2\int_0^L \rho A x^2 dx + \int_0^L \rho A \dot{v}^2 dx + 2\int_0^L \rho A \dot{v}\dot{\varphi}x dx + \dot{\varphi}^2\int_0^L \rho I dx + \int_0^L \dot{\theta}^2 \rho I dx + 2\int_0^L \rho I \dot{\varphi}\dot{\theta} dx\,\Big] \qquad (5)$$

The terms $2\int_0^L \rho I\dot{\varphi}\dot{\theta}dx$ and $\int_0^L \dot{\theta}^2\rho I dx$ are coupling and rotary inertia terms, respectively, which are neglected in Bernouli-Euler beam theory. Moreover, the term $\dot{\varphi}^2\int_0^L \rho I dx$ is negligible in comparison to the first term on the right hand side of Eq. (5). Thus, assuming Bernouli-Euler beam theory, the kinetic energy of the system simplifies to

$$T = \frac{1}{2}\Big[\dot{\varphi}^2\int_0^L \rho A x^2 dx + \int_0^L \rho A \dot{v}^2 dx + 2\int_0^L \rho A \dot{v}\dot{\varphi}x dx\Big] \qquad (6)$$

Consider the four-element finite element model of the arm shown in Fig. 2. It is worth noting here that for the pin-pin model the vertical displacement at the fixed end O and the tip A are zero. For a C^1 beam, introducing Hermite interpolation functions $H(x)$ given in Eq. (8) [3] and the element discrete nodal displacements shown in Fig. 3, the beam displacement $v(x)$ is obtained from

$$v(x) = H(x)\bar{v} \qquad (7)$$

where

$$\bar{v} = \begin{bmatrix}\theta_1 & v_1 & \theta_2 & v_2\end{bmatrix}$$

and

$$
\begin{aligned}
H_1 &= x - 2L_e\left(\frac{x}{L_e}\right)^2 + L_e\left(\frac{x}{L_e}\right)^3 \\
H_2 &= 1 - 3\left(\frac{x}{L_e}\right)^2 + 2\left(\frac{x}{L_e}\right)^3 \\
H_3 &= -L_e\left(\frac{x}{L_e}\right)^2 + L_e\left(\frac{x}{L_e}\right)^3 \\
H_4 &= 3\left(\frac{x}{L_e}\right) - 2\left(\frac{x}{L_e}\right)
\end{aligned}
\tag{8}
$$

From Eq. (7) we obtain

$$
\begin{aligned}
\dot{v}(x) &= H(x)\,\dot{\bar{v}} \\
[\dot{v}(x)]^2 &= \dot{\bar{v}}^T[H(x)]^T H(x)\dot{\bar{v}}
\end{aligned}
$$

Substituting into Eq. (6), the kinetic energy is obtained to be

$$
\begin{aligned}
T = \frac{1}{2}\,[I_0\dot{\varphi}^2 &+ \\
&\dot{\bar{v}}^T\left(\int_0^L [H(x)]^T\rho A[H(x)]dx\right)\dot{\bar{v}} + \\
&2\dot{\varphi}^2\left(\int_0^L \rho Ax[H(x)]dx\right)\dot{\bar{v}}]
\end{aligned}
$$

Or,

$$
T = \frac{1}{2}I_o\dot{\varphi}^2 + \frac{1}{2}\dot{\bar{v}}^T M_f\,\dot{\bar{v}} + \dot{\varphi}M_c\dot{\bar{v}} \tag{9}
$$

where the finite element mass M_f is given by

$$
M_f = \int_0^{L_e}\rho A[H(x)]^T[H(x)]dx
$$

and the finite element coupling mass is

$$
M_c = \int_0^{L_e}\rho Ax[H(x)]dx
$$

Figure 2 Four-element model of the pin-pin arm

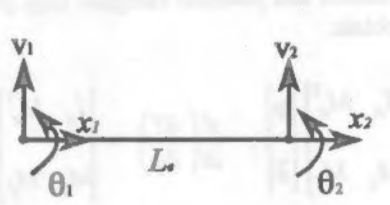

Figure 3 Element nodal displacements

For the beam element shown in Fig. 2, the displacement x is obtained from $x = H(x)\bar{r}$ where $\bar{r} = [x_1 \;\; 1 \;\; x_2 \;\; 1]$ is the vector of discrete element displacements at the nodes for the rigid body motion. Substituting for x in the element coupling mass, it can be demonstrated that the element coupling mass $M_c^{(e)}$ is

$$
M_c^{(e)} = \int_0^{L_e}\rho AxH(x)dx = \bar{r}^T M_f^{(e)} \tag{10}
$$

The potential energy of the arm, V, neglecting the deformation in the u-direction, is given by

$$
V = \frac{1}{2}\int_0^{L_e} EI[v(x)'']^2 dx \tag{11}
$$

where $v(x)''$ is the derivative of $v(x)$ with respect to x. Substituting for $v(x)''$ and simplifying, we obtain

$$
V = \frac{1}{2}\bar{v}^T K_f\,\bar{v} \tag{12}
$$

where K_f, the element stiffness matrix, is given by

$$
K_f = \int_0^{L_e}[H''(x)]^T EI[H''(x)]dx \tag{13}
$$

EQUATION OF MOTION

Equation of motion of the system is obtained from energy considerations by applying Lagrange equations.

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q \tag{14}
$$

where $q = \begin{bmatrix}\varphi \\ - \\ \bar{v}\end{bmatrix}$, and $\dot{q} = \begin{bmatrix}\dot{\varphi} \\ - \\ \dot{\bar{v}}\end{bmatrix}$.

From the kinetic and potential energies, Eqs. (9) and (12), we obtain:

$$\frac{\partial T}{\partial \dot{q}} = \begin{bmatrix} I_A & M_c^T \\ M_c & M_f \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\bar{v}} \end{bmatrix}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \begin{bmatrix} I_A & M_c^T \\ M_c & M_f \end{bmatrix} \begin{bmatrix} \ddot{\varphi} \\ \ddot{\bar{v}} \end{bmatrix}$$

$$\frac{\partial T}{\partial q} = 0 \ , \quad \frac{\partial V}{\partial q} = K_f \bar{v}$$

Substituting into Eq. (14), the equation of motion in matrix form is written as

$$\begin{bmatrix} I_A & M_c^T \\ M_c & M_f \end{bmatrix} \begin{bmatrix} \ddot{\varphi} \\ \ddot{\bar{v}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_f \end{bmatrix} \begin{bmatrix} \varphi \\ \bar{v} \end{bmatrix} = [Q] \qquad (15)$$

where

M_f is $n \times n$ mass matrix as obtained from the finite element model;

I_O is the mass moment of inertia of the arm about O;

K_f is $n \times n$ stiffness matrix;

Q is $(n+1) \times 1$ load vector;

v is the vector of nodal point displacements; and

φ is the rigid body rotation of the arm.

Or

$$M\ddot{q} + Kq = Q \qquad (16)$$

where

M is $(n+1) \times (n+1)$ mass matrix, and

K is $(n+1) \times (n+1)$ stiffness matrix.

NUMERICAL EXAMPLE

Consider the flexible arm OA shown in Fig. 4(a) subjected to the torque T applied at pin O for the first $0.5s$ with the magnitude shown in Fig. 4(b). The dynamic behaviour and time history response of the arm are analyzed for $2s$.

Arm characteristics are:

$$l = 1.5m, \quad I = 10^{-10} m^4, \quad A = 10^{-4} m^2,$$

$$E = 2 \cdot 10^{11} \frac{N}{m^2}, \quad \rho = 8000 \frac{kg}{m^3}$$

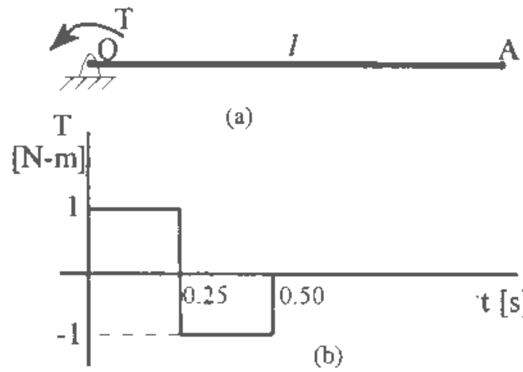

Figure 4 A flexible pin-pin arm subjected to a torque

The flexible arm OA is modelled by a four-element FEM model shown in Fig. 2. For each element of the modelling, the element mass matrix is obtained by numerical integration using Gauss quadrature with four sampling points from the equation

$$M_f^{(e)} = \int_0^{L_e} \rho A H(x)^T H(x) dx$$

The element coupling mass vector $M_c^{(e)}$ is calculated from

$$M_c^{(e)} = \int_0^{L_e} \rho A x H(x) dx - \bar{r}^T M_f^{(e)}$$

Using four sampling points for calculating the element mass improves accuracy.

For element 1,

$$\bar{r}^T = [0 \quad 1 \quad 0.375 \quad 1]$$

Hence, $M_c^{(e)}$ for element 1 is obtained to be

$$M_c^{(1)} = [0 \quad 1 \quad 0.375 \quad 1] \left[M_f^{(e)}\right] - [0.0169 \quad 0.0014 \quad 0.0349 \quad -0.0021]^T$$

rrying out this operation for the remaining three elements and assembling, we obtain the global coupling mass matrix M_c to be

$$M_c = [0.0014 \quad 0.1125 \quad 0.0028 \quad 0.2250 \quad 0.0028 \quad 0.3375 \quad 0.0028 \quad -0.0126]^T$$

The element stiffness matrix is obtained again by numerical integration using Gauss quadrature with two sampling points from the equation

$$K_f = \int_0^{L_e} [H''(x)]^T EI [H''(x)] dx$$

The global mass and stiffness matrices are obtained by assembling the element matrices. Thus the equation of motion in a discretized form can now be written as

$$M\ddot{q} + Kq = Q \qquad (17)$$

where M and K are the global mass and stiffness matrices; Q is the forcing vector; and q is the vector of the nodal displacements and rigid body rotation. The stiffness matrix K has zero entries in the first row and first column. Thus to be able to determine the natural frequencies of the system, the first element of K is given a very small magnitude to eliminate singularity conditions.

Eq. (17) is solved by using Newmark algorithm [2] with the initial conditions given by $q(0) = \dot{q}(0) = \ddot{q}(0) = 0$.

RESULTS

1. Linear displacement of the tip A is obtained from rigid-body motion and elastic deformation of the arm, and is given by

$$v_t = L\varphi_t.$$

Flexibility of the arm is included in the determination of φ_t.

2. Angular displacement of the arm is composed of rigid body motion and the elastic deformation and is given by

$$\varphi_t = \varphi + \theta.$$

Rigid-body motion characteristics of the arm are shown in Fig. 5. Plot of the linear displacement of the tip of the flexible arm is shown in Fig. 6. Figure 7 represents the hub rotation of the flexible model.

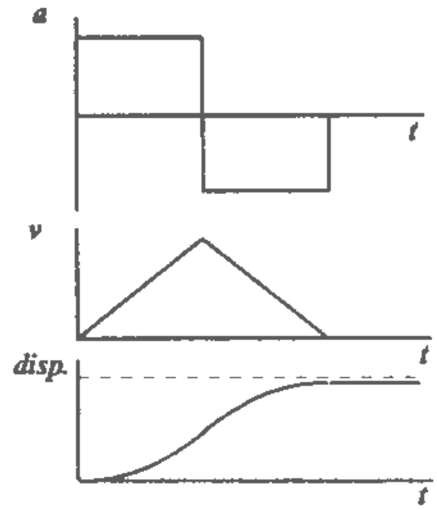

Figure 5 Motion characteristics of the rigid model

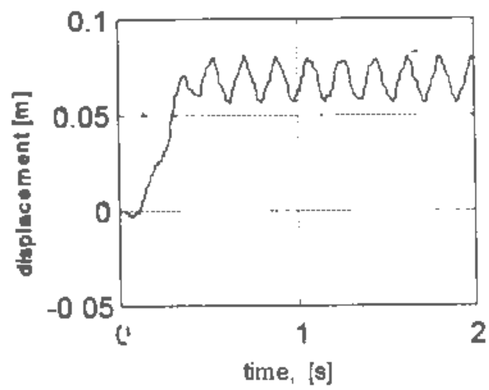

Figure 6 Total displacement of the tip of the arm

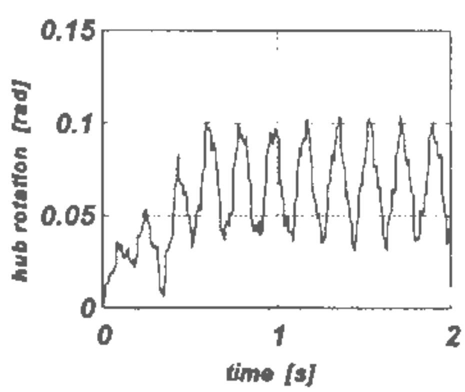

Figure 7 Total hub rotation

CONCLUSION

The vibratory responses of the flexible arm, shown in Figures 6 and 7, include both rigid-body motion and elastic deformations. These responses indicate that the arm undergoes free-vibration about the deflected position which results from the flexibility of the arm. This result is expected from physical considerations of the arm. For the first few hundredths of a second, due to the inertia of the arm, the linear displacement tends to be negative with a very small magnitude which then increases to vibrate about the equilibrium position.

In robotic applications, this analysis forms the basis toward vibration control of the tip and any other point of interest on the arm, where unwanted vibrations may introduce errors in the determination of the exact positions of points of interest.

Further, the approach presented herein can be used for the analysis of flexible arms with variable cross-sections and various boundary conditions.

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