A tensor approach to the estimation of hydraulic conductivities in Table Mountain Group aquifers of South Africa

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Abstract

Based on the field measurements of the physical properties of fractured rocks, the anisotropic properties of hydraulic conductivity (*HC*) of the fractured rock aquifer can be assessed and presented using a tensor approach called hydraulic conductivity tensor. Three types of *HC* values, namely point value, axial value and flow direction one, are derived for their possible applications. The *HC* values computed from the data measured on the weathered or disturbed zones of rock outcrops tend to give the upper limit values. To simulate realistic variations of the hydraulic property in a fractured rock aquifer, two correction coefficients, i.e. the fracture roughness and combined stress conditions, are adapted to calibrate the tensor model application. The application results in the Table Mountain Group (TMG) aquifers show that the relationship between the *HC* value and fracture burial depths follows an exponential form with the power hyperbola.

Keywords: hydraulic conductivity tensor, roughness, combined stress, hydraulic aperture, Table Mountain Group (TMG)

Introduction

Darcy's law is always used to estimate the groundwater flow in both porous and fractured media, depending upon realistic estimates of aquifer hydraulic conductivities (viz. *k*) and hydraulic gradients (viz. *J*) at the scale of problem. In the case of fractured rock aquifers, presentation and determination of the hydraulic conductivity prove to be challenges. With respect to a fracture set with a mean aperture of *b* and a parallel face distance of *d*, the following classic expression is adopted for flow through the set of conduits (Talobre, 1957; Jaeger, 1972):

$$q = \frac{gb^3}{12\mu d} \cdot J \qquad (1)$$

where:

ρ is the density of fluid

μ is dynamic viscosity of fluid, which is 10^{-6} m^2/s for water at 20°C

g the acceleration of gravity

J hydraulic gradient

Eq. (1) represents an idealised type of flow behaviour that has been intensively studied, both experimentally and numerically, by many researchers. The term $gb^3/12\mu d$ in Eq. (1) is usually referred to as the hydraulic conductivity (*HC*) *K* for the set of fractures involved. For the determination of *K* value, many theories and methods have been developed. A series of results for one of the intrinsic properties of fractured rock aquifers have been obtained to various extents for more than 30 years. As a summary, there are thus far three approaches to the estimation of hydraulic conductivity of fractured rock aquifers, namely:

- *HC* tensor approach based on statistic or stochastic methods of *in situ* fracture geometry and physical measurements
- Fracture property field and laboratory tests for the parameter *K* evaluation
- Inverse analysis on continuous or discontinuous problems dependent on numerical models and parametric calibrations.

The estimation of *K* values using either pumping or packer test is based on the assumption that the groundwater is flowing through a geological continuum. It is often an expensive exercise to estimate and predict the regional aquifer properties (viz. *K* and *J*, etc.) from local-scale hydraulic tests. Also the large variation of *HC,* both along borehole sections and in between holes, usually makes it difficult to determine the representivity of the parameters in terms of groundwater assessments. Even where a representative elementary volume (*REV*) can be defined, it may not be appropriate to directly apply the local test results to a regional aquifer. In porous media the *REV* can be very small, whereas in fractured media the *REV* may be very large or even does not exist in some cases (Kulatilake and Panda, 2000; Wang et al., 2002).

The statistic methods for calculating *HC* tensor were developed in 1980s (Hsieh and Neuman, 1985; Hsieh et al., 1985; Oda, 1985; Tian, 1988). The results from these methods can successfully indicate 3-D principle *HC* values and directions by means of coordinate rotation of the incorporation of input data that derived from the surface measurements. The basic assumptions of the tensor approach are:

- Groundwater flow is exclusively governed by fractures
- The fractures through a rock matrix are well-connected
- Flows between fracture sets do not interfere, or no deflection flow occurs.

For the ideal flow pattern with *M* sets of fractures involved in a study area, the hydraulic conductivity of the fracture sets is expressed in the form of matrix which reflects a sort of flow superposition:

$$\begin{cases} \overline{\overline{K}} = \sum_{i=1}^{M} \left(\frac{gb_i^3}{12\mu d}\right)_i \cdot K_{jl}^i, \quad j,l = x,y,z \\ K_{jl}^i = [I - \overline{n}^T \cdot \overline{n}] \end{cases} \qquad (2)$$

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where:

$\bar{\bar{K}}$ is the *HC* tensor matrix accounting for the anisotropic nature of studied media

I is the unit matrix

$\bar{n}$ is the direction cosine vector whose components are expressed in terms of the dip azimuth β and dip angle α of the fracture sets in the coordinate system where the x, y axes are pointing to the north and east direction, respectively, while z axis is pointing upward.

The elements of the matrix are dependent on the geometric and physical parameters of fractures, which were studied by many others (Tian, 1989). From Eq. (2) two key *HC* parameters, namely principal *K* values and orientations and their corresponding composite *K* value can be achieved by employing the techniques of linear algebra arithmetic.

However, the complexity of *K* in fractured rocks is far beyond what the existing models could handle since there are many geometrical and mechanical factors that impact on the flow through fracture gaps (Snow, 1969; Peter, 2001). Among them, fracture aperture is known to be most critical in controlling the quantity of flow, and the study of the aperture doubtlessly becomes a main focus in this paper. The factors affecting the aperture of fractures include geometrical and mechanical properties of the fracture walls such as rigidity, roughness, mineral fillings, and stress levels surrounding the fractures, in which the roughness has a crucial impact on the aperture around the surface zone (Lomize, 1951; Louis, 1974; Patir and Cheng, 1978). Therefore, an expression of equivalent aperture due to roughness is suggested to calibrate the original *HC* tensor model. Furthermore, by taking into account crustal stress, lithostatic and hydraulic pressures that act on fractures, together with the equivalent aperture, an expression of hydraulic aperture is accordingly developed for model calibration purposes.

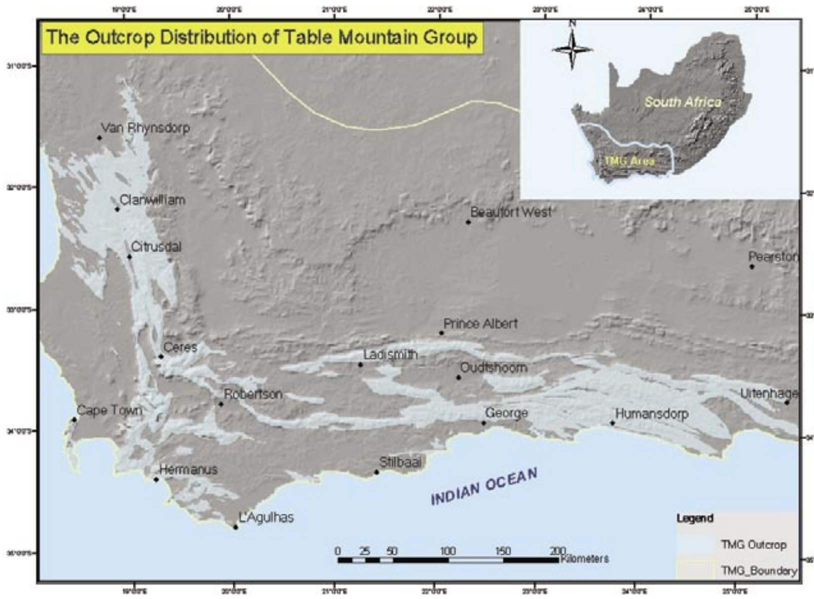

Figure 1
The TMG outcrop basically stretches along mountain ranges in Western and Eastern Cape of South Africa

HC tensor approach to the TMG fractured rocks

Background of the Table Mountain Group aquifer

The Table Mountain Group (TMG) comprises a sequence of sedimentary units that extends from Vanrhynsdorp in the north-west to the Cape Peninsula in the south and then incurves eastward to Port Elisabeth (Fig. 1). As a part of African Craton, the distribution of TMG extends to an area of about 248 400 km^2 with the outcrops of 37 000 km^2. It forms the backbone of the Cape Fold Belt that was produced in Permo-Triassic period extending from Australia through Antarctica and South Africa to South America (McCathy and Rubidge, 2005). The structural frame was modified as an effect of the break-up of Gondwanaland during the Jurassic to Cretaceous time periods, which led to a series of tensile and dextral displacements.

Due to the experiences of groundwater usage by borehole abstractions, together with the cognitions of lithological characteristics, stratigraphic build-ups and structural fabrics of the Table Mountain Group, it has been concluded that the TMG is a regional aquifer system which may extend to great depths (2 000~5 000 m b.g.l). It has also been recognised that the fractured rocks mainly consist of sandstones, siltstones, and sandwiched shales and mudstones that were formed during the Ordovician to the Silurian period, 500 to 400 Ma ago, with sedimentation along a south-eastward trough (Rust, 1967, 1973; De Beer, 2001).

As underlain by Precambrian metamorphic rocks, overlain by mid- to neopalaeozoic basin deposits and bounded by some regional faults such as Kango and Worcester faults, the southernmost aquifer system on the African continent has the potential to become a major source of bulk water supply for both agricultural and urban requirements in the Western and Eastern Cape Provinces of South Africa. Extensive exploration and exploitation of the groundwater resource in the aquifer system have been done for about 30 years; and more than 45 Mm^3 of groundwater is annually abstracted in about 30 locations for the requirements of municipalities, irrigation farmers and holiday resorts. Minor users are the smaller scale farmers, homesteads and stock farms. Currently, major problems faced are the lack of information on the properties of the huge fractured rock aquifers and shallow and deep groundwater circulation. With regard to the determination of key aquifer parameters, such as hydraulic conductivity, transmissivity and storativity, the estimates at different scales are largely interpreted from borehole tests. In terms of the hydraulic conductivity, a wide range of *K* values from 1.99 to 1.99×10^{-3}m/d has been given in various locations of the TMG area (Rosewarne, 2001). In some well-fields the overestimation of the aquifer parameters including the *K* value leads to unrealistic recommendations on water supply capacity, causing continuous decreases of borehole water levels or even a water scheme failure (Jolly, 2001).

Adaptation of hydraulic conductivity tensor theory

It is well established that the direction of groundwater velocity (*V*) and hydraulic gradient (*J*) is usually not coincident with each other over time in fractured rock media. There is an included angle θ between *V* and *J*, and the *J* component on flow direction is:

$$J_V = J \cdot \cos\theta \tag{3}$$

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In an anisotropic medium, the *HC* is not a scalar any more compared with the isotropic one. The flow velocity is correspondingly expressed as follows:

$$\left.\begin{aligned}\vec{V}_x &= -(K_{xx}\frac{\partial H}{\partial x} + K_{yx}\frac{\partial H}{\partial x} + K_{zx}\frac{\partial H}{\partial x})\boldsymbol{i} \\ \vec{V}_y &= -(K_{xy}\frac{\partial H}{\partial y} + K_{yy}\frac{\partial H}{\partial y} + K_{zy}\frac{\partial H}{\partial y})\boldsymbol{j} \\ \vec{V}_z &= -(K_{xz}\frac{\partial H}{\partial z} + K_{yz}\frac{\partial H}{\partial z} + K_{zz}\frac{\partial H}{\partial z})\boldsymbol{k}\end{aligned}\right\}. \tag{4}$$

where:

$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors on coordinates *x, y, z* respectively.

Its parameter term is generally presented in the form:

$$\overline{\overline{K}} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \tag{5}$$

Eq. (5) is the hydraulic tensor matrix, in which $K_{xy}=K_{yx}$, $K_{yz}=K_{zy}$, and $K_{xz}=K_{zx}$. Note that in Eq.(2) the expression of the row matrix $\overline{n}$ is:

$$\overline{n} = [\cos\beta_i \sin\alpha_i \quad \sin\beta_i \sin\alpha_i \quad \cos\alpha_i], \quad i = 1, \cdots, M\,. \tag{6}$$

The above $\overline{\overline{K}}$ is a symmetric square matrix with three different eigenvalues and corresponding orthogonal eigenvectors satisfying the following relation:

$$(K_{ij} - \lambda_i \delta_{ij})U_i = 0, \quad U_i = [U_{ix} \quad U_{iy} \quad U_{iz}]^T, \quad i = 1,2,3. \tag{7}$$

where:

δ_{ij} is Kronecker's symbol

λ_i the eigenvalues

U_i is the corresponding eigenvectors associated to λ_i

Eq. (7) is the representative of a homogeneous equation group and the solutions of λ_i and U_i (*i*=1, 2, 3) may be obtained from Eqs. (8) and (9) respectively.

$$|K - \lambda I| = 0 \tag{8}$$

$$\begin{cases}(k_{xx} - \lambda)U_x + k_{xy}U_y + k_{xz}U_z = 0 \\ k_{yx}U_x + (k_{yy} - \lambda)U_y + k_{yx}U_z = 0 \\ k_{zx}U_x + k_{zy}U_y + (k_{zz} - \lambda)U_z = 0 \\ U_x^2 + U_y^2 + U_z^2 = 1\end{cases} \tag{9}$$

Using U_i (*i*=1,2,3) as the basic vectors, the *HC* tensor reduces to the diagonal matrix:

$$\overline{\overline{K'}} = diag[\lambda_1, \quad \lambda_2, \quad \lambda_3] \tag{10}$$

Hydrogeologically it is easy to understand that the λ_i (*i*=1,2,3) represent three principal *HC* values, namely $K_{p1}=\lambda_1$, $K_{p2}=\lambda_2$ and $K_{p3}=\lambda_3$, and the direction of K_{pi} is given by *Ui*. Whereby the composite *HC* value of a given sample site can be represented by the geometric mean of the three principal *HC*:

$$K_{comp} = \sqrt[3]{K_{p1} \cdot K_{p2} \cdot K_{p3}} \tag{11}$$

Furthermore, the relationship between *HC* components K_x, K_y and K_z along x, y, and z axes defined above and principal *HC* tensor can be established via U_i:

$$\begin{cases}K_x = K_{p1}U_{1x}^2 + K_{p2}U_{2x}^2 + K_{p3}U_{3x}^2 \\ K_y = K_{p1}U_{1y}^2 + K_{p2}U_{2y}^2 + K_{p3}U_{3y}^2 \\ K_z = K_{p1}U_{1z}^2 + K_{p2}U_{2z}^2 + K_{p3}U_{3z}^2.\end{cases} \tag{12}$$

Assuming that the flow direction is known (Fig.2), using Eqs. (3) and (4) we have the following expression for *HC* along flow direction:

$$K_V = \frac{\vec{V}\cdot\vec{V}}{\vec{V}\cdot\vec{J}} = \frac{(K_x\frac{\partial H}{\partial x}\boldsymbol{i})^2 + (K_y\frac{\partial H}{\partial y}\boldsymbol{j})^2 + (K_z\frac{\partial H}{\partial z}\boldsymbol{k})^2}{K_x(\frac{\partial H}{\partial x}\boldsymbol{i})^2 + K_y(\frac{\partial H}{\partial y}\boldsymbol{j})^2 + K_z(\frac{\partial H}{\partial z}\boldsymbol{k})^2}$$

$$= \frac{1}{\frac{1}{V^2}(\frac{V_x^2}{K_x} + \frac{V_y^2}{K_y} + \frac{V_z^2}{K_z})} \tag{13}$$

Alternatively, using the projective relations of $V_x/V=cos\beta \cdot sin\alpha$, $V_y/V=sin\beta \cdot sin\alpha$ and $V_z/V=cos\alpha$ (see Fig.2), one may also write that:

$$\frac{1}{K_V} = \frac{\cos^2\beta\sin^2\alpha}{K_x} + \frac{\sin^2\beta\sin^2\alpha}{K_y} + \frac{\cos^2\alpha}{K_z}. \tag{14}$$

The above analyses indicate that the three types of *K* values, namely composite *HC* K_{comp}, axial *HC* K_i (*i*=x,y,z) and flow direction *HC* K_V, are physically different. However, they are related through the *HC* tensors and vector projections. The former (*HC* K_{comp}) may be accounting for the *K* value at a measuring site in the form of scalar that is averaged from three anisotropic principal *HC*. The *HC* K_i (*i*=x,y,z) are the projections of three principal *HC* along original or user-defined coordinate system x,y,z. This is important in practice, for it is more convenient to use the K_i (*i*=x,y,z) than to use the three principal *HC* directly in groundwater modelling processing. Note that the quantity of *HC* K_V is not simply the quadratic sum of K_i (*i*=x,y,z) as it used to be because the flow direction *HC* is more physically determined by the components of flow velocity and head gradient than geometrically determined by axial *HC*. There exists a non-linear relationship between K_V and K_i (*i*=x,y,z). Accordingly, if there exist not less than two sets of foregone values of K_V and K_i (*i*=x,y,z) obtained from hydraulic tests and surface measurements respectively, it is possible to deduce the flow direction by using the following linear equation:

$$(K_x^{re} - K_y^{re})x_1 + (K_y^{re} - K_z^{re})x_2 = K_V^{re} - K_z^{re} \tag{15}$$

where:

$x_1 = \cos^2\beta\sin^2\alpha$

$x_2 = \sin^2\alpha$

$K_V^{re} = 1/K_V$

$K_i^{re} = 1/K_i, i = x, y, z$

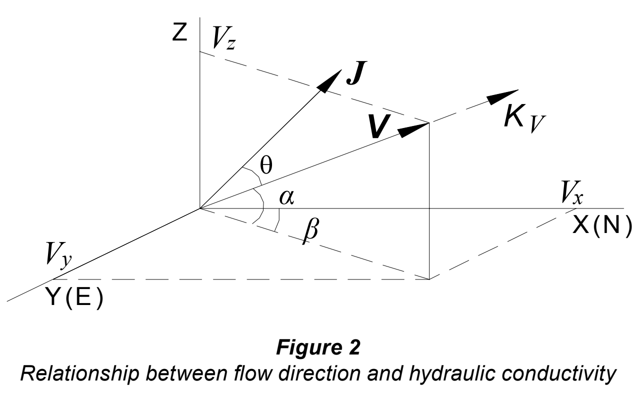

Figure 2
Relationship between flow direction and hydraulic conductivity

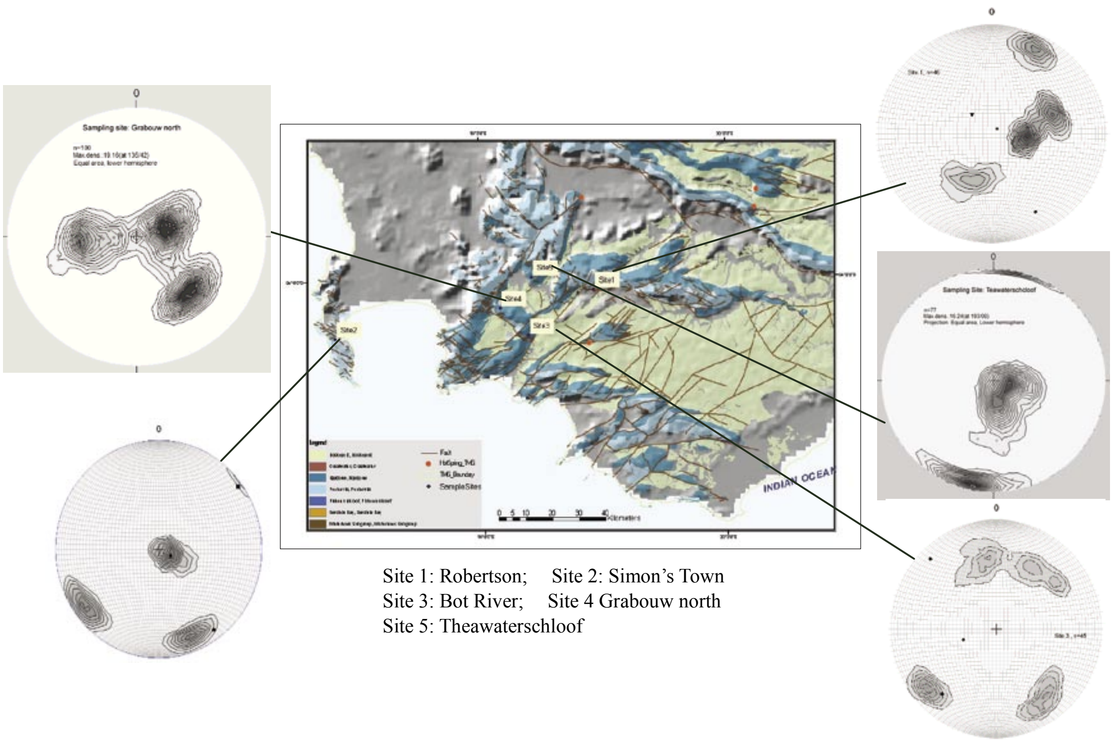

Figure 3
Showing geological settings of the TMG in the study area of structural syntax zone. The sizes of sampling sites are in the range of 20×30m ~ 60×60m.

The analysis on the thin sections sampled from TMG sandstones reveals that the porosity of intact rocks is very low or even null. This implies that the motion of groundwater in those aquifers is basically controlled by various types of discontinuities in the form of bedding planes, joints, faults, unconformities and weathered fractures. There are at least three sets of fractures in TMG rocks. Those are bedding fractures, conjugate joints, and the others structural and weathered fractures.

The above-mentioned relations have been programmed in MS Excel Workbook. In datum preprocessing, the basic data were measured at sites of the TMG outcrops from which the aperture (b) and distance (d) are geometric mean values and the angles arithmetic ones. Figure 3 shows the site locations and distributions of fracture orientations in the form of fracture density. The computed results, using the above-mentioned models, are listed in Table 1. The composite *HC* in Table1 are ranging from 1 to 21 m/d which are much higher than those of borehole tests mostly ranging from 5×10^{-2} to 1.5×10^{-3}m/d. This is because of the inevitable magnification of results when applying the measured data to the smooth plate model. Particularly, the apertures, measured at road cuttings and open quarries where the fractures that have long been undergoing disturbance and stress release, tend to be dilated compared with their original status.

Determination of hydraulic aperture

Equation (1) for groundwater flow in fractures is derived from the Navier-Stokes differential equation for pressure-induced laminar flow through the gap of two flat parallel plates where the hydraulic aperture is assumed to be uniform. The actual condition of the TMG is quite different, as the rocks including fractures have undergone phases of deformations or even distortions. What can be measured at rock outcrops is the mechanical aperture (Olsson and Barton, 2001) mostly ranging from 1 to 10^{-3}mm. The hydraulic aperture discussed here is regarded as the effective aperture for groundwater flow and can be obtained or inferred from both tracer tests (Charles, 1988) and laboratory experiments. Taking into account the main influencing factors, i.e. roughness and stress conditions, we rewrite Eq. (2) by adding the correction coefficients due to fracture roughness and stress condition respectively:

$$\overline{\overline{K}} = \sum_{i=1}^{M} \frac{g b_i^3}{12 \mu d}_i \, C_{er} \cdot C_{es} \cdot K_{jl}^i , \quad j,l = x,y,z \qquad (16)$$

where:

C_{er} is the correction coefficient of roughness

C_{es} is the correction coefficient of stress condition.

TABLE 1
Calculated *HC* values from surface fracture measurements

HC Values / Site No.	Principal *HC* values (m/d)				Axial *HC* values (m/d)		
	K_{p1}	K_{p2}	K_{p3}	Composite *HC*	K_x	K_y	K_z
1. Robertson	2.313	1.880	0.437	1.239	1.940	2.029	0.662
2. Simon's Town	13.816	11.858	2.176	7.091	13.039	12.261	2.550
3. Bot River	38.538	31.257	8.622	21.818	23.741	26.521	28.153
4. Grabouw	13.258	11.804	1.869	6.639	11.121	10.263	5.549
5. Theewaterskloof	6.355	6.007	0.543	2.748	1.554	5.507	5.844

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