COMPUTATIONALLY EFFICIENT ANALYSIS PROCEDURE FOR FRAMES WITH SEGMENTED AND LINEARLY-TAPERED CROSS SECTIONS

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ABSTRACT

Computationally-efficient analytical procedure that provides high-quality analysis results for two-dimensional skeletal structure with segmented (stepped) and linearly-tapered non-prismatic flexural members has been developed based on the stiffness method of structural analysis. A computer program coded in FORTRAN 90 has been developed to facilitate the computational procedure. Concentrated and uniformly-distributed loads have been taken into consideration in the development of the computational procedure. The stiffness coefficients have been formulated employing both closed-form and numerical techniques. High quality computational procedures have been established to handle closed-form analytical solutions that have been observed to produce results that are close to theoretical values and that generally surpass those from commercially available software products.

A numerical example has been presented to demonstrate the accuracy of the proposed computational procedure and related computer routines. The accuracy of the proposed analysis technique has been verified by comparing the results with theoretical results and those obtained using a commercial structural analysis software system.

Keywords: *Computer-oriented analysis, Linearly-tappered cross-sections, Step-wise segmented cross-sections, Rigid frames, Computational efficiency, High-quality computations.*

INTRODUCTION

Beams, columns and other flexural structural members of large-span systems, such as industrial facilities, assembly halls, aircraft maintenance hangars and bridges call for the use of relatively deeper cross-sections in order to provide effective resistance to the effects of loads and to keep deflections and other structural response quantities within acceptable limits. Efficient and economical response to such demands can be met by using non-prismatic members that may have segmented or tapered cross sections or combinations of both types. The use of non-prismatic members in civil engineering structures usually results in reduced structural self-weight thereby alleviating the need to provide undesirably large cross sections that would have otherwise resulted in heavy and uneconomical structures in addition to loss of head-rooms or hydraulic clearances, among others.

The shapes of cross-sections along the respective length of members significantly influence the analysis of structural systems that are composed of non-prismatic members, among others. This work explores the concept behind analysis methods and techniques of structures with non-prismatic members with the view of providing design engineers with a tool for solving such structural systems and, thus, to come up with safe, efficient and economical designs.

STEPPED- AND LINEAR-HAUNCH NON-PRISMATIC MEMBERS

There are a number of non-prismatic shape types and members that are employed in various structural engineering applications. In view of the ease with which they can be constructed, the most frequent types are those with stepped haunches and those with linearly-varying haunches as shown in Fig. 1.

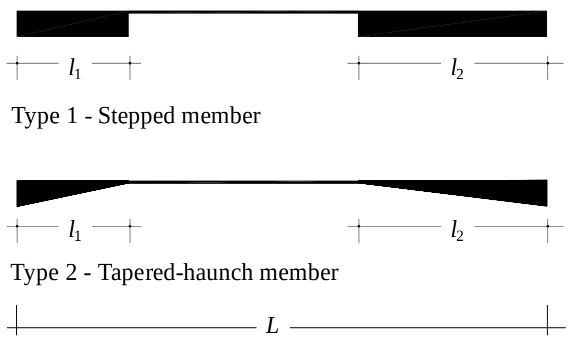

Figure 1 Non-prismatic members

Depending on the length of the left and right haunches – designated l_1 and l_2, respectively – one may get various forms of structural elements employing different haunch topologies as shown in Table 1.

Table 1: Types of non-prismatic member topologies (for notations, refer to Fig. 1)

Type 1: Stepped	
l_1 , l_2	Shape
$l_1 \neq 0$. $l_2 \neq 0$, L-l_1- $l_2 > 0$	
$l_1 = 0$ $l_2 \neq 0$, $l_2 < L$	
$l_1 \neq 0$ $l_2 = 0$, $\alpha < L$	
$l_1 = 0$ $l_2 = 0$	

Type 2: Linearly tapered	
l_1 , l_2	Shape
$l_1 \neq 0$ $l_2 \neq 0$, L- l_1- $l_2 > 0$	
$l_1 = 0$ $l_2 \neq 0$, $l_2 <$ L	
$l_1 \neq 0$ $l_2 = 0$, $l_1 <$ L	
$l_1 = 0$ $l_2 = L$	
$l_1 = L$ $l_2 = 0$	
L- l_1- $l_2 = 0$ $l_1 \neq 0$, $l_2 \neq 0$	

ANALYSIS OF STRUCTURES WITH NON-PRISMATIC MEMEBERS

In implementing the stiffness method of structural analysis [1], member stiffness terms and matrices and, subsequently, the structural stiffness matrices, play a central role in the formulation of equilibrium equations. Member-end actions and, subsequently, the structural nodal-load vectors, are the other components that enter into the equilibrium equations in the solution process. Both the structural stiffness matrix and the corresponding load vectors are influenced by the cross-sectional geometries of the non-prismatic members contained in the structural system. Accordingly, subsequent sections of this paper present the formulation of both stiffness matrices and load vectors as related to structures with non-prismatic members of both stepped- and tapered-haunch types.

Member stiffness matrix in local coordinates

Forces and deformations in structures are related to one another by means of stiffness influence coefficients.

Figure 2 shows a frame element, Member *i*, that is fully restrained at both ends, which are denoted as ends *j* and *k*. Member-oriented axes are designated by x_m, y_m, z_m. The x_m axis is in the direction of the member and is positive in the sense from *j* to *k* while the orientations of the other axes can be established using the right-hand rule as shown in Fig. 2.

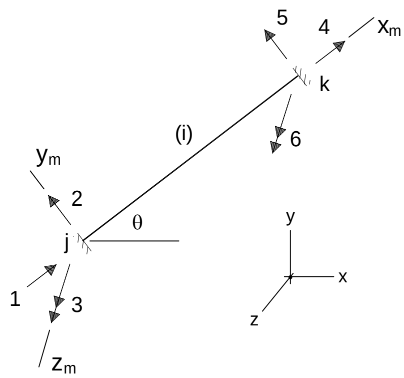

Figure 2 Spatial frame element, local and global coordinates

The various member stiffness terms for the restrained member are also indicated in Fig. 2. They consist of actions exerted on the member by the restraints when unit displacements (translations or rotations) are imposed at each end of the member and are assumed to be positive in the x_m, y_m, z_m directions. Thus, the arrows in Fig. 2 indicate the positive senses of the three translations and rotations at each end of the member; the single headed arrows denote translations while the double-headed arrows represent rotations.

The stiffness matrix in the member coordinate for any plane-frame member may be designated by $\mathbf{k}_m$ and can be presented as follows [2].

$$k_m = \frac{E}{\int_0^L dx/I_{z.x} \int_0^L x^2 dx/I_{z.x} - \left(\int_0^L x dx/I_{z.x}\right)^2} \begin{bmatrix} k_{m11} & 0 & 0 & k_{m14} & 0 & 0 \\ & k_{m22} & k_{m23} & 0 & k_{m25} & k_{m26} \\ & & k_{m33} & 0 & k_{35} & k_{m36} \\ & & & k_{m44} & 0 & 0 \\ & & & & k_{m55} & k_{m56} \\ & & & & & k_{m66} \end{bmatrix} \quad (1a)$$

where each of the stiffness terms is given by the following expressions:

$$k_{m11} = \frac{\int_o^L dx/I_{z.x} \int_0^L x^2 dx/I_{z.x} - \left[\int_0^L x\,dx/I_{z.x}\right]^2}{\int_0^L dx/A_x} \qquad k_{m14} = -k_{m11}$$

$$k_{m22} = \frac{1}{L^2}\left[\int_0^L x^2 dx/I_{z.x} + 2\int_0^L (L-x)x\,dx/I_{z.x} + \int_0^L (L-x)^2\,dx/I_{z.x}\right]$$

$$k_{m23} = \frac{1}{L}\left[\int_0^L x^2 dx/I_{z.x} + \int_0^L (L-x)\,x\,dx/I_{z.x}\right] \quad (1b)$$

$$k_{m33} = \int_0^L x^2 dx/I_{z.x} \qquad k_{m36} = \int_0^L (L-x)\,x\,dx/I_{z.x}$$

$$k_{m14} = -k_{m11} \qquad k_{m25} = -k_{m22} \qquad k_{m26} = k_{m23}$$

$$k_{m35} = k_{23} \qquad k_{m44} = k_{m11} \qquad k_{m55} = k_{m22}$$

$$k_{m56} = -k_{m23} \qquad k_{m66} = k_{m36}$$

In Eqs. (1), A_x and $I_{z.x}$ represent the cross-sectional area and moment of inertia, respectively, of the member under consideration and they are expressed as a function of the variable cross-sectional dimensions of the non-prismatic section at the point of interest along the member axis x_m.

The expression given by Eq. (1) are perfectly general and may be used for any cross section as long as expressions for the moment of inertia $I_{z.x}$ and cross-sectional area A_x can be explicitly established and where the integral equation can be evaluated in closed analytical format (see Annex A).

The integral values that one needs for evaluating the stiffness terms are the following:

$$U_0 = \int_0^L \frac{1}{A_x} dx \qquad U_1 = \int_0^L \frac{dx}{I_{z.x}} \quad (2)$$

$$U_2 = \int_0^L \frac{x\,dx}{I_{z.x}} \qquad U_3 = \int_0^L \frac{x^2 dx}{I_{z.x}}$$

$$U_4 = \int_0^L \frac{(L-x)\,x\,dx}{I_{z.x}} = L\int_0^L \frac{xdx}{I_{z.x}} - \int_0^L \frac{x^2 dx}{I_{z.x}}$$

$$U_5 = \int_0^L \frac{(L-x)^2 dx}{I_{z.x}} = \int_0^L \frac{L^2 dx}{I_{z.x}} - \int_0^L \frac{2Lx\,dx}{I_{z.x}} + \int_0^L \frac{x^2 dx}{I_{z.x}}$$

$$U_6 = \int_0^L \frac{dx}{I_{z.x}} \int_0^L \frac{x^2 dx}{I_{z.x}} - \left(\int_0^L \frac{xdx}{I_{z.x}}\right)^2$$

(2 cont'd)

Member stiffness terms and, subsequently, the member stiffness matrix can then be determined easily once the above integrals are evaluated. Details of evaluation procedure for these expressions are given in Annex A.

The element stiffness matrix is now ready to enter the assembly process to form the structural stiffness matrix as will be discussed in subsequent sections.

Member fixed-end equivalent actions in local coordinates

Structural elements are usually subjected to various kinds of loads including generalized forces and displacements both in terms of distribution and direction. In the formulation of analysis equations of structural elements of the bending type as in the case of plane frames, the effect of loads are taken care of by converting all span loads into equivalent joint loads that may consist of direct forces and flexural moments. The latter will then be assembled into nodal load vectors with reference to the member axis for an eventual expression in structural coordinates. Before formulating the nodal load vectors, however, one needs to find fixed-end actions when such generalized set of span loads are applied to different types of non-prismatic members.

This paper concentrates on such loads of most common types for the setup of analysis equations; these are concentrated and distributed loads; in the latter case, both uniformly distributed and other types will be considered. Detailed formulation with reference to other loading types may be found elsewhere [2].

Concentrated loads

The member shown in Fig. 3 has variable flexural rigidity along its length while its two ends are fixed and it is subjected to one concentrated load as shown in the figure. The case of multiple concentrated loads will not introduce any difficulty as they will be taken care of by the superposition principle of elastic analysis.

When a beam is fixed at both ends, the slopes of the tangents to the elastic curve at the ends equal to zero and, according to conjugate beam principles [3], they may be expressed as the end reactions of a simply supported beam loaded with *M*/*EI* diagram.

While fixed-end moments can be determined in this manner, the limit of various parts of the integrals will, however, depend upon position of the load; that is, whether it is located within one region of the haunches or in the intervening straight part of the member. Figure 3 shows the moment diagram for a fixed end beam with a concentrated load P, placed at a distance dL from support *j* and d_1L from support *k*.

In Fig. 3, the following parameters have been used to designate various ordinates in the moment diagram:

$$h_1 = \frac{M_j x_1}{L} \qquad h_2 = \frac{M_k x}{L} \qquad h_3 = P\, d_1 x \qquad h_4 = P\, d\, x_1$$

The reaction at the *k*-end of the member due to the "*M*/*EI* loading" should satisfy the following relationships:

$$\int_j^k \frac{M_j}{EI_{z.x}} \frac{x_1}{L} x\mathrm{d}x + \int_j^k \frac{M_k}{EI_{z.x}} \frac{x}{L} x\mathrm{d}x + \int_j^{dL} \frac{Pd_1 x^2}{EI_{z.x}} \mathrm{d}x + \int_{dL}^k \frac{Pd(L-x)x}{EI_{z.x}} \mathrm{d}x = 0 \tag{3a}$$

$$M_j \int_0^L \frac{(L-x)x}{I_{z.x}} \mathrm{d}x + M_k \int_0^L \frac{x^2}{I_{z.x}} \mathrm{d}x + Pd_1 L \int_0^{dL} \frac{x^2}{I_{z.x}} \mathrm{d}x + PdL \int_{dL}^L \frac{(L-x)x}{I_{z.x}} \mathrm{d}x = 0 \tag{3b}$$

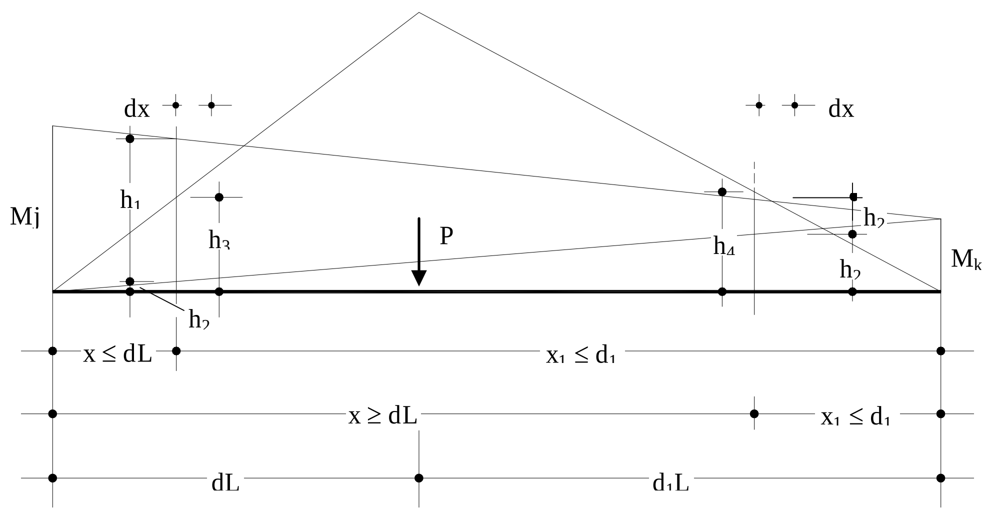

Figure 3 Bending moment diagram under a concentrated load

Similarly, the reaction or slope at *j* of *M*/*EI* diagram should satisfy:

$$\int_k^j \frac{M_j}{EI_{z.x}} \frac{x_1^2}{L} dx_1 + \int_k^j \frac{M_k}{EI_{z.x}} \frac{x}{L} x_1 dx_1 + \int_k^{d_1L} \frac{Pdx_1^2}{EI_{z.x}} dx_1 + \int_{d_1L}^{L} \frac{Pd_1x_1x}{EI_{z.x}} dx_1 = 0 \tag{3c}$$

$$M_j \int_0^L \frac{(L-x)^2}{I_{z.x}} dx + M_k \int_0^L \frac{(L-x)x}{I_{z.x}} dx + PdL \int_0^{dL} \frac{x_1^2}{I_{z.x}} dx + Pd_1L \int_{dL}^{L} \frac{(L-x_1)x_1}{I_{z.x}} dx_1 = 0 \tag{3d}$$

We have seen before the first two terms of Eqs. (3) and their evaluations. The third and fourth terms must be integrated in parts depending upon the location of the load in reference to the haunches.

Therefore, the fixed end moments at the two ends become:

$$M_j = -PL\,(b_2c_1 - b_1c_2)/(a_1b_2 - a_2b_1) \tag{4a}$$

$$M_k = -PL\,(a_1c_2 - a_2c_1)/(a_1b_2 - a_2b_1) \tag{4b}$$

where

$$a_1 = \int_0^L \frac{(L-x)\,x}{I_{z.x}} dx \qquad b_1 = \int_0^L \frac{x^2}{I_{z.x}} dx \qquad c_1 = (d_1e_2 + df_1)$$

$$a_2 = \int_0^L \frac{(L-x)^2}{I_x} dx \qquad b_2 = \int_0^L \frac{x\,(L-x)}{I_{z.x}} dx \qquad c_2 = (d_1e_2 + d_1f_1)$$

$$e_1 = \int_0^{dL} \frac{x^2}{I_{z.x}} dx \qquad f_1 = \int_0^{d_1L} \frac{(L-x_1)\,x_1}{I_{z.x}} dx_1$$

$$e_2 = \int_0^{dL} \frac{x_1^2}{I_{z.x}} dx_1 \qquad f_2 = \int_0^{dL} \frac{(L-x)x}{I_{z.x}} dx \tag{4c}$$

These integrals a_1, a_2, b_1, b_2, e_1, e_2, f_1, and f_2 must be carried out in parts depending up on the location of the load.

Uniformly- and variably-distributed loads

Other common loading types are the uniformly distributed load acting fully or partially on the member and to a lesser extent variably distributed loads either over the entire member or on part of it.

The derivations provided earlier for concentrated loads can easily be extended to distributed loads either making use of closed-form analytical solutions or by numerical integration techniques [4].

Thus, for example, for a member subjected to uniformly distributed direct load across its entire length, the fixed-end moments will become [2]:

$$M_j = \frac{W}{2} \frac{\left[\int_0^L \frac{x^2}{I_{z.x}} dx\right]^2 - \int_0^L \frac{x}{I_{z.x}} dx \int_0^L \frac{x^3}{I_{z.x}} dx}{\int_0^L \frac{x^2}{I_{z.x}} dx \int_0^L \frac{dx}{I_x} - \left[\int_0^L \frac{x}{I_{z.x}} dx\right]^2} \tag{5a}$$

$$M_k = \frac{W}{2} \frac{\left[L^2 \int_0^L \frac{x}{I_{z.x}} dx\right]^2}{\int_0^L \frac{x^2}{I_{z.x}} dx \int_0^L \frac{dx}{I_{z.x}} - \left[\int_0^L \frac{x}{I_{z.x}} dx\right]^2}$$

$$+ \frac{W}{2} \frac{\int_0^L \frac{x^2}{I_{z.x}} dx \left[\int_0^L \frac{x^2}{I_{z.x}} dx - L\int_0^L \frac{x}{I_{z.x}} dx - L^2 \int_0^L \frac{dx}{I_{z.x}}\right]}{\int_0^L \frac{x^2}{I_{z.x}} dx \int_0^L \frac{dx}{I_{z.x}} - \left[\int_0^L \frac{x}{I_{z.x}} dx\right]^2}$$

$$+ \frac{W}{2} \frac{\int_0^L \frac{x^3}{I_{z.x}} dx \left[L\int_0^L \frac{dx}{I_{z.x}} - \int_0^L \frac{x}{I_{z.x}} dx\right]}{\int_0^L \frac{x^2}{I_{z.x}} dx \int_0^L \frac{dx}{I_{z.x}} - \left[\int_0^L \frac{x}{I_{z.x}} dx\right]^2} \tag{5b}$$

Fixed-end moments for general forms of loading can be obtained in a similar manner [2].

Once the fixed end moments are determined, direct reactions along and perpendicular to the member axis, can be easily determined from conditions of static equilibrium.

We are now ready to establish the vector of member-end actions in local coordinates. Thus, member end actions $\mathbf{A}_m$ as shown in Fig. 1 can be established as follows:

$$\mathbf{A}_m = \{A_{m1} \quad A_{m2} \quad A_{m3} \quad A_{m4} \quad A_{m5} \quad A_{m6}\}^T \tag{6}$$

where A_{m1} and A_{m2} are forces in the x_M and y_m directions (axial and shear), respectively, and A_{m3} is the flexural moment in the z_m sense, all at the *j* end. Similarly, A_{m4}, A_{m5} are A_{m6} are the corresponding values at the *k* end of the member.

Analysis Procedure

The analysis procedure for plane frames with non-prismatic members is identical to that of its counterparts with prismatic members and, hence, will not introduce any new procedure at structural level. Thus, the basic equation for relating actions and displacement at joints is

$$\mathbf{A}_J = \mathbf{K}_J \, \mathbf{D}_J \qquad (7)$$

The matrix $\mathbf{K}_J$ can be identified as the *joint stiffness matrix* relating the action $\mathbf{A}_J$ to the displacement $\mathbf{D}_J$.

The member stiffness matrices and the member end actions, both transformed into structural coordinate as presented in Annex B, will be assembled to form structural joint stiffness matrix and the structural nodal load vector, respectively, following which the various structural response quantities are determined based on the procedures of matrix structural analysis [5].

COMPUTER PROGRAM LOGIC

A computer program for the analysis of two-dimensional plane frame structures with non-prismatic members of the types shown in Fig. 1 and for a wide variety of external loads has been developed in FORTRAN 90 and tested [6]. The significant differences for prismatic and non-prismatic structural elements lie in the establishment of member stiffness matrices and equivalent nodal-load vectors. The relevant programming routines for stiffness and member-end actions computations for both stepped (segmented) and linearly-varying (straight or tapered) flexural members have been provided in Ref. 6. Computational details for the establishment for stiffness terms and fixed-end actions for such members are further provided in Annex B.

The flow chart given in Fig. 4 provides the overall programming logic implemented for the coding of the program.

NUMERICAL EXPERIENCE

The capabilities and output qualities of the computer program developed based on the procedures outlined this far will now be demonstrated on a two-way concrete bridge frame with linearly varying (tapered) girder members in both spans as shown in Fig. 5 [7]. The result will be compared with those obtained from theoretical manual calculations based on the Cross method of moment distribution [8] and those from a commercial software system [9].

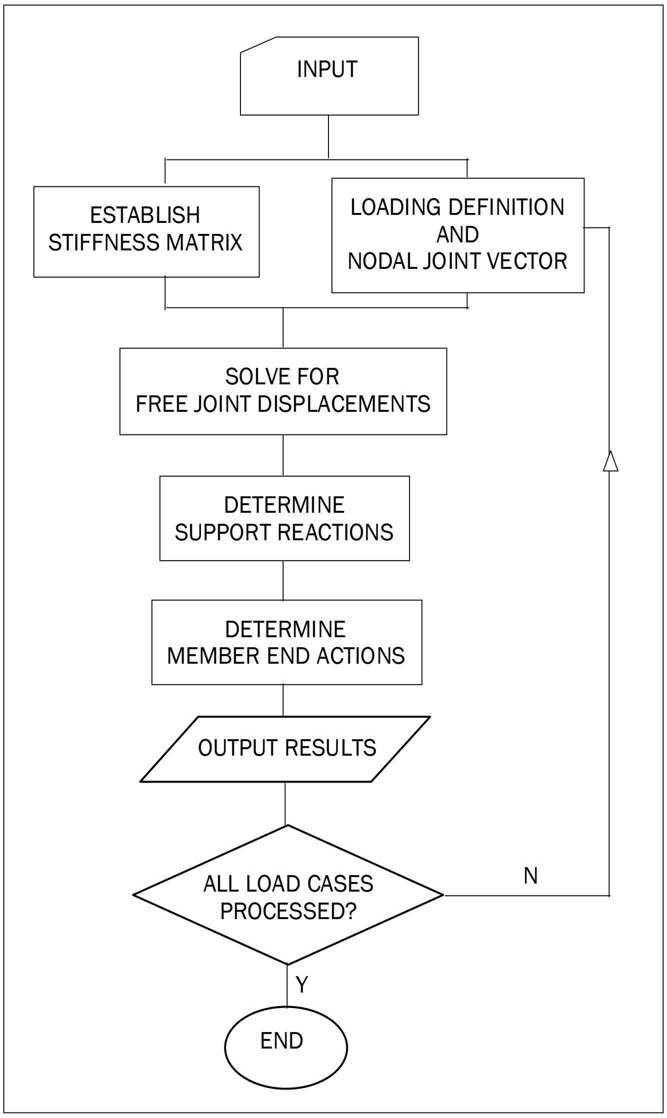

Figure 4 Computation flow chart

Flexural moments at each end of a particular member as obtained from the three analysis techniques are summarized in Table 2. For the sake of comparison, the bending moment diagrams by all the three approaches have been sketched in Fig. 6.

Table 2: Summary of flexural moments

Joint	Member	Theory [7]	Present work	STAADPro
A	AD	12.59	13.74	19.20
B	BE	367.75	370.70	318.90
C	CF	30.73	31.99	31.75
D	DA	-368.53	-367.92	-366.38
	DE	368.53	367.92	366.38
E	ED	-757.81	-759.07	-718.25
	EB	563.39	565.69	521.89
	EF	194.42	193.38	196.37
F	FE	19.73	18.91	15.39
	FC	-19.73	-18.91	-15.39

One can easily observe that the results obtained by all the three approaches are comparable; indeed the procedure proposed in this work produces results that are closer to the theoretical ones compared to those obtained using the commercial software [9]. One reason could be that the present work is base on finding the solution of various integral equations in a closed analytical format, thus, providing much better results.

Other structural response quantities such as axial and shear forces as well as the various types of displacements are also in close agreement.

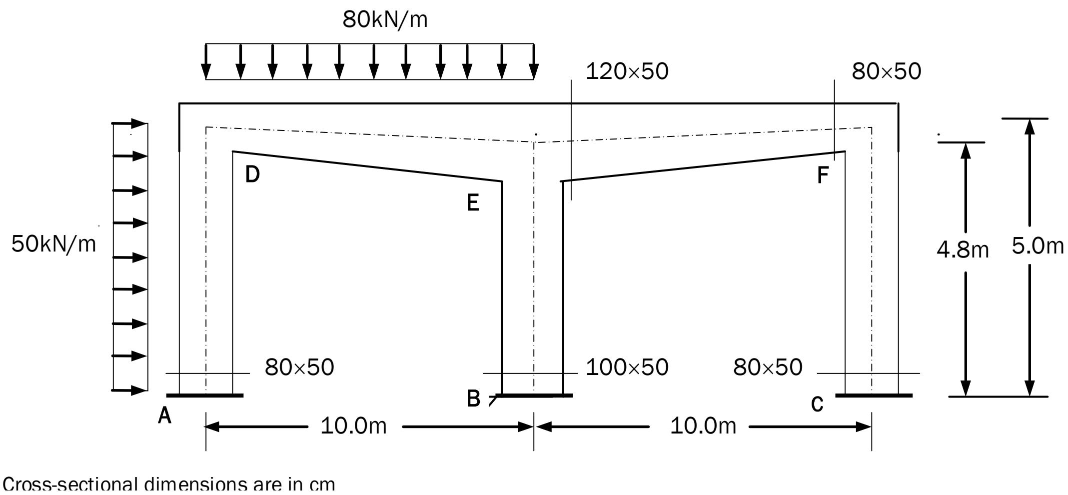

Figure 5 A bridge frame for numerical example

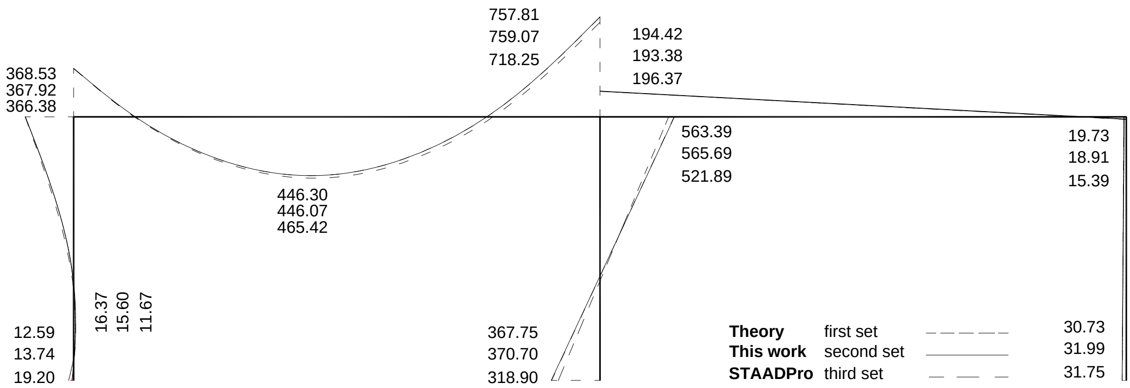

Figure 6 Moment diagrams by each of the three analysis options

CONCLUSION

Economical design of frame-type structures may be attained by reducing member cross-sectional dimensions in the low-moment regions by introducing non-prismatic members whose shape follows the pattern (and, thus, the magnitude) of the flexural moment. In this work, a computer program for the analysis of non-prismatic members with step-wise and linearly-varying cross-sections has been developed and tested. Automatic generation of generalized member-end actions in the form of axial and shear forces as well as flexural moments has been implemented for frequently encountered load cases. The later may also be extended and developed to incorporate a variety of other load cases. The program is easy to use and input-output information is printed in a separate file.

The procedure and software developed in this work not only avail useful tools that meet a particular necessity, but will also serve as a basis for future developments of computerized analysis tools at least for the local structural engineering community that, in most cases, cannot afford to subscribe to high-end software systems. A number of application areas such as, for example, for the analysis of flat slab systems where the design is based on the equivalent frame analysis techniques may also be envisaged as a result of this work.

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REFERENCES

[1] Przemieniecki, J.S., *"Theory of Matrix Structural Analysis"*, McGraw-Hill, 1968.

[2] Chen, W.F., *"Handbook of Frame Constants"*, Portland Cement Association, Illinois, 1958.

[3] Kinney, J.S., *"Indeterminate Structural Analysis"*, Addison-Wesley, 1957.

[4] Jennings, A., *"Matrix Computation for Engineers and Scientists"*, Wiley, 1977.

[5] Weaver, W. and Gere, J.M., *"Matrix Analysis of Framed Structures"*, Van Nortrand, New York, 1980.

[6] Samson Takele, "Computerized Analysis of Frames with Non-Prismatic members", *A thesis submitted for partial fulfillment of M. Sc. degree in Civil Engineering*, Addis Ababa University, 2002.

[7] Hirschfeld, K., *"Baustatik – Theorie und Beispiele"*, Springer Verlag, Berlin, 1969.

[8] Cross, H., "Analysis of Continuous Frames by Distributing Fixed-End Moments", *Transactions of the American Society of Civil Engineers*, USA, 1932.

[9] STAADPro Users' Manual, Research Engineers International, CA, USA, 2008.

[10] Golub, G.H. and van Loan, C.F., *"Matrix Computations"*, The John Hopkins University Press, 1983.

Annex A

A-1 Evaluated integral for members with stepped (segmented) haunches

Int. No.	I_x, A_x arbitrary $I_{z.x} = I_c\,[(1+r(1-x/\eta))]$ $A_x = A_c\,[(1+r(1-x/\eta)]$ where $\eta = l_1$ or l_2		Transformed integral $y = 1 - x/\eta$	Evaluated integrals $\eta = l_1$ or l_2
1	$\int_0^{L-l_2} \frac{dx}{A_x}$	when $A_x = A_c$ $\frac{1}{A_c}\int_{l_1}^{L-l_2} dx$		$\frac{1}{A_c}(L - l_1 - l_2)$
2	$\int_\alpha^{L-l_2} \frac{dx}{I_{z.x}}$	when $I_{z.x} = I_c$ $\frac{1}{I_C}\int_{l_1}^{L-l_2} dx$		$\frac{1}{I_c}(L - l_1 - l_2)$
3	$\int_{l_1}^{L-l_2} \frac{xdx}{I_{z.x}}$	when $I_{z.x} = I_c$ $\frac{1}{I_c}\int_{l_1}^{L-l_2} xdx$		$\frac{1}{2I_c}\left[(L - l_2)^2 - l_1^2\right]$
4	$\int_{l_1}^{L-l_2} \frac{x^2dx}{I_{z.x}}$	when $I_{z.x} = I_c$ $\frac{1}{I_c}\int_{l_1}^{L-l_2} x^2dx$		$\frac{1}{3I_c}\left[(L - l_2)^3 - l_1^3\right]$

A-2 Evaluated integral for members with linearly-varying (tapered) haunches

No.	I_x, A_x arbitrary $I_{z.x} = I_c\,[(1+r(1-x/\eta)]$ $A_x = A_c\,[(1+r(1-x/\eta)]$ where $\eta = l_1$ or l_2		Transformed integral $y = 1 - x/\eta$	Evaluated integrals $\eta = l_1$ or l_2
1	$\int_{X_1}^{X_2} \frac{dx}{A_x}$	$\frac{1}{A_c}\int_{X_1}^{X_2} \frac{dx}{(1 + r(1 - x/\eta))}$	$-\frac{al}{A_c}\int_{Y_1}^{Y_2} \frac{dy}{1 + ry}$	$-\frac{\eta}{A_c r}\left[\ln(1 + ry)\right]_{Y1}^{Y2}$
1a	$y = 1 - x/\eta$	$\frac{1}{A_c}\int_0^\eta \frac{dx}{\left[1 + r(1 - x/\eta)\right]}$	$-\frac{\eta}{A_c}\int_1^0 \frac{dy}{1 + ry}$	$\frac{\eta}{A_c r}\ln(1 + r)$
2	$\int_{X_1}^{X_2} \frac{dx}{I_{z.x}}$	$\frac{1}{I_c}\int_{X_1}^{X_2} \frac{dx}{\left[1 + r(1 - x/\eta)\right]^3}$	$-\frac{al}{I_c}\int_{Y_1}^{Y_2} \frac{dy}{(1 + ry)^3}$	$\frac{\eta}{rI_c}\left[\frac{1}{2(1 + ry)^2}\right]_{Y1}^{Y2}$
2a	$\int_0^{al} \frac{dx}{I_{z.x}}$	$\frac{1}{I_c}\int_0^\eta \frac{dx}{\left[1 + r(1 - x/\eta)\right]^3}$	$-\frac{\eta}{I_c}\int_1^0 \frac{dy}{(1 + ry)^3}$	$\frac{\eta}{2I_c}\left[\frac{2 + r}{(1 + r)^2}\right]$
3	$\int_{X_1}^{X_2} \frac{xdx}{I_{z.x}}$	$\frac{1}{I_c}\int_{X_1}^{X_2} \frac{xdx}{\left[1 + r(1 - x/\eta)\right]^3}$	$\frac{\eta^2}{I_c}\int_{Y_1}^{Y_2} \frac{(y - 1)dy}{(1 + ry)^3}$	$\frac{\eta^2}{r^2 I_c}\left[\frac{1 + r}{2(1 + ry)^2} - \frac{1}{1 + ry}\right]_{Y1}^{Y2}$
3a	$\int_0^\eta \frac{xdx}{I_{z.x}}$	$\frac{1}{I_c}\int_0^\eta \frac{xdx}{\left[1 + r(1 - x/\eta)\right]^3}$	$-\frac{\eta^2}{I_c}\int_1^0 \frac{(y - 1)dy}{(1 + ry)^3}$	$\frac{\eta^2}{2I_c}\left[\frac{1}{(1 + r)}\right]$

A-2 Evaluated integral for members with linearly-varying (tapered) haunches (cont'd)

No.	I_x, A_x arbitrary $I_{z.x} = I_C\,[1+r(1-x/\eta)]$ $A_x = A_C\,[(1+r(1-x/\eta)]$ where $\eta = l_1$ or l_2		Transformed integral $y = 1-\frac{x}{\eta}$	Evaluated integrals $\eta = l_1$ or l_2
4	$\int_{X_1}^{X_2}\frac{x^2dx}{I_{z.x}}$	$\frac{1}{I_c}\int_{X_1}^{X_2}\frac{x^2dx}{[1+r(1-x/\eta)]^3}$	$-\frac{\eta^3}{I_c}\int_{Y_1}^{Y_2}\frac{(y-1)^2dy}{(1+ry)^3}$	$-\frac{\eta^3}{r^3I_c}\left[-\frac{(1+r)^2}{2(1+ry)^2}+\frac{2(1-r)}{1+ry}+\ln(1+ry)\right]_{Y1}^{Y2}$
4a	$\int_0^{\eta}\frac{x^2dx}{I_{z.x}}$	$\frac{1}{I_c}\int_0^{\eta}\frac{x^2dx}{[1+r(1-x/\eta)]^3}$	$-\frac{\eta^3}{I_c}\int_1^0\frac{(y-1)^2dy}{(1+ry)^3}$	$-\frac{\eta^3}{r^3I_c}\left[\frac{r(2-r)}{2}-\ln(1+r)\right]$
5	$\int_{X_2}^{X_1}\frac{x^3dx}{I_{z.x}}$	$\frac{1}{I_c}\int_{X_1}^{X_2}\frac{x^3dx}{[1+r(1-x/\eta)]^3}$	$-\frac{\eta^4}{I_c}\int_{Y_1}^{Y_2}\frac{(y-1)^3dy}{(1+ry)^3}$	$\frac{\eta^4}{r^4I_C}\left[\frac{(1+r)^3}{2(1+ry)^2}-\frac{3(1+r)^2}{(1+ry)}+(1+ry)-3(1+r)\ln(1+ry)\right]$
5a	$\int_0^{\eta}\frac{x^3dx}{I_{z.x}}$	$\frac{1}{I_c}\int_0^{\eta}\frac{x^3dx}{[1+r(1-x/\eta)]^3}$	$\frac{\eta^4}{I_c}\int_1^0\frac{(y-1)^3dy}{(1+ry)^3}$	$\frac{\eta^4}{r^4I_c}\left(r(r^2-3r-6)/2+3(1+r)\ln(1+r)\right)$

Annex B

B.1 Element Stiffness Matrix in Structural Coordinates

Structural or system coordinates are generally different from most member or local coordinates and are established in a manner suitable to facilitate the structural input data and also interpretation of computed end results, among others.

Element stiffness matrices, established in local coordinates, need be expressed with reference to the structural coordinate in order to analyze the structure. To this effect, coordinate transformation [10] is employed and this constitutes one of the most essential steps in structural analysis. Through coordinate transformation, physical quantities of interest such as forces and displacements expressed in one of the coordinate systems are manipulated back and forth into corresponding quantities in the other system.

Components of any form of actions and responses $\mathbf{A}_s$ in the structure coordinate system and those with reference to member coordinate system $\mathbf{A}_m$ are related as follows [5].

$$\mathbf{A}_m = \mathbf{R}\,\mathbf{A}_s \tag{a}$$

where matrix $\mathbf{R}$ is referred to as the *rotation matrix*. A matrix such as $\mathbf{R}$ has the property that its determinant is unity; i.e $|\mathbf{A}| = 1$. This result is a very useful property, referred to as the orthogonality property [10] and exhibits the relationship given by $\mathbf{R}^{-1} = \mathbf{R}^T$.

Thus, solving for $\mathbf{A}_s$ from Eq. (a),

$$\mathbf{A}_S = \mathbf{R}^{-1}\,\mathbf{A}_m = \mathbf{R}^T\,\mathbf{A}_m \tag{b}$$

The transformation matrix $\mathbf{R}$ for a plane frame member has the following form:

$$\mathbf{R} = \begin{bmatrix} C_x & C_y & 0 \\ -C_y & C_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{c}$$

where $C_x = \cos\theta$ and $C_y = \sin\theta$ and θ is the orientation of an element to the horizontal as shown in Fig. 2 .

The relationship given by Eq. (a) is perfectly general and applies to both actions and responses. Thus, displacements expressed in one coordinate system may posed in the other using this same relationship.

The action – displacement relationships at the two ends of member i in the local coordinate system $x_{\mathrm{m}}, y_{\mathrm{m}}$ may be expressed by the following

$$\mathbf{A}_{\mathrm{mi}} = \mathbf{k}_{\mathrm{mi}}\,\mathbf{D}_{\mathrm{mi}} \tag{d}$$

in which $\mathbf{A}$ and $\mathbf{D}$ denote generalized actions and displacements, respectively. The member stiffness matrix $\mathbf{k}_{\mathrm{mi}}$ for a plane frame structure is presented in Eqs. (1). In order to proceed with the analysis, it is first necessary to transform $\mathbf{k}_{\mathrm{mi}}$ into the corresponding member stiffness matrix $\mathbf{k}_{\mathrm{si}}$ in the structural axis.

For this, we can write writing (d) in a partitioned form [5] as follows:

$$\begin{bmatrix}\mathbf{A}_{\mathrm{mj}}\\ \mathbf{A}_{\mathrm{mk}}\end{bmatrix} = \begin{bmatrix}\mathbf{k}_{\mathrm{mjj}} & \mathbf{k}_{\mathrm{mjk}}\\ \mathbf{k}_{\mathrm{mkj}} & \mathbf{k}_{\mathrm{mkk}}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{\mathrm{mj}}\\ \mathbf{D}_{\mathrm{mk}}\end{bmatrix} \tag{e-1}$$

with reference to the two ends j and k of the member. By using the appropriate rotation formulas from Eqs. (a) and (b) and substituting in to the above, one obtains:

$$\begin{bmatrix}\mathbf{R}\,\mathbf{A}_{\mathrm{sj}}\\ \mathbf{R}\,\mathbf{A}_{\mathrm{sk}}\end{bmatrix} = \begin{bmatrix}\mathbf{k}_{\mathrm{mjj}} & \mathbf{k}_{\mathrm{mjk}}\\ \mathbf{k}_{\mathrm{mkj}} & \mathbf{k}_{\mathrm{mkk}}\end{bmatrix}\begin{bmatrix}\mathbf{R}\,\mathbf{D}_{\mathrm{sj}}\\ \mathbf{R}\,\mathbf{D}_{\mathrm{sk}}\end{bmatrix} \tag{e-2}$$

an equivalent form of Eq. (e-2) is:

$$\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{\mathrm{sj}}\\ \mathbf{A}_{\mathrm{sk}}\end{bmatrix} = \begin{bmatrix}\mathbf{k}_{\mathrm{mjj}} & \mathbf{k}_{\mathrm{mjk}}\\ \mathbf{k}_{\mathrm{mkj}} & \mathbf{k}_{\mathrm{mkk}}\end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{\mathrm{sj}}\\ \mathbf{D}_{\mathrm{sk}}\end{bmatrix} \tag{e-3}$$

Equation (e-3) can be written in a simplified form as:

$$\mathbf{T}\,\mathbf{A}_{\mathrm{s}} = \mathbf{k}_{\mathrm{m}}\,\mathbf{T}\,\mathbf{D}_{\mathrm{s}} \tag{f}$$

where $\mathbf{T} = \begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}$, referred to as the *action-displacement rotation transformation matrix* and the elements of $\mathbf{A}_{\mathrm{s}}$ and $\mathbf{D}_{\mathrm{s}}$ are actions and displacements at the ends of the member in the direction of structure axes.

Pre multiplying both sides by $\mathbf{T}^{-1}$ and noting that $\mathbf{T}^{-1} = \mathbf{T}^{\mathrm{T}}$ one obtains,

$$\mathbf{A}_{\mathrm{s}} = \mathbf{T}^{\mathrm{T}}\,\mathbf{k}_{\mathrm{m}}\,\mathbf{T}\,\mathbf{D}_{\mathrm{s}} \tag{g-1}$$

Action displacement equation of a particular member in the structural axis, thus, becomes,

$$\mathbf{A}_{\mathrm{s}} = \mathbf{k}_{\mathrm{ms}}\,\mathbf{D}_{\mathrm{s}} \tag{g-2}$$

in which $\mathbf{k}_{\mathrm{ms}}$ is the member stiffness matrix for structure axes and is given by:

$$\mathbf{k}_{\mathrm{ms}} = \mathbf{T}^{\mathrm{T}}\,\mathbf{k}_{\mathrm{m}}\,\mathbf{T} \tag{h}$$

Now, carrying out the matrix multiplications in Eq. (h) and the definition of $\mathbf{k}_{\mathrm{mi}}$ in Eq. (1a) provides the following member stiffness matrix $\mathbf{k}_{\mathrm{ms}}$ in the structural coordinate:

$$\mathbf{k}_s = \begin{bmatrix} k_{s11} & k_{s12} & k_{s13} & k_{s14} & k_{s15} & k_{s16} \\ & k_{s22} & k_{s23} & k_{s24} & k_{s25} & k_{s26} \\ & & k_{s33} & k_{s34} & k_{s35} & k_{s36} \\ & & & k_{s44} & k_{s45} & k_{s46} \\ & & & & k_{s55} & k_{s56} \\ & & & & & k_{s66} \end{bmatrix} \tag{i-1}$$

where the following expressions are represented by respective stiffness terms:

$$\begin{array}{llll} k_{s11} = k_{cm1}c_x^2 + k_{cm5}c_y^2 & k_{s12} = (k_{cm1} - k_{cm5})c_x c_y & k_{s13} = \text{-}k_{cm6}c_y & k_{s16} = \text{-}k_{cm7}c_y \\ k_{s22} = k_{cm1}c_y^2 + S_{cm5}c_x^2 & k_{s23} = k_{cm6}c_x & k_{s25} = \text{-}k_{cm1}c_y^2 - k_{cm5}c_x^2 & \\ k_{s33} = k_{cm2} & k_{s34} = k_{cm6}c_y & k_{s36} = k_{cm3} & \\ k_{s66} = k_{cm4} & & & \\ k_{s14} = -k_{s11} & k_{s15} = -k_{s21} & k_{s24} = k_{s15} & k_{s25} = -k_{s22} \\ k_{s36} = -k_{s23} & k_{s44} = k_{s11} & k_{s45} = -k_{s15} & k_{s46} = -k_{s16} \\ k_{s55} = k_{s22} & k_{s55} = -k_{s26} & & \end{array} \tag{i-2}$$

and, further,

$$\begin{array}{ll} k_{cm1} = EA_c/U_0L & k_{cm5} = EI_c(U_3 + 2U_4 + U_5)/U_7L^3 \\ k_{cm2} = EI_cU_3/U_6L & k_{cm6} = EI_c(U_3 + U_4)/U_6L^2 \\ k_{cm3} = EI_cU_4/U_6L & k_{cm7} = EI_c(U_4 + U_5)/U_6L^2 \\ k_{cm4} = EI_cU_5/U_6L & \end{array} \tag{i-3}$$

in which U_0, U_1, U_2, U_3, U_4, U_5 and U_6 are given earlier by Eqs. (2).

B.2 Member Fixed-End Actions in Structural Coordinates

In order to proceed with structural analysis, it is further important to carry out the transformation of the member end actions given by Eq. (6) from member axes to structure axes. For this purpose, the concept of rotation transformation concept presented earlier can be applied in order to transform $\mathbf{A}_m$ into the corresponding values $\mathbf{A}_s$ in the structural coordinates. Thus, for any members, the fixed end forces $\mathbf{A}_S$ in structure axes can be obtained from those established in the member axis $\mathbf{A}_m$ using the relationships:

$$\mathbf{A}_S = \mathbf{T}^T \mathbf{A}_m \tag{j}$$