A CLOSED-FORM SOLUTION PROCEDURE TO THE VIBRATION OF NON-CLASSICALLY DAMPED SYSTEMS SUBJECTED TO HARMONIC LOADS

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ABSTRACT

The dynamic response of a multi-degree-of-freedom (MDF) system with non-proportional damping subjected to harmonic loads is considered. Modal substitution is employed to transform the coupled differential equations of motion from geometric to modal coordinates. As might be expected, the modal transformation does not uncouple the differential equations of motion because of the inherent non-proportional damping, but transforms them into a system of coupled algebraic equations of convenient form for solution. The modal coordinates are then easily determined with the help of conventional techniques for solving systems of coupled equations. The method presented gives a closed form solution without the need to resort to iterative procedures, which would otherwise be necessary for other types of more irregular excitations like earthquake ground motions. Besides, it presented a technique of compiling the damping matrix of MDF systems exhibiting different mechanisms of energy dissipation in different regions. The application of the method is illustrated on a two-mass system, whose foundation interacts with the supporting soil - a phenomenon resulting totally to a four-degree-of-freedom system.

Keywords:
Non-proportional damping, geometric damping, modal transformation, harmonic loading

INTRODUCTION

The method of modal transformation is well recognized as a powerful technique to transform and uncouple the differential equations of motion of classically damped linear MDF systems. This is possible, because one may set the damping matrix proportional to either the mass or the stiffness matrix or both. A type of damping proportional to both the stiffness and mass matrices is called Rayleigh damping. A proportional damping is appropriate as far as the energy dissipating mechanism is more or less of uniform nature throughout the system.

This, however, is not always the case. There are vibrating systems, in which it is not possible to employ a global proportional damping matrix for the whole system. Examples of such systems include structures, whose foundations interact with the soil. Another group of examples includes structures made of different materials in distinctly different regions - say a building structure with steel frames up to a certain height and concrete frames above that or vice versa [1,2].

Unlike in classically damped systems, the damping matrix of such systems cannot be set proportional to the mass and stiffness matrices. This is because of the inherent different mechanisms of energy dissipation in the different regions of the system. Generally speaking, such systems may not always have orthogonal modes. These systems are known as non-proportionally or non-classically damped systems.

The dynamic analysis of non-classically damped systems is generally more demanding than that of classically damped systems. This is particularly true when one deals with structures, whose foundations interact with the underlying soil. Whereas the damping of the superstructure can for most practical purposes be considered to be of viscous nature, and thus a proportional damping, the damping in the soil is of different nature that cannot be set as such. Damping in soils is generally composed of geometric and material damping. The geometric damping is a result of energy dissipation due to waves propagating into the soil mass and away from the foundations. It can be the most important source of energy dissipation in thick deposits of soils. The material damping, on the other hand, represents energy dissipation due to the cyclic nature of the loading. This damping increases with increasing level of strains. As a result of this, material damping could be a good source of energy dissipation in systems subjected to strong earthquakes, whereas the geometric damping is more important in the vibration of foundations providing support to machines, in which the deformations are in the order of fractions of a millimeter.

In this paper, the dynamic response of an MDF to a set of harmonic loads is considered. The system has a non-proportional damping. Modal superposition is employed to transform coordinates. It has been shown that the modal substitution results in a transformation of a system of differential equations into a set of linear algebraic equations that can be solved easily. The number of equations in the latter is double the number of modes used for the coordinate transformation. The modal coordinates are then easily determined making use of simple matrix algebra. The technique presented is illustrated by an example consisting of a reinforced concrete frame subjected to harmonic loads at the roof level. Two different damping mechanisms are considered. A proportional damping is employed for the superstructure and a geometric damping is used for the substructure. The latter is based on recent works on foundation dynamics [3,4,5].

The material presented is expected to shade light on the importance of accounting for different sources of damping, a seemingly illusive property of vibrating systems that lessens damages, in the analysis of structures subjected to dynamic loads. Specifically, the technique presented could be of special interest to engineers dealing with vibrating systems that have damping difficult to deal with using approaches that are limited to classical damping only. This is the case, for example, in structures, whose foundations interact with the surrounding soil.

THEORY

Equations of Motion

The governing differential equations of motion of elastically behaving MDF systems subjected to externally applied dynamic loads are given by

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{f(t)\} \qquad (1)$$

where $[m]$, $[c]$ and $[k]$ are the mass, damping and stiffness matrices of the system, and $\{u\}$ and $\{f(t)\}$ are the displacement and external force vectors, respectively.

The solution of Eq. (1) for common systems with proportional damping, where the damping matrix can be set proportional to the mass and/or stiffness matrices, involves a relatively straight forward operation. In this work, the solution of this equation is sought for non-proportionally damped systems subjected to a set of harmonic loads through the modal transformation of

$$\{u\}=[\phi]\{z_n\} \qquad (2)$$

where $[\phi]$ is the modal matrix with the natural modes , $\{\phi_n\}$, as its columns and $\{z_n\}$is the vector of modal coordinates. Very often, inclusion of only a few numbers of the natural modes in Eq. (2) gives sufficiently accurate results of the response.

Free Vibration – Natural Frequencies and Natural Modes

The natural modes, $\{\phi\}$, required in Eq. (2) are determined by solving the eigenvalue problem associated with the undamped system and given by

$$([k]-\omega_n^2[m])\{\phi\}=\{0\} \qquad (3)$$

in which ω_n are the natural frequencies.

For nontrivial solution of Eq. (3), the determinant of the coefficient matrix in the round brackets on the left side is set to zero:

$$\left|[k]-\omega_n^2[m]\right|=0 \qquad (4)$$

The solution of Eq. (4) yields the natural frequencies of the system. Substitution of the frequencies in turn in Eq. (3) yields all the natural modes.

The solution of Eq. (4) for non-proportional damped systems is generally not as straight forward as for proportionally damped systems. In some cases of such systems, the stiffness matrix itself can be frequency dependent. This is, for example, the case in structures, whose foundations interact with the underlying soil. The solution of Eq. (4) in such cases may generally demand iterative procedures, in which the predominant frequency can be used as the controlling parameter. It can be shown that the natural modes so obtained satisfy the orthogonality condition with respect to the stiffness matrix as in the case of proportionally damped systems [5]. The orthogonality condition with respect to the mass matrix is always satisfied.

For the special case of systems subjected to a set of harmonic loads of single frequency, ω, which is considered in this work, the stiffness matrix can be uniquely determined using the resonance frequency. The natural modes satisfy thus both orthogonality conditions with respect to the mass and stiffness matrices.

Modal Transformation

Once the eigenvalue problem of Eq. (3) is solved, the coordinate transformation is accomplished by substituting Eq. (2) into Eq. (1) to obtain

$$[m][\phi]\{\ddot{z}_n\}+[c][\phi]\{\dot{z}_n\}+[k][\phi]\{z_n\}=\{f\} \qquad (5)$$

Pre-multiplying Eq. (5) by $[\phi]^T$ and noting the orthogonality conditions, one obtains

$$[M]\{\ddot{z}\}+[C]\{\dot{z}\}+[K]\{z\}=\{F\} \qquad (6a)$$

where, the generalized mass and stiffness matrices in Eq. (6a) are diagonalized having the diagonal elements $M_m=\{\phi_m\}^T[m]\{\phi_m\}$ and $K_m=\{\phi_m\}^T[k]\{\phi_m\}$, respectively.

The modal damping matrix, $[C]=[\phi]^T[c][\phi]$, remains, on the other hand, still non- diagonalized because of the non-proportional nature of the damping of the system considered. This is attributed to the different nature of damping mechanisms in different parts of the system. The modal damping matrix has $C_{mn}=\{\phi_m\}^T[c]\{\phi_n\}=\sum_i\sum_j\phi_{im}\phi_{jn}c_{ij}$ as its elements, in which the subscripts m and n refer to the m^{th} and n^{th} modes, whereas i and j are summation indices. The differential equations of (6a) remain thus generally coupled through the damping terms despite the transformation [5].

The modal force vector has $F_m=\{\phi_m\}^T\{f\}$ as its elements. The vector $\{z\}$ is the vector of modal coordinates.

Equation (6a) can also be written as

$$M_m\ddot{z}_m+\sum_n C_{mn}\dot{z}_n+K_m z_m=F_m;$$
$$m=1,\ 2,\ 3,\ \dots,\ N \qquad (6b)$$

where N is the number of modes included in Eq.(2) and the subscript m stands for the m^{th} mode. The coupling is once again evident in the second term of Eq. (6b).

Response to Harmonic Loading

The special case of a set of harmonic loads is now considered with the excitation frequency, ω, but different amplitudes given by

$$\{f(t)\}=\{f_0\}e^{i\omega t} \qquad (7)$$

where f_0 are the force amplitudes. Dropping the transient response, the steady-state solution to Eqs.(6) for at-rest initial condition is also harmonic and given by

$$\{z(t)\}=\{z_0\}e^{i\omega t} \qquad (8)$$

In Eq. (8), z_o are the amplitudes of the modal coordinates. Due to the shift in phase of the response with respect to the excitation, these amplitudes are complex quantities.

Substituting Eq. (8) and its time derivatives in Eqs.(6), one obtains

$$([K]-\omega^2[M]+i\omega[C])\{z_0\}=\{F_0\} \qquad (9)$$

where $\{F_0\}$ is the vector of the amplitudes of the modal forces, F. It is to be noted that the coupled differential Eqs. (6) are now transformed into the algebraic Eqs. (9) for the complex amplitudes, z_{m0}, coupled through the damping terms.

Noting that the complex vector $\{z_0\}$ can be written as

$$\{z_0\}=\{z_{0R}\}+i\{z_{0I}\} \qquad (10)$$

and separating the real and imaginary parts, one obtains the following system of 2N equations:

$$\begin{aligned}&[A]\{z_{0R}\}-\omega[C]\{z_{0I}\}=\{F_0\}\\&[A]\{z_{0I}\}+\omega[C]\{z_{0R}\}=\{0\}\end{aligned} \qquad (11)$$

In Eq. (11), the matrix $[A]$ is a diagonal matrix with $K_m-\omega^2M_m$ as its elements.

After some rearrangement, the 2N equations of (11) can be concisely written in the following form:

$$[B]\{\hat{z}\} = \omega^{-1}\{\hat{F}\} \tag{12}$$

in which the modified vectors in Eq. (12) are given by

$$\{\hat{z}\} = \begin{Bmatrix} \{z_{0R}\} \\ \{z_{0I}\} \end{Bmatrix}, \text{ and } \{\hat{F}\} = \begin{Bmatrix} \{F_0\} \\ \{0\} \end{Bmatrix} \tag{13}$$

It can be easily shown that the 2N by 2N coefficient matrix $[B]$ is given by

$$[B] = \begin{bmatrix} [\bar{A}] & [-C] \\ [C] & [\bar{A}] \end{bmatrix} \tag{14}$$

where $[\bar{A}]$ is a diagonal matrix with its elements given by $M_m\left(\omega_m^{\ 2} - \omega^2\right)/\omega$ for the m^{th} row. The submatrix $[C]$ is the generalized damping matrix explained under Eq. (6a). The assembly of this matrix will be discussed in the next section.

The modified modal coordinates are then easily determined through inversion of the coefficient matrix $[B]$ as

$$\{\hat{z}\} = \omega^{-1}[B]^{-1}\{\hat{F}\} \tag{15}$$

The amplitudes of the actual modal coordinates are determined from the argument of the complex modal displacement. For the m^{th} mode, this is given by

$$\left|(z_0)_m\right| = \sqrt{(z_{0R})_m^{\ 2} + (z_{0I})_m^{\ 2}} \tag{16}$$

Finally, the displacement amplitudes of interest in the geometric coordinates are obtained from Eq.(2).

It is important to remind the reader that the solution can also be found directly from Eq. (1), which, for harmonic loading, transforms itself directly into 2N coupled algebraic equations without the need for the modal transformation explained above. With the substitution $\{u\} = \{u_0\}e^{i\omega t}$ and after separating the real and imaginary parts Eq. (1) takes the following form:

$$\begin{aligned} ([k] - \omega^2[m])\{u_{0R}\} - \omega[c]\{u_{0I}\} &= \{f_0\} \\ \omega[c]\{u_{0R}\} + ([k] - \omega^2[m])\{u_{0I}\} &= \{0\} \end{aligned} \tag{17}$$

in which $\{u_{0R}\}$ and $\{u_{oI}\}$ are the real and imaginary parts of the vector $\{u\}$. The damping matrix needed in Eq. (17) is assembled in a similar manner to the one in the previous procedure. It is to be noted, however, that the free vibration analysis is mandatory in order to assemble the damping matrix.

This alternative procedure has its own shortcoming in that it does not provide for a means of identifying the contributions of the individual modes to the overall response. Besides, it does not have the advantage of estimating the response on the basis of the first few modes only that contribute most. This is especially important in systems of large numbers of degrees of freedom and taking into account the fact that the proposed method involves the simultaneous solution of a system of equations of double the number of the degrees of freedom. These shortcomings will be evident in the simple illustrative example treated at the end, in which the fundamental mode is almost the sole contributor to the overall response.

Assembly of the Damping Matrix

It is clear from Eqs. (14) and (15) that the damping matrix should first be established in order to determine the modal displacements. This task is relatively demanding in the dynamic analysis of non-classically damped systems. Depending on the inherent mechanisms, different regions of such systems could generally dissipate energy in different form and proportion.

One way of assembling the damping matrix in such systems involves partitioning the matrix into sub-matrices corresponding to the distinctly identified regions. Depending on the specific nature of the given system, part or all of these sub-matrices are then established as proportional damping matrices separately. These are later on assembled to obtain the non-proportional damping of the entire original system [1].

One commonly employed type of proportional damping is Rayleigh damping, which is established as a linear combination of the mass and stiffness matrices. It is of the form [1,2]

$$[c] = \alpha[k] + \beta[m] \tag{18}$$

where $[c]$, $[k]$ and $[m]$ are the damping, stiffness and mass matrices, respectively. The coefficients α and b are proportionality constants that are determined by exploiting the two orthogonality conditions, which the natural modes satisfy with respect to the stiffness and mass matrices. For this purpose, two modal damping ratios for two selected modes are specified. It can be easily shown that these constants are given by

$$\alpha = \frac{2(\xi_s\omega_s - \xi_r\omega_r)}{\omega_s^2 - \omega_r^2} \tag{19a}$$

$$\beta = \frac{2\omega_s\omega_r(\xi_r\omega_s - \xi_s\omega_r)}{\omega_s^2 - \omega_r^2} \tag{19b}$$

in which ξ_r and ξ_s are the specified damping ratios for the selected two modes – the r^{th} and s^{th} modes – with the corresponding controlling frequencies ω_r and ω_s. If the two modal damping ratios are specified to be one and the same, say ξ, then Eqs. (19) simplify to

$$\alpha = \frac{2\xi}{\omega_r + \omega_s} \tag{20a}$$

$$\beta = \frac{2\omega_r\omega_s\xi}{\omega_r + \omega_s} = \omega_r\omega_s\alpha \tag{20b}$$

The value of ξ in the above technique can be easily selected from experimental results depending on the nature of the material and structural system. It is recommended in the literature to use the fundamental frequency as one of the controlling frequencies. The other controlling frequency is recommended to be one among the higher frequencies [1].

To the category of non-classically damped systems belong structures, whose foundations interact with the surrounding soil. In such systems, the Rayleigh damping explained above can be used for the superstructure as far as this is made up of the same material and structural system, say reinforced concrete framed/dual system or steel structure. On the other hand, the foundation damping is of different nature involving both hysteresis and geometric damping. A proportional damping cannot be employed in this case. Instead, the principles of foundation dynamics are employed. This is strictly adhered to in the example problem solved below to illustrate the application of the method presented in the foregoing section.

ILLUSTRATIVE EXAMPLE

A 30cm-thick and 3.5m by 10.5m concrete platform shown in Fig. 1 carries at its center a vibrating machine inducing a harmonic horizontal force of 120 kN amplitude and 50 Hz frequency due to an unbalanced mass. The platform is supported by six 30cm by 30cm and 2.5m-high reinforced concrete columns arranged in two rows. The columns are in turn supported by a 40cm-thick and 4.3m by 11.3m rigid mat embedded 1.0m into the underlying deep uniform clay formation. It is required to determine the vibration amplitudes of the roof slab and the base shear and overturning moments at the base of the columns using a simple planar (2D) two-mass oscillator model with due consideration of the interaction of the rigid foundation with the surrounding soil. This results in a four-degree-of-freedom system. The clay formation may be assumed to have a uniform behavior with depth and exhibiting a unit weight of 16.8 kN/m³, a Poisson ratio of 0.42 and an initial-tangent shear modulus of 60 MPa. For the concrete in the superstructure and the foundation, a unit weight of 24 kN/m³ and an elastic modulus of 25 GPa may be used.

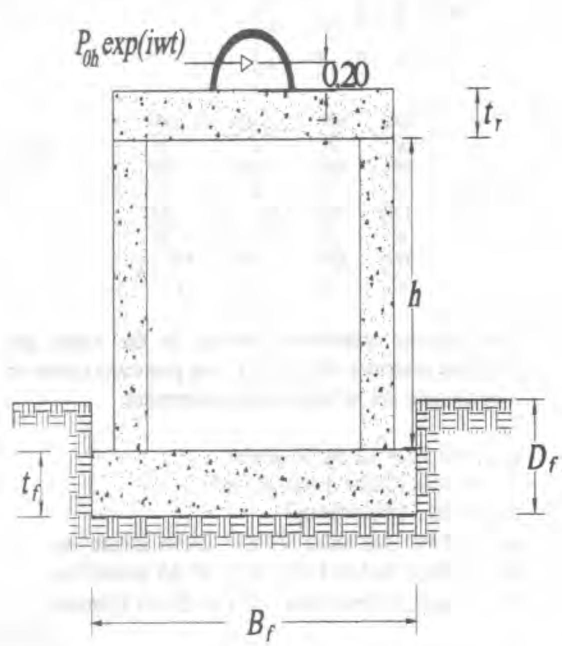

Figure 1 A concrete platform providing support to a vibrating machine

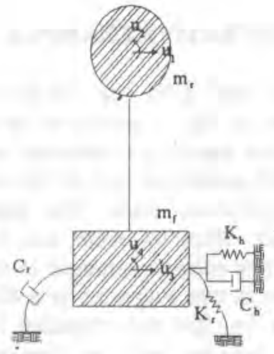

Figure 2 A four-degree-of-freedom oscillator model for the system of Figure 1

Solution

In the first place, the stiffness and mass matrices are established. As the degrees of freedom indicated in Fig. 2 are selected at points, where the masses are lumped, the mass matrix is diagonalized, whereas the stiffness matrix is not. These matrices can be easily determined using the direct stiffness approach as follows:

$$[m]=\begin{bmatrix} m_r & 0 & 0 & 0 \\ 0 & I_{\theta r} & 0 & 0 \\ 0 & 0 & m_f & 0 \\ 0 & 0 & 0 & I_{\theta f} \end{bmatrix}$$

$$[k]=\begin{bmatrix} \frac{12EI}{h^3} & \frac{6EI}{h^2} & \frac{-12EI}{h^3} & \frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{4EI}{h} & \frac{-6EI}{h^2} & \frac{2EI}{h} \\ \frac{-12EI}{h^3} & \frac{-6EI}{h^2} & \frac{12EI}{h^3}+K_h & \frac{-6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{2EI}{h} & \frac{-6EI}{h^2} & \frac{4EI}{h}+K_r \end{bmatrix} \quad \text{(E1)}$$

The various quantities needed in the mass and stiffness matrices of Eq. (E1) are provided below in a consistent set of units of measurement:

E_{eq} $= 0.5E_c = 12.5\text{x}10^6$ kN/m^2
I_c $= 6\text{x}0.3^4/12 = 4.05\text{x}10^{-3}$ m^4
$E_{eq}I_c = 50.625\text{x}10^3$kNm^2
m_r $= 24\text{x}0.3\text{x}3.5\text{x}10.5/9.81 = 26.98$ kNsec^2/m
m_f $= 24\text{x}0.4\text{x}4.3\text{x}11.3/9.81 = 47.55$ kNsec^2/m
$I_{\theta r}$ $= m_rB_r^2/12 = 26.98\text{x}3.5^2/12 = 27.54$ kNmsec^2 (E2)
$I_{\theta f}$ $= m_fB_f^2/12 = 47.55\text{x}4.3^2/12 = 73.27$ kNmsec^2
h $= 2.5$m
G $= 60\text{x}10^3$kN/m^2
$B = B_f = 4.3$m; $L_f = 11.3$m
$L_f/B_f = 11.3/4.3 = 2.628$

E_{eq} in Eq. (E2) is the equivalent stiffness, and $I_{\theta r}$ and $I_{\theta f}$ are the mass moments of inertia of the roof slab and the rigid foundation mat about their respective centroidal horizontal axes perpendicular to the plane of the page.

The following parameters are also needed in the determination of the stiffness and damping coefficients of the foundation:

$$\upsilon = 0.42$$
$$a_0 = \bar{\omega}B/2v_s = \pi fB\sqrt{\frac{\gamma}{Gg}} = 3.61 \quad \text{(E3)}$$
$$a = c_L/c_S \approx c_P/c_S = \sqrt{2(1-\upsilon)/(1-2\upsilon)} = 2.693$$

The parameter a_0 in Eq. (E3) is a dimensionless frequency parameter; the dimensionless parameter a is the ratio of the shear wave velocity, c_L, to the longitudinal wave velocity, c_S, of the foundation soil.

The static stiffness coefficients, $K_{h,st}$ and $K_{r,st}$, corresponding to the horizontal and rocking degrees of freedom of a rectangular foundation embedded in a homogenous half-space are determined as follows [3]:

$$K_{h,st} = \frac{GB}{2(2-\upsilon)}\left[6.8\left(\frac{L}{B}\right)^{0.65} + 2.4 + 0.8(L/B-1)\right]$$
$$\left[1+\left(0.33+\frac{1.34}{1+(L/B)}\right)\left(\frac{2D}{B}\right)^{0.8}\right]$$
$$= 1.85\text{x}10^6 \text{ kN/m}$$

$$K_{r,st} = \frac{GB^3}{8(1-\upsilon)}\left[3.73\left(\frac{L}{B}\right)+0.8\right]$$
$$\left[1+\frac{2D}{B}+\left(\frac{1.6}{0.35+(L/B)}\right)\left(\frac{2D}{B}\right)^2\right]$$
$$= 17.23\text{x}10^6 \text{ kNm}$$

The dynamic stiffness coefficients are obtained from [3]

$$k_h = 1.0;\quad k_r = 1 - \frac{\left(0.55+0.1\sqrt{L/B-1}\right)a_0^{\,2}}{2.4-\frac{0.4}{(L/B)^3}+a_0^{\,2}} = 0.427$$

The dynamic spring coefficients become then

$$K_h = k_hK_{h,st} = 1.85\times10^6 \text{ kN/m};$$
$$K_r = k_rK_{r,st} = 0.427(17.23\times10^6)$$
$$= 7.357\times10^6\, kNm$$

The mass and stiffness matrices of the entire system can now be compiled as given below:

$$[m]=\begin{bmatrix}26.98 & 0 & 0 & 0\\ 0 & 27.54 & 0 & 0\\ 0 & 0 & 47.55 & 0\\ 0 & 0 & 0 & 73.27\end{bmatrix};$$

$$[k]=\begin{bmatrix}38.88 & 48.60 & -38.88 & 48.6\\ 48.60 & 81 & -48.60 & 40.5\\ -38.88 & -48.60 & 1888.88 & -48.6\\ 48.6 & 40.5 & -48.6 & 7438\end{bmatrix}\times 10^{3} \qquad \text{(E4)}$$

Similarly, the force vector takes the form:

$$\{f\}^{T}=\{120 \quad -42 \quad 0 \quad 0\} \qquad \text{(E5)}$$

The eigenvalues and the eigenvectors are determined and the subsequent matrix operations performed using MATLAB 7.0. The resulting spectral and modal matrices with $\lambda=\omega^{2}$ are thus

$$[\lambda]=\begin{bmatrix}0.0025 & 0 & 0 & 0\\ 0 & 0.0402 & 0 & 0\\ 0 & 0 & 0.3979 & 0\\ 0 & 0 & 0 & 1.0155\end{bmatrix}\times 10^{5}\,\mathrm{sec}^{-2} \qquad \text{(E6a)}$$

$$[\Phi]=\begin{bmatrix}0.1607 & -0.1058 & -0.0057 & 0.0022\\ -0.1049 & -0.1589 & -0.0071 & 0.0018\\ 0.0006 & -0.0069 & 0.1448 & -0.0020\\ -0.0005 & 0.0016 & 0.0017 & 0.1168\end{bmatrix} \qquad \text{(E6b)}$$

It is evident from the spectral matrix in Eq. (E6) that none of the eigenvalues (the quantities in the main diagonal of [λ]) is identically zero verifying the fact that the natural modes of the non-classically damped system considered here are orthogonal with respect to the mass and stiffness matrices.

The natural circular frequencies are easily obtained as the square roots of the eigenvalues:

$\omega_1=15.81$ sec^{-1}; $\omega_2=63.40$ sec^{-1}; $\omega_3=199.47$ sec^{-1}; $\omega_4=318.67$ sec^{-1};

The natural mode shapes corresponding to the four vectors of the modal matrix in Eq. (E6) are sketched in Fig. 3.

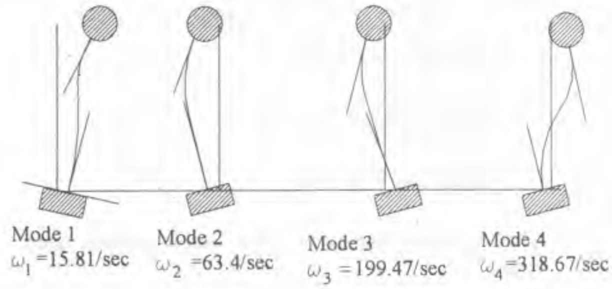

Figure 3 The natural modes and circular frequencies of the system

The damping matrix is next established, in which Rayleigh damping is employed for the superstructure only. The two proportionality constants needed for this purpose are evaluated in accordance with Eq. (20) as

$$\alpha=2\xi/(\omega_1+\omega_3)=4.645\times10^{-4}\,\mathrm{sec},\ \text{and}$$

$$\beta=2\omega_1\omega_3\xi/(\omega_1+\omega_3)=1.465\,\mathrm{sec}^{-1}$$

With these coefficients, Eq. (18) yields the damping of the superstructure as follows:

$$[c]_L=4.645\times\begin{bmatrix}38.88 & 48.60\\ 48.60 & 81.00\end{bmatrix}\times10^{-1}+1.465\times\begin{bmatrix}26.98 & 0\\ 0 & 27.54\end{bmatrix}=\begin{bmatrix}57.585 & 22.575\\ 22.575 & 77.971\end{bmatrix}$$

The dashpot coefficients for the horizontal and rocking motions of the foundation are determined in accordance with recently published materials [3,4] on foundation dynamics using

$$C=\frac{a_0 c K_{st}}{\omega}$$

The dynamic coefficients c_h and c_r for the horizontal and rocking motions of the foundation are given by

$$c_h=\frac{4L}{B}=4\times2.628=10.512,\ \text{and}$$

$$c_r=\frac{4\alpha L}{3B}\frac{a_0^{2}}{2.2-0.4/(L/B)^{3}+a_0^{2}}=8.085$$

This yields for the dashpot coefficients

$$C_h = \frac{3.61x10.512}{314.15} \times 1.85 \times 10^6, \text{ and}$$
$$= 223.48 \times 10^3 \frac{kN\,\text{sec}}{m}$$
$$C_r = \frac{3.61x8.085}{314.15} \times 7.357 \times 10^6$$
$$= 683.52 \times 10^3\, kNm\,\text{sec}$$

The damping matrix for the entire system then becomes

$$[c] = \begin{bmatrix} 0.0576 & 0.0226 & 0 & 0 \\ 0.0226 & 0.078 & 0 & 0 \\ 0 & 0 & 223.48 & 0 \\ 0 & 0 & 0 & 683.52 \end{bmatrix} \times 10^3 \qquad \text{(E7)}$$

The generalized mass, stiffness and damping matrices are calculated next to obtain:

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [I]$$

$$[K] = \begin{bmatrix} 0.0025 & 0 & 0 & 0 \\ 0 & 0.0402 & 0 & 0 \\ 0 & 0 & 0.3979 & 0 \\ 0 & 0 & 0 & 1.0155 \end{bmatrix} \times 10^5 \text{ sec}^{-2}$$

$$[C] = \begin{bmatrix} 0.0018 & -0.0014 & 0.0189 & -0.0383 \\ -0.0014 & 0.0158 & -0.2223 & 0.1287 \\ 0.0189 & -0.2223 & 4.6901 & 0.0699 \\ -0.0383 & 0.1287 & 0.0699 & 9.3259 \end{bmatrix} \times 10^3 \qquad \text{(E8)}$$

As expected, the generalized mass and stiffness matrices in Eq. (E8) are diagonalized due to the orthogonality conditions. Furthermore, these two matrices are identical to the unit and spectral matrices, respectively. This indicates that the modal vectors in [Φ] are normalized with respect to the mass matrix. Such modes are also called orthonormal modes. In contrast, the generalized damping matrix is not diagonalized.

The generalized force vector is also found to be

$$\{F\}^T = \{148796 \;\; -193726 \;\; -0.9838 \;\; 0.3359\} \qquad \text{(E9)}$$

The submatrix $[\bar{A}]$ in Eq. (14) takes the form

$$[A] = \begin{bmatrix} -313.364 & 0 & 0 & 0 \\ 0 & -301.364 & 0 & 0 \\ 0 & 0 & -187.505 & 0 \\ 0 & 0 & 0 & 9.083 \end{bmatrix}$$

After compiling the matrix [B] according to Eq.(14), the vector of the modified modal coordinates is calculated using Eq. (15) to obtain

$$\{\bar{z}\}^T = \begin{Bmatrix} -0.2405 & 0.0637 & 0.004 & -0.0019 \\ -0.0012 & 0.0007 & 0.0001 & -0.0001 \end{Bmatrix} \times 10^{-3}$$

The first four elements are the real parts and the remaining four elements are the imaginary parts of the corresponding complex modal displacement amplitudes. The vector of the arguments of the modal coordinates is obtained by applying Eq. (16). This gives

$$\{|z_0|\}^T = \{0.2405 \;\; 0.0637 \;\; 0.0004 \;\; 0.0019\} \times 10^{-3} \qquad \text{(E10)}$$

It is noted that in this particular case, the imaginary components are all negligibly small. Finally, the vector of the displacement amplitudes in the geometric coordinates is obtained from Eq. (2):

$$\{u_0\}^T = \{0.0386 \;\; -0.0252 \;\; 0.0002 \;\; -0.0001\} \qquad \text{(E11)}$$

Note that the displacement amplitudes in the directions of 1 and 3 are in millimeters, whereas the rotations in the directions of 2 and 4 are in radians.

It is of interest to compare this to the contributions of the individual modes, which can be easily obtained by multiplying each eigenvector by the corresponding modal displacement amplitude. These are given below

$$\{u_0\}_1^T = \{0.0386 \;\; -0.0252 \;\; 0.0001 \;\; -0.0001\}$$
$$\{u_0\}_2^T = \{-0.1058 \;\; -0.1589 \;\; 0.1448 \;\; 0.0017\} \times 10^{-4}$$
$$\{u_0\}_3^T = \{-0.0228 \;\; -0.0284 \;\; 0.5792 \;\; 0.0068\} \times 10^{-4} \qquad \text{(E12)}$$
$$\{u_0\}_4^T = \{0.0042 \;\; 0.0034 \;\; -0.0038 \;\; 0.2219\} \times 10^{-6}$$

It is evident from Eqs. (E11) and (E12) that the contribution to the overall displacements or rotations in the respective degrees of freedom comes almost entirely from the first (fundamental) mode. The remaining three modes play no significant role at least in this example. Particularly worth noting is the insignificant participation of the fourth mode despite the closeness of the corresponding frequency to the excitation frequency. This observation would not be possible, if the alternative direct solution procedure without modal transformation was followed. Furthermore, one could have determined the response accurately enough had the first mode only been employed. This is a clear justification for the preference of the presented solution procedure based on modal analysis to the direct one.

The vector of equivalent peak static forces along the defined degrees of freedom can also be computed as follows:

$$\{f_{s0}\}^T = [k]\{u_0\} = \{262.2361 \quad -181.0844 \quad 111.2680 \quad 1.5066\}$$

The first and third elements of the vector are forces in kN, whereas the second and forth are moments in kNm.

The peak base shear and overturning moment at the base of the columns are finally computed from considerations of static equilibrium, which result in

$$V_{b0} = (f_{s0})_1 = 262.24\ kN\ ;$$
$$M_{b0} = (f_{s0})_1 h + (f_{s0})_2 = 262.24 \times 2.5 - (-181.08) = 836.68\,kNm$$

These final results, which include the contributions of all the natural modes, are the basis for the design of the structure. It is important to note the significant amplification of the base shear from 120 kN to 262.24 kN by well over 150% – a fact attributable to the dynamic nature of the loading despite the inherent damping both in the superstructure and in the soil.

CONCLUSION

In the foregoing, it has been shown that the coupled differential equations of motion of non-classically damped systems subjected to harmonic loads can be transformed easily into a system of coupled algebraic equations. The number of the algebraic equations is double that of the number of degrees of freedom. It has also been shown that the modal transformation technique is superior to the direct solution procedure if judiciously used.

The method presented leads to a closed form solution and thus does not demand any iterative procedure, which would otherwise be the case for excitations of more general and non-harmonic nature. Furthermore, a technique of compiling the damping matrix of systems exhibiting different mechanisms of energy dissipation in different regions has been presented.

With the intention of staying within the scope of the work, the application of the proposed method is illustrated using a relatively simple example, but to a sufficient detail. The method has also applications in more practical problems including framed foundations providing support to heavy rotary machines like turbo-generators that induce harmonic loads and in slender structures subjected to wind-induced vortex shedding, cases in which the foundations interact with the underlying soil. A detailed treatment of the applications of the method to such practical cases is deferred to future works.

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