PARAMETRIC ANALYSIS OF THE DYNAMIC PROPERTIES OF SECTORIAL SHELLS

Dereje Seyoum and Shifferaw Taye
Department of Civil Engineering,
Addis Ababa University

ABSTRACT

The behavioral responses of sectorial shells as related to the number of modes and certain geometric properties of the shell have been studied. By varying certain dimensionless geometric properties of the shell and considering a series of undamped vibration modes, the influences of these properties on displacements, edge reactions, bending moments and other responses have been plotted and relevant conclusions reached.

Two very important observations from this work are that, unlike in conventional regular building structures where the first mode is dominant in influencing various dynamic response behaviors and high-order modes affect less, the principal mode in sectorial shells fails to influence these responses while higher modes contribute progressively and significantly to those effects.

INTRODUCTION

The behavior of shell-type structures under a variety of dynamic loads, such as seismic, blast and wind loads among others, is a phenomenon of interest in view of the relatively large sizes of such structures and the nature of forces they are usually subjected to in practice. Any linearly elastic continuum will have well-defined natural frequencies and vibration modes that can be investigated by considering the mass of the body and its stiffness. Sectorial shells, otherwise known as gronoids, are among reasonably complicated continuum structural systems that call for extensive modeling requirements in order to predict their response behavior under dynamic loads. The nature of discontinuity in their geometry layout makes them unique among various shell-type structures. Closed analytic solution in this regard is particularly demanding, if not impossible, as their response behavior, as in any general shell-type structures, require determination of all types of structural responses including those emanating from membrane or in-plane deformations. In this respect, therefore, a sound approximate numerical approach to their solution – the finite elements method – plays a significant role in facilitating the study of the various dynamic behaviors of such shell structures.

In recent years analysis of static and dynamic problems has been the focus of intense research efforts mainly due to the emphasis placed by manufacturers, contractors and certifying agencies on realistic modeling that call for accurate analysis of critical structural components [1]. This endeavor has prompted the development of versatile and powerful finite element descretization methods as well as improved numerical methods and programming techniques for linear and non-linear static and dynamic analysis of structures [2, 3, 4].

An examination of the static load-deflection characteristic and dynamic response time histories of a number of simple structural systems reveals that they are generally not more complicated than those of presumably complex structures [5]. The large number of degrees of freedom in large-scale structures is often dictated by their topology rather than by the expected complexity of their behavior. This fact has been recognized and techniques for reducing the degrees of freedom have long been proposed in vibration analysis and automated optimum design.

The objective of this paper is to assess the influence of the profile geometry of sectorial shells given by a dimensionless ratio of overall rise h1 to the vault rise (or depression) h_2 as shown in Fig. 1 on the various structural dynamic response behavior with due consideration to the number of vibration modes in the process. In order to minimize the number of variables, the plan dimension of the shell has been kept constant.

GEOMETRY OF SECTORIAL SHELLS AND THEIR DYNAMIC ANALYSIS MODELS

Sectorial shells represent types of structures where shells are combined to form a roof covering on a polygonal area. They are usually employed to cover large space without employing additional stiffening elements. If the base of the structure is a uniform polygon, each sector of the shell will cover an isosceles triangular area (Fig. 1a). Each lobe of a sectorial shell can assume various geometric shapes whose middle surfaces may generally be expressed as:

$$z = h_1 \frac{x^{\alpha_1}}{a^{\alpha_1}} + h_2 \frac{y^{\beta_1}}{b^{\beta_2}} \qquad (1a)$$

The most frequently employed shape employed in such shells is a second-order curve. Accordingly, the middle surface of a second-order sectorial shell may be represented by the following general form:

$$z = h_1 \frac{x^2}{a^2} + h_2 \frac{y^2}{b^2} \qquad (1b)$$

where h_1, h_2 are parameters as shown in Fig. 1 whose values will be varied in the parametric analysis and $h_1 > 0$. This equation represents an elliptic paraboloid if h_2 is positive (Fig. 1b), a parabolic cylinder if $h_2 = 0$ (Fig. 1c), a hyperbolic paraboloid if h_2 is negative (Fig. 1d).

The sectors of the shell are supported only along their lines of intersections by arched girders to which the actions of the sectors, in the form of tangential shear forces, are transmitted. It should be mentioned here that such shells are not supported along their individual outer edges.

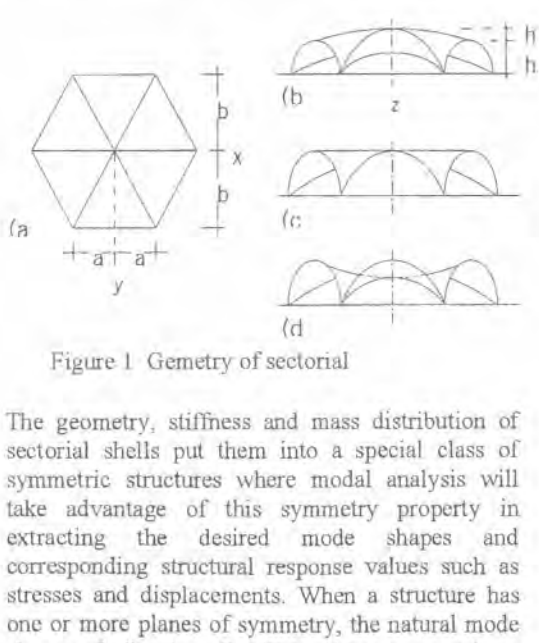

Figure 1 Gemetry of sectorial

The geometry, stiffness and mass distribution of sectorial shells put them into a special class of symmetric structures where modal analysis will take advantage of this symmetry property in extracting the desired mode shapes and corresponding structural response values such as stresses and displacements. When a structure has one or more planes of symmetry, the natural mode shapes of vibration all will be either symmetric or antisymmetric with respect to these planes [1, 6]. In such types of systems, one also needs only analyze a portion of the original structure. The reduction to a smaller sized problem may be accomplished by introducing artificial restraints at joints located along the planes of symmetry. In addition, the relevant properties of members that lie in those planes must be altered. Thus, depending on the number of planes of symmetry, only half, a quarter or even less portion of the structure may need only to be analyzed.

Sectorial shells generally exhibit multiple planes of symmetry and the present work makes advantage of this property to reduce problem size and yet increase the number of nodal points in the analytical model for improved approximating quality of the finite element model. When a vibration mode is symmetric with respect to a plane of structural symmetry, the nodal displacements, strains, stresses and reactions as well as other structural responses will also be symmetric with respect to the same plane. To this goal, nodal points located on a plane of symmetry must be restrained in such a manner that the structure deforms symmetrically with respect to that plane. In general, the component of nodal translation normal to the plane of symmetry and the components of rotational displacement in the plane must be prevented in order to enforce a symmetric pattern of deformation.

If a vibration mode is antisymmetric with respect to a plane of structural symmetry, the nodal displacements, strains, stresses and reactions will also be antisymmetric with respect to the same plane. The components of nodal translations in the plane of symmetry and the component of rotation normal to the plane must be prevented to give a pattern of deformation that is antisymmetric with respect to the plane.

If an element of a discretized continuum lies along a plane of symmetry, we must divide its relevant rigidities entering calculations by two in order to cut the structure into equal parts. In the case where a finite element is normal to and bisected by a plane of symmetry, one must divide it into equal parts and introduce new nodes on the bisecting plane that are restrained as noted earlier. These modeling concepts will be implemented in the numerical experience for the problem at hand.

Continuum structures, such as sectorial shells, subjected to dynamic loads respond with a combination of rigid-body and flexible-body motion. If a structure is analyzed as an elastic continuum, its flexible-body response to dynamic actions would consist of the sum of an infinite number of vibrational motions. However, if the structure is discretized by the finite number, the resulting analytical model will have only a finite number of nodal degrees of freedom and a finite number of natural modes of vibration. Therefore, such a model has only a finite vibration modes and motions that contribute to its dynamic response.

Structural continua that are subjected to arbitrary dynamic loads become extremely difficult, if not impossible, to analyze in their original physical coordinates. To alleviate such shortcomings, we can use the natural modes of vibrations as generalized coordinates as a result of which the equations of undamped motion, Eq. (2), become uncoupled [3, 4]. In these coordinates, each equation may be solved as if it pertained to a system with only one degree of freedom. Superposition of these single-degree-of-freedom results is accomplished through a transformation back to the original coordinates. By this means, one can evaluate time-varying nodal displacements, internal stresses, and support reactions for the established analytical model.

An important advantage of the normal-mode method is that only the significant modal responses need be included in a dynamic analysis thereby making the method more efficient than other available methods such as the numerical-integration methods [4]. The other modal responses may often be omitted without much loss of accuracy.

Modal superposition technique is a very effective reduction method for linear dynamic problem when only few vibration modes are exited by the external loading. The basis vector in this technique consists of a limited number of vibration mode shapes. Several studies have been made on improving the accuracy and efficiency of modal superposition technique in linear problems [1].

The semi-discrete form of the governing equation for undamped, linear structural system at time t can be written in the following form:

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{P}(t) \qquad (2)$$

where **M** and **K** are, respectively, the structural mass and stiffness matrices while **U(t)** and $\ddot{U}(t)$ represent vectors of nodal displacements and accelerations, respectively. **P(t)** is the vector of externally applied loads.

Equation (2) has got a known solution that may be satisfied by:

$$U_i(t) = \Phi_i \sin(\omega_i t + \alpha_i) \qquad i = 1,2,\cdots,n \qquad (3)$$

where

- Φi is the vector of nodal amplitudes or mode shapes of the i-mode
- ωi is the circular frequency of the i-th mode
- αi denotes the phase angle
- n represents the degree of freedom

By differentiating Eq. (3) twice with respect to the time variable t, one finds:

$$\ddot{U}_i(t) = -\omega_i^2\,\Phi_i\,\sin(\omega_i t + \alpha_i) \qquad (4)$$

Substitution of Eqs. (3) and (4) into Eq. (2) allows cancellation of the $\sin(\omega_i t + \alpha_i)$ term, leaving:

$$\left(\mathbf{K} - \omega_i^2\,\mathbf{M}\right)\mathbf{\Phi}_i = 0 \qquad (5)$$

Equation (5) exhibits the condition in which the time variable is separated from that of space and, as a result, leaves a set of n homogeneous algebraic equations. This equation has also got the form of an algebraic eigenvalue problem [7].

From the theory of homogenous algebraic equations [8], non-trivial solutions exist only if the determinant of the coefficient matrix vanishes. Thus,

$$\left|\mathbf{K} - \omega_i^2\,\mathbf{M}\right| = 0 \qquad (6)$$

Expansion of this determinant produces a polynomial of order n called the characteristic equation. When the system is positive definite, n real and distinct eigenvalues ω_i^2 will be obtained as the roots of the polynomial.

The n roots of this polynomial are the characteristic values or eigenvalues. Substitution of these roots (one at a time) into the homogeneous equations, Eq. (4), produces the characteristic vectors, or eigenvectors Φ_i, within arbitrary constants.

The choice of an appropriate method for the computation of eigenvectors depends on a number of criteria [4] among which the number of degrees of freedom and the number of required eigenvalues are dominant. In the latter case, Chopra [8] presented the number of modes to be considered in relation to minimization of error-propagation in modal responses.

With a specification of the initial conditions, Eq.(2) can be integrated to produce the time-history response of the structure. A wide variety of explicit and implicit techniques have been proposed for integrating Eq. (2) and obtaining the response time-histories of the structure [4]. However, the computational effort involved in applying these techniques to large-scale continuum structures such as multiple-lobed sectorial shell can be quite substantial. Therefore, the reduction of degrees of freedom in dynamic problems is an important approach for practical application of analysis methods.

The use of modal methods in linear problems appears, at a first glance, to conform to the well known fact that superposition principles are applicable to the system. However, the number of modes to be considered in predicting the response of the structure has got considerable influence on the behavior values to be identified [8].

If the Lagrangian displacement formulation is used, the vector of nodal displacement is expressed as a linear combination of the lowest vibration modes [9] as follows:

$$U(t) = \Phi\,\eta(t) = \sum_{i=1}^{n} \phi_i\,\eta_i(t) \qquad (8)$$

where the column matrix Φ are the basis vectors, otherwise known as the modal matrix, which consists of the lowest vibration modes and $\eta(t)$ are the vector of generalized coordinates η_i.

The first step in a mode superposition solution is to obtain the natural frequencies and natural modes of the system. The important topic of truncation that makes use of fewer than n modes could be studied here.

The natural frequencies and modes satisfy Eq. (5); the semi-discrete governing equation of the structure, Eq. (2), is then approximated by the following reduced system of ordinary differential equation:

$$\overline{M}\,\ddot{\eta}(t) + \overline{K}\,\eta(t) = \overline{P}(t) \qquad (8)$$

where

$$\overline{M} = \Phi^T M \Phi$$
$$\overline{K} = \Phi^T K \Phi$$
$$\overline{P}(t) = \Phi^T P(t)$$

Due to the orthogonality condition, both the and are $\overline{K}$ diagonal matrices. The total response $\eta(t)$ can be obtained by superposing of the response due to initial conditions alone and response due to the excitation alone.

The effectiveness of the modal superposition techniques in dynamic problem depends on three major parameters; these are the number of vibration modes required to accurately simulate the response, the frequency of updating vibration modes, and the efficiency of the algorithms used in extracting the initial eigenmodes. The number of vibration modes required depends on the structural properties of the system as well as on the spatial distribution and frequency content of the loading [8].

The computational cost of extracting the vibration modes can be reduced by applying one of the condensation schemes-mass condensation method [3, 10] – to discrete system prior to extracting the eigenvectors.

The solution of three-dimensional continuum structural systems, such as sectorial shells, that are subjected to dynamic actions is best-handled universally by the finite elements method. The finite element method enables one to convert a dynamic problem with an infinite number of degrees of freedom to one with a manageable, finite number in order to simplify the solution process. The basic concept in the method is to divide a continuum into sub-regions having simpler geometries than the original problem. Nodes on each such sub-region, or finite element, are the key points that control the behavior of the element. By making the displacements and other behavioral responses such as stresses, velocities and accelerations at any point in an element dependant on those at the nodes, one need only write a finite number of differential equations of motion for such nodes. For good accuracy in the solution, the number of nodal degrees of freedom usually must be fairly large; and the details of element formulations for closed-form analytical solution are rather complicated.

The primary objectives of dynamic analysis by finite elements are, therefore, to calculate approximately the responses at such nodes or at other selected points for subsequent interpolation at all desired points in the structure. Making use of the underlying concepts for the various dependencies, the differential equations of motion for the nodes of the discretized continuum, such as sectorial shells, can be developed by assembling the finite element contributions.

From the homogeneous equations of motion, one can then perform a vibration analysis of the sectorial shell. This type of analysis consists of finding undamped frequencies and corresponding mode shapes for the discretized analytical model of the gronoids. These and related findings are important and very useful to the study of the dynamic behavior of such systems and are essential for the normal-mode method of dynamic analysis [1, 4].

The principle of virtual work can be used to develop the finite element formulation of the dynamics of sectorial shells [1, 2]. Such equations include energy-equivalent stiffnesses, masses, and nodal loads for a typical element. Once these relationships are developed, one can implement the principle of virtual work to establish the various parameters in the equation of motion such that:

$$\mathbf{M}\ddot{\mathbf{U}}(t)+\mathbf{K}\mathbf{U}(t)=\mathbf{P}(t)$$

in which:

$$K=\int_V \boldsymbol{B}^T \boldsymbol{E}\boldsymbol{B}\,dV$$

$$M=\int_V \rho \boldsymbol{f}^T \boldsymbol{f}\,dV$$

and

- **B** = strain-displacement transformation matrix
- **E** = stress-strain transformation matrix
- **f** = nodal-to-generic displacement transformation matrix
- ρ = material density
- $\mathbf{P}(t)$ = vector of nodal loads.

The theoretical background on closed-form analytical solutions of discretized-mass system has bee presented by several authors [1, 2, 11]; these will form the basis for the parametric analysis presented in this paper.

The techniques for reducing the degrees of freedom are referred to as reduction methods [1]. The essence of reduction methods for linear analysis is limited to the deformation modes of the descritized structure to some known modes which are considerably fewer in number than the number of the degrees of freedom in the original descritized system. Due to the high potential of the reduction method for linear analysis, increasing interest has recently been shown in the application of these methods to linear and non linear dynamic problems.

The principles presented so far will now be applied to study the dynamic behavioral responses of a four-lobe sectorial shell. The influence of the number of modes on these responses by varying the parameter h_2/h_1 as shown in Fig. 1 will be studied.

PARAMETRIC STUDIES

Numerical studies which demonstrate the influence of the mode on the response computation of the sectorial shell are presented in this section for the solution of modal superposition method.

A series of sectorial-shell geometries [6] with variable depth ratios of the form:

$$z(x,y)=h_1\frac{x^2}{a^2}+h_2\frac{y^2}{b^2}$$

have been examined. The roof of the sectorial shell consists of a cylindrical vault composed of thin self-supporting double shells with simple curvature, crossing orthogonally with generating lines parallel to the diagonal of the base square. The effect of variable vault rise h_2 to depth h_1 ratio on various structural responses is evaluated.

The structure in the shape of a double shell with shell units fanning out from the support may be imagined to be composed of a series of adjacent arches with triangular plan.

For the purpose of investigation of the dynamic response of the sectorial shell, a square base width of 30.0x30.0m ($a = b = 30$m), rise of vault, $c_1 = 1.4$m fixed at four legs were considered. A series of ratios of h_2/h_1 in the range of 0.02 to 0.2 has been employed and all analytical models have been processed with STAAD/Pro [11]. Due to symmetry, only one fourth of the sectorial shell has been processed using both triangular and quadrilateral isoparametric element shown in Fig. 2. Appropriate boundary conditions, accounting for structural symmetry, have been implemented in the analysis model.

Langrangian interpolation functions were used to approximate each of the displacement and rotation components and a total of 3720 shell isoparametric elements and 22302 displacements degrees of freedom were considered.

The material of the shell structure has linear elastic properties with density 24kN/m^3, Young's modulus 29GPa and Poisson's ratio of 0.2.

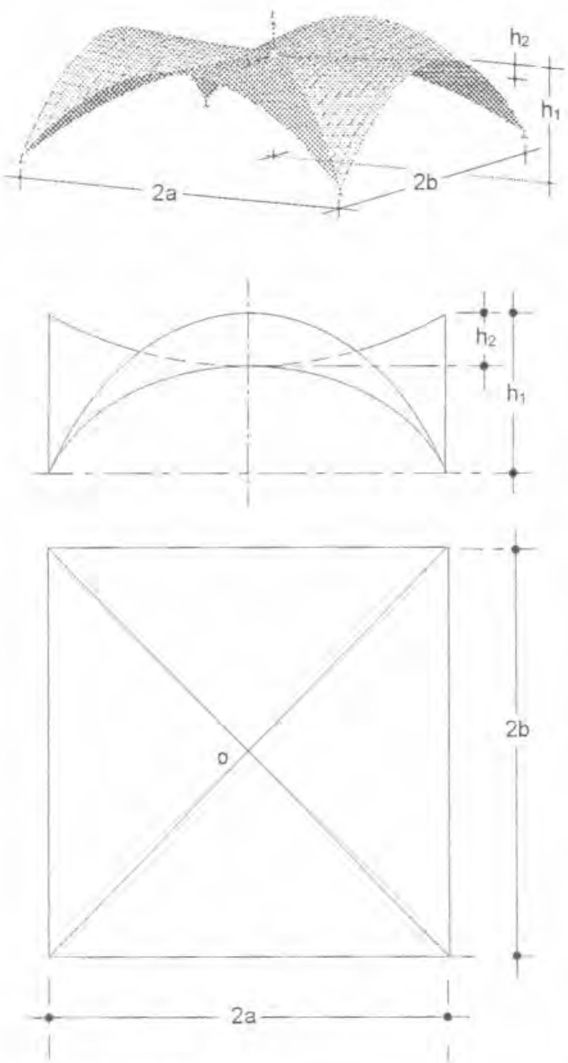

Figure 2 Analysis model

The model has been subjected to spectral cases of the load defined in EBCS-8 [12] for an arbitrarily selected soil type B. The ordinates of the response spectrum have been directly supplied to the implemented program [11] and the analysis was conducted to determine the overall response of the sectorial shell for various combinations of mode shapes. Accordingly, the variations of spectral reaction and bending moment at the edges of the shell with reference to the number of modes, h_2/h_1 ratios and stiffnesses have been investigated. The results of these analyses are presented subsequently in Fig. 3 to Fig. 7.

RESULTS AND DISCUSSION

Analysis of the various response behaviors indicate that there are generally two-way variation since they are influenced both by the number of modes and h_2/h_1 parameter. The degrees to which the responses are affected by both variables differ significantly as can be seen from Figs. 3 through Fig. 7. For instance, edge stresses and corresponding edge moments, Fig.3 and Fig. 5, respectively, indicate that with increased h_2/h_1 parameter, the values tend to converge to specific values as the number of modes taken into account increases. The increasing arching effect with h_2/h_1 ratio is responsible for the convergence of both response behaviors.

The various behavioral responses as related to the first mode are worth assessing. In all these cases, the corresponding structural responses are barely affected and remain insignificant under the first mode of vibration. This is because the first mode of vibration was not appreciably influenced in view of less mass participation in the process. This is a very important observation as it is not the case, for example, with frame-type structures where the first mode is dominant in many cases.

Increasing the number of modes to be accounted for, the structural response has generally shown tremendous variations as a function of the parameters considered. Again, this is contrary to what is generally observed in regular frame-type structures where only the first few modes dominate the structural response. The number of modes required to strictly simulate the actual response behavior of the system bending stresses as well as edge moments under a variety of stiffening and edge conditions is about 50 modes while to analyze the reaction, it generally required more than 50 modes.

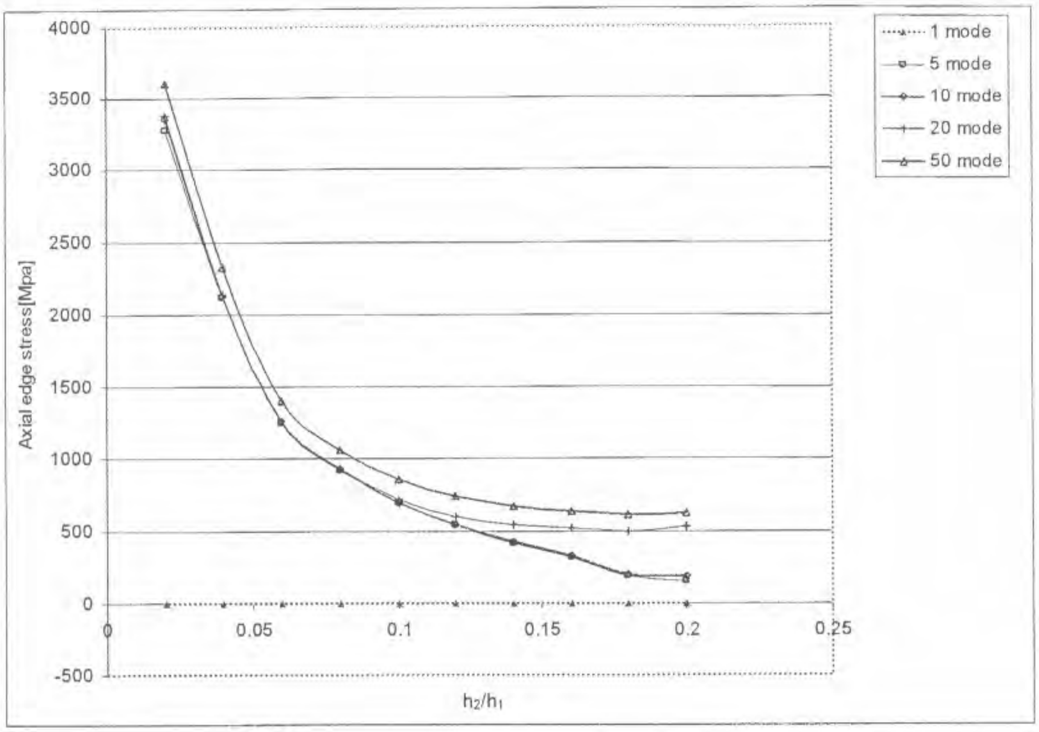

Figure 3 Axial stress versus h_2/h_1 for constant thickness

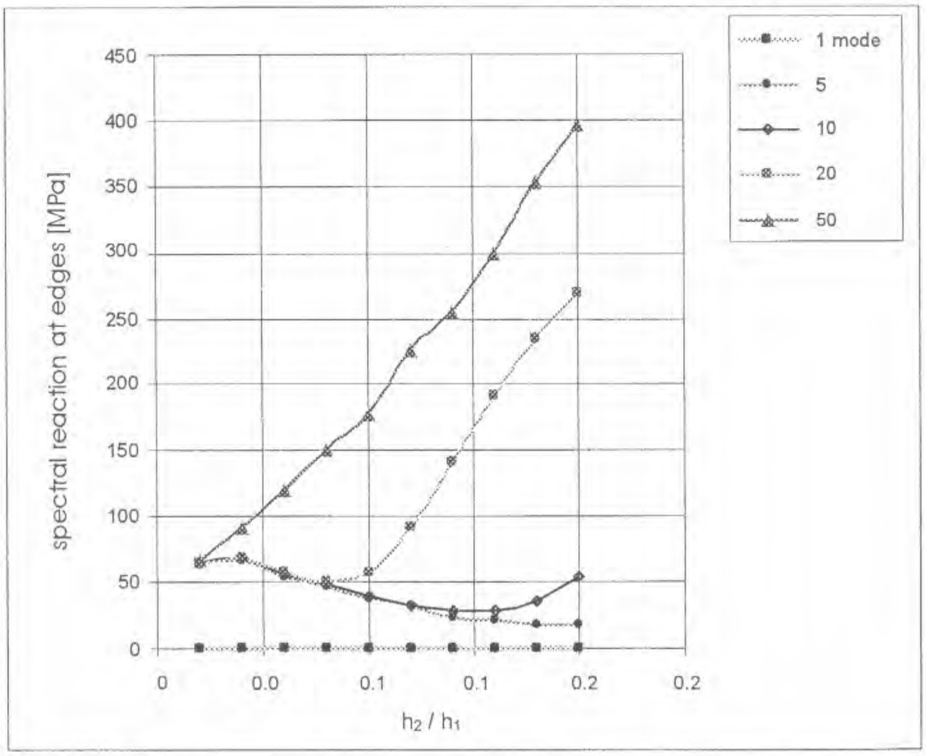

Figure 4 Spectral reaction at the edge of the sectorial shell

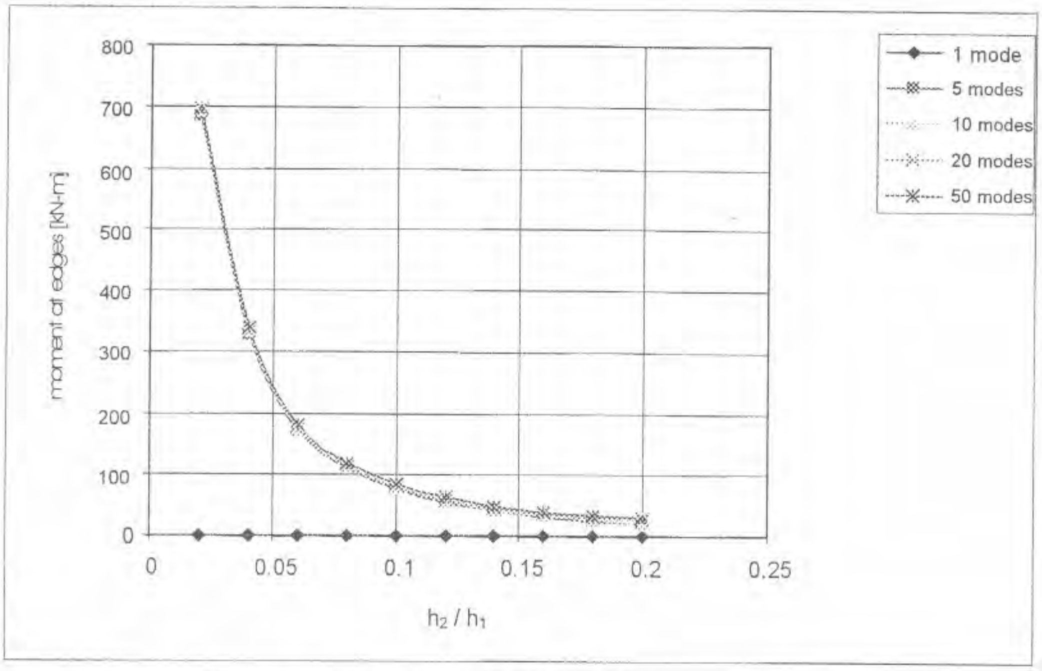

Figure 5 Edge moment for different mode contribution

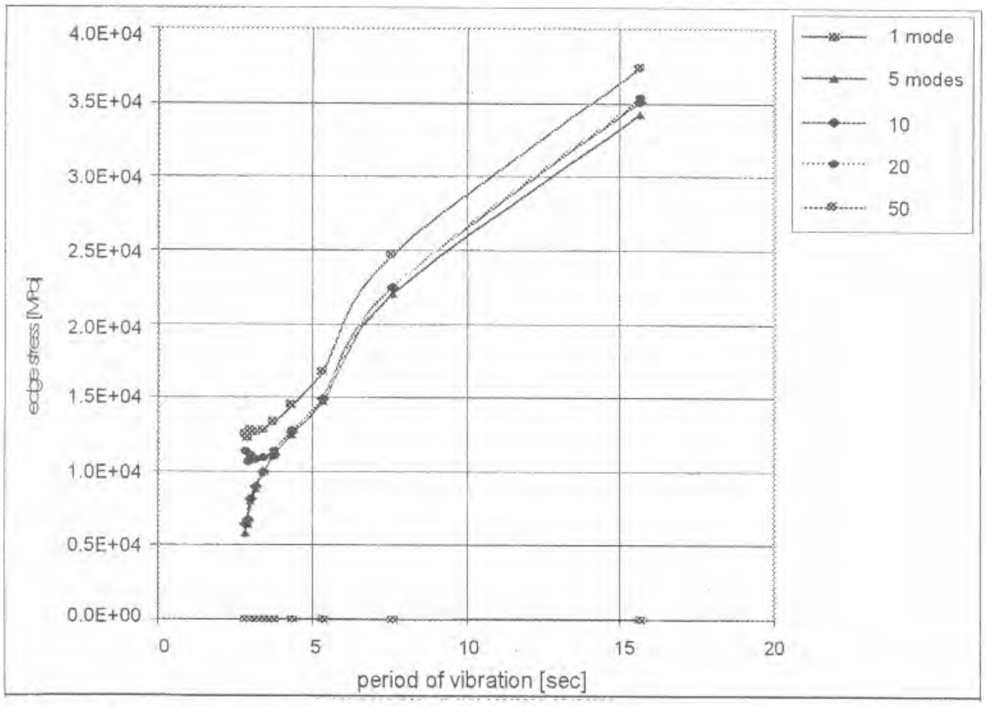

Figure 6 Period of vibration versus edge stress for constant thickness

Increased number of modes generally results in elevated behavioral responses. However, as can be noted from Fig. 3 and Fig. 5, certain responses tend to be unaffected by the number of modes especially at the higher ranges.

participation. Since the geometric nature of the sectorial shell considered is symmetrical, the first mode of vibration of this system is also symmetrical about its crown and, thus, the response leads to contribute to tensional effects, however

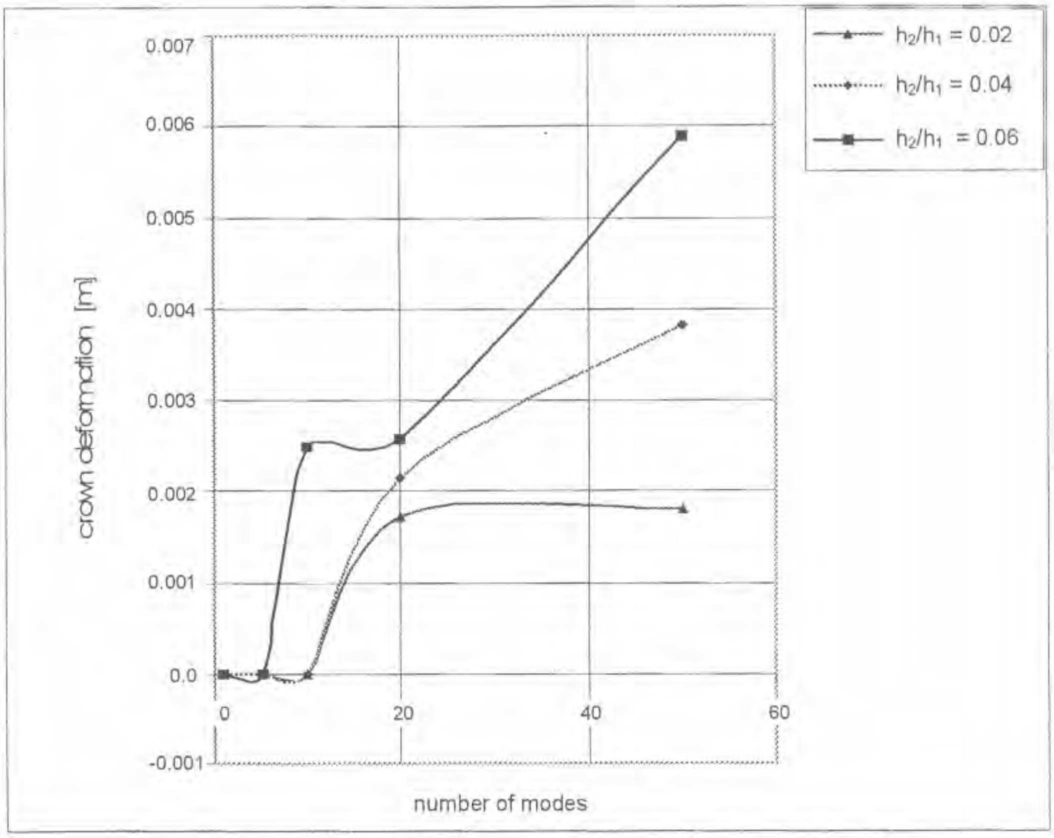

Figure 7 Crown deformation due to spectral load

CONCLUDING REMARKS

Various structural response behaviors of the sectorial shell as influenced by the number of modes considered and shell geometry have been studied. A very important observation from this work is that, unlike in conventional regular building structures where the first mode is dominant in influencing the dynamic response behavior, it can be seen that the edge axial stress, spectral reaction, edge moment and edge stresses of sectorial shells, the contribution of the first mode – which is supposed to be the principal mode – is insignificant.

In the dynamic analysis of such systems, the behavior of the mode of vibration depends on the geometric nature of the structure and, thus, the mode varies with the percentage of the total mass little that might be, rather than to compressive ones. This shows that the total mass of the system participates to influence the rest of the parameters at the later vibration modes rather than in the earlier ones.

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REFERENCES

[1] Soedel, W, Vibrations of Shells and Plates, Third Ed., CRC Press, 2005.

[2] Zienkiewicz, O.C. and Taylor, R.L., The Finite Element Method, Fifth Edition, Vol. 1: The Basics, Butterworth-Heinemann, Oxford, England, 2000.

[3] Weaver, W. and Johnston, P. R., Structural dynamics by Finite Elements, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.

[4] Wilson, E. L., Three Dimensional Dynamic Analysis of Structures with Emphasis on Earthquake Engineering, Computers and Structures, Inc., Berkeley, CA, USA, 1997.

[5] Geradin, M. and Rixen, D., Mechanical Vibrations – Theory and Application to Structural Dynamics, John Wiley, 1994

[6] Glockner, P. G.; Symmetry in structural mechanics, J. Structural Div., ASCE, Vol. 99, No. ST1, pp 71-89, 1973.

[7] Jennings, A., Matrix Computation for Engineers and Scientists, Wiley, 1977.

[8] Chopra, A. K., Dynamics of Structures: Theory and Application to earthquake Engineering, Prentice-hall, New Jersey, 1995.

[9] Craig, R. R., Structural Dynamics: An Introduction to Computer Methods, John Wiley and Sons, 1981.

[10] Clough, R. W. and Penzien, J., Dynamics of Structures, McGraw-Hill, New York, 1993.

[11] STAAD/Pro 2002 User's Manual, Research Engineers International, California, USA, 2002.

[12] EBCS 8 1995, Ethiopian Building Code Standard for Design again Earthquake, Ministry of Works and Urban Development, Addis Ababa, Ethiopia, 1995.