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NEARNESS OF FRACTIONAL OPERATORS FOR SOLVABILITY OF NATURAL PERTURBATIONS OF FRACTIONAL DYNAMIC OPERATORS

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ABSTRACT

Variation of parameters is obtained on the application of nearness properties of operators to a natural perturbation of nonlinear fractional operators. The nearness principle yields solvability of difficult perturbed nonlinear fractional integral equations via variation of parameters.

KEYWORDS: Fractional integral equations, nearness of operators, small perturbations, variation of parameters
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INTRODUCTION

The nearness concept in operator theory (Barbagallo *et al*, 2019) generalizes the concept of contracting mappings by identifying small perturbations of bijective mappings to obtain fixed points, zeros and solutions of associated problems. The introduction of the principle of nearness of operators by Campanato, (1994) and Annamaria *et al* (2019). in the nineties turned out a tremendous unifier of several existence methods. In this praxis, we have applied properties of the nearness of functions to obtain hitherto unresolved natural perturbation problems in fractional are dynamic problems that used to defy the methods of variation of parameters. Here, we undertake investigations of the solvability of fractional Volterra integral equations of the form:

(1) $u(t) = h(t) - \int_0^t a(t-s) - g\big(t, u(s)\big)ds$

where the kernel $a(t-s)$ is singular and completely monotone and the nonlinearity $g(t,u(t))$ needs not be contractive nor a perturbation of identity by a contraction mapping. The cases when $g(t,u(t))$ is perturbation $g(t,u) = \eta u + g_0(t,u(t))$ of identity by a contraction mappings $g_0(t,u(t))$ have been investigated by (Burton-Zhang, 2009), (Burton, 2011) and many others in the references therein; all of them circumventing the case of the most natural perturbation $g(t,u) = \eta u + g_0(t,u(t))$, where $g_0(t,u(t)) = g(t,u(t)) \pm u(t)$.

This type of perturbation in an equation defies so many solvability techniques in that neither Schauder fixed point theorem Granas-Dugundj, (2003)) nor the popular contraction mapping principle is applicable. Worst still, due to the fact that the nonlinearity $g(t,u)$ is not the usualsmallperturbation of identity the Miller's variation of parameters method (Miller, 1971) fails to be useful. But on application the of nearness principle, the method of variation of parameters can be recovered if the nonlinearity $g(t,u)$ is near to identity in the sense of (Companato, 1994). In this case, following the methodology of R. K. Miller, we obtain variation of parameters in the following steps:

(A) We use the natural perturbation to separate the integral non-linearnon-linearequations into linear and non-linear parts.
(B) We use the nearness properties of the nonlinearity to nonself-contractionobtain a nonself contraction mapping.
(C) We apply Rothe's fixed point theorem to the associated operator.

It is important to observe that since the nonlinearity is not a contraction mapping, it may be difficult to apply (Krasnoselskii, 1958) fixed point theorem for sum of two mappings in this instance. But it will be of advantage to formulate conditions under which the problem can be adapted to conform with applicability of Krasnoselskii fixed point theorem. For the rich development on Krasnoselskii-type fixed point theorem we make reference to (Demling, 1985), (Edelstein, 1966), (Garcia and Latrach, 2012). Other related important works can be found in (Granas and Dugundj, 2003), (Oregan, 1996) and (Smart, 1974). Due to the recurrent and cyclic nature of this natural perturbation, investigations of solvability via this case still remain shallow or fallow. In this work we apply certain types of nearness to identity function to obtain a desirable variation of parameters for associated fractional integral equations.

As remarked in (Burton, 2009, 2011) when the kernel $a(t-s)$ in (1) is singular and completely monotone its resolvent $r(t-s)$ is also completely monotone and $\int_0^\infty r(t)ds = 1$ if the integral of the kernel $a(t)$ diverges (i.e. $\int_0^\infty a(t)dt = \infty$) which is the case with kernels $a(t-s) = (t-s)^{\alpha-1}$; $\alpha \in (0,1)$ of fractional dynamic problems of the form given below:

(2) $u(t) = h(t) - \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\big(t, u(s)\big)ds$

Here, the fractional integral above is defined in the sense of Riemann-Liouville. Though there are many variant definitions of fractional calculus, that of Riemann-Liouville remains a very generic definition. But, in general, no definition of fractional derivative is optimal since each is customized to specific problems at hand. In this study we restrict our investigations with respect to the Riemann Liouville definition because the Riemann-Liouville derivative is popularly used by physicists and engineers in automation, control theory and signal processing especially for image enhancement and texture analysis. Secondly, we prefer formulation our investigations in terms of Riemann Liouville fractional integral because its theory is more well developed than that of its derivative. The Riemann derivative poses generic problems in its fractional differential equations due to presence of not well-defined initial conditions. Lastly, after converting Riemann-Liouville differential equation into its equivalent fractional integral equation this drawback disappears.

For $u \in L^1[a,b]$ and $\alpha \in (0,1)$, we define the left Riemann-Liouville fractional integral by

$$J_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

Where $\Gamma(\alpha)$ denotes the gamma function; and the left Riemann-Liouville fractional integral is given by

$$I_{b^-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\, ds$$

METHODS

Our methodology requires application of the following proposition

Theorem 1:
For $\alpha \in (0,1)$ we have
(i) $J_{a^+}^{s}$ is a continuous operator from $L^p(a,b)$ into $L^r(a,b)$ for $p \in \left[1, \frac{1}{\alpha}\right)$ and $r \in \left[1, \frac{p}{1-\alpha p}\right)$

(ii) For $p > \frac{1}{\alpha}$, $J_{a^+}^{s}$ is a continuous operator from $L^p(a,b)$ into $L^r(a,b)$ with $r \in [0,\infty)$

(iii) The fractional integral $J_{a^+}^{\alpha} u(t)$ is a continuous operator from $L^\infty(a,b)$ into $C^\alpha(a,b)$. Where C^α denotes the space of Holder's continuous function of order α.

Note

$$J_{a^+}^{\alpha}(BV[a,b]) \subset C^\alpha(a,b)$$

Hence, $J_{a^+}^{\alpha}(AC[a,b]) \subset C^\alpha(a,b)$
Given the nonlinearity $g(t,u)$ our methodology consists in obtaining the linear part $u_0(t) = h(t) - \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} u(s)ds$ and the nonlinear part $-\frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[u(s) - g(t,u(s))]ds$ which yields the desired variation of parameters below provided $g(t,u)$ is near identity function:

$$u(t) = u_0(t) + \int_0^t r(t-s)\,\big(g(s,u(t) - u(s)\big)ds$$
$$= u_0(t) - \int_0^t r(t-s)[u(s) - g\big(s,u(s)\big)ds$$

where $u_0(t) = h(t) + \int_0^t r(t-s)h(s)\,ds$ is the well-known solution of the linear part $u_0(t) = h(t) - \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} u(s)ds$ and the resolvent $r(t-s)$ is defined by $r(t) = a(t) - \int_0^t a(t-s)r(s)ds$.

Many problems of this nature often defy application of the Krasnoselskii fixed point theorem which informs research for newer principles like perturbation and nearness properties introduced by (Campanato, 1994) studied herein.

Theorem 2.
Krasnoselskii fixed point theorem.
Let M be a closed convex non-empty subset of a Banach space X. Suppose that A and B
map M into X and that

(i) $Ax + By \in M$ $(\forall x, y \in M)$,

(ii) A is compact and continuous.

(iii) B is a contraction mapping.

Then there exists $y \in M$ such that $Ay + By = y$.
The nearness principle follows from the following generalization of Neuman's lemma by Campanato

Theorem 3: Campanato, 1994)
Let E ba a real Banach space. $T: E \to E$ a nonlinear mapping such that there exist $\lambda > 0$ and $\alpha \in [0,1)$ and

(3) $\big||(u-v) - \lambda(Tu - Tv)|\big| \le c\big||u-v|\big|$ then $Lip(I - \lambda A) \le c$ and $Lip(A^{-1}) \le \frac{\lambda}{1-c}$.

Where $Lip(T)$ is the Lipchitz norm of T.
We recall a mapping $A: X \to E$ of a subset X of a Banach space E is said to be near a mapping $B: X \to E$ if there exist two constants $\lambda > 0$ and $c \in [0,1)$ such that the following property holds:

(4) $\big||(Bu - Bv) - \lambda(Au - Av)|\big| \le c\big||Bu - Bv|\big|$

It follows from (4) that when B is the identity mapping i.e. $B = I$ then A is said to be near identity mapping I. Our investigation is limited to the property of nearness of A to identity mapping given below:

(5) $\big||(u-v) - \lambda(Au - Av)|\big| \le c\big||u-v|\big|$
Our main result is an application of Rothe's fixed point theorem below.

Theorem 4: (Smart 1974)
Let E be a Banach space, X a closed unit ball in E with boundary ∂X. Let T be a countable compact mapping of X into E such that $T(\partial X) \subset X$, then T has a fixed point.

MAIN RESULT

Theorem 5:
Let $u(t) = h(t) + \int_0^t a(t-s)g\big(s,u(s)\big)ds$ be a given integral equation with singular kernel $a(t-s)$ which is

completely monotone with $\int_0^t a(t)dt = \infty$. If the nonlinearity $g(t, u(t))$ is near identity operator, then the integral equation has a unique solution provided $g(t, 0) = 0$.

Proof
We start by rewriting the nonlinearity $g(t, u(t))$ as

(6) $g(t, u(t)) = u(t) + g_0(t, u(t))$; where $g_0(t, u(t)) = g(t, u(t)) - u(t)$

Then by variation of parameters, we obtain the solution of the equation as

$$u(t) = u_0(t) + \int_0^t r(t-s)\,(g(s, u(t) - u(s))ds$$

(7) $= u_0(t) - \int_0^t r(t-s)[u(s) - g(s, u(s))ds$

(8) where $u_0(t) = h(t) + \int_0^t r(t-s)h(s)\,ds$

Let $(Tz)(t) = u_0(t) - \int_0^t r(t-s)[z(s) - g(s, z(s))ds$

Then $T: B_1(u_0) \to E$ is a contraction mapping and to apply Rothe's Theorem 4, we need to prove that $T: \partial B_1(u_0) \to B_1(u_0)$ that is in particular, $||Tz - u_0|| < 1$.

But, $||Tu - u_0||_{L^1} = \int_0^\infty |\int_0^t r(t-s)[(g(s, u(s)) - g(s, 0)]ds\,|dt$

$$\leq c\int_0^\infty \int_s^\infty r(t-s)dt|u(s)|ds$$

$$= c\int_0^\infty |u(s)|ds = c||u||_{L^1}$$

Therefore, $Tu \in B(u_0)$ whenever $u \in \partial B(u_0)$ and by Rothe's Theorem 4, there is a unique solution. Hence, the integral equation has a unique solution. End of proof □

APPLICATIONS
The next theorem illustrates applications of nearness principle to resolution of hitherto difficult solvability of fractional dynamic problems with degenerate perturbation of identity by nonlinearities which are neither small nor contractive.

Theorem 6:
Let $u(t) = h(t) - \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, u(s))ds$; $\alpha \in (0,1)$ be a given fractional integral equation such that

(HI) The nonlinearity $g(t, u(t))$ is near identity mapping.
(HII) $g(t, 0) = 0$.
(HIII) $h \in L_1[0, \infty)$
Then the fractional integral equation has a unique solution in the space $L_1[0, \infty)$.

Proof
As mentioned above, the fractional integral equation can be decomposed into the linear part $u_0(t) = h(t) - \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$ with the well known solution $u_0(t) = h(t) + \int_0^t a(t-s)h(s)\,ds$ where the resolvent $r(t-s)$ is defined by $r(t) = a(t) - \int_0^t a(t-s)r(s)ds$ followed with the nonlinear part $-\frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[u(s) - g(s, u(s))]ds$.

These two parts are combined into one formulation via variation of parameters since It is well known that the kernel $a(t-s) = (t-s)^{\alpha-1}$ is completely monotone with $\int_0^\infty t^{\alpha-1}\,dt = \infty$. Hence by R. K. Miller's results ((Miller,1971) -see (Burton-Zhang, 2009) and (Burton, 1998, 2011)) it follows that the resolvent $r(t-s)$ is also completely monotone with $\int_0^\infty r(t)\,dt = 1$.

Therefore, by Theorem 5, the associated operator $(Tz)(t) = u_0(t) - \int_0^t r(t-s)[z(s) - g(s, z(s))ds$ satisfies the contraction mapping condition:

$$||Tu - Tv|| = \int_0^\infty |\int_0^t r(t-s)[\{u(s) - v(s)\} - (g(s, u(s)) - g(s, v(s))]ds|\,dt \leq c\int_0^\infty |\int_0^t r(t-s)(v(s) - u(s))ds\,|\,dt$$

since g is near identity.

$$\leq c||u - v||.$$

From Theorem 5 above, we know that T is a mapping from the ball $B_1(u_0)$ into the Banach space $L_1[0, \infty]$ such that the boundary $\partial B_1(u_0)$ satisfies $T(\partial B) \subset L_1[0, \infty]$. So T has a unique fixed point which is a solution of the integral equation. End of proof. □

Example:
The versatility of the nearness principle of operators is that, given any two functions A and B in a given Banach space, we have the following alternatives either:

(a) A is near B
(b) Or $-A$ is near B
(c) Or A is orthogonal to B.

So, this informs the motivation of various researches into diverse connections between the nearness property and orthogonality property. These connections yield many positive possibilities leading to various generalizations of the concept of orthogonality in very general spaces like locally convex spaces and arbitrary metric spaces. Next, using the fact that it either e^{-u} is near identity or $-e^{-u}$ is near identity, we attempt solvability of the fractional integral equation below.

The equation

(9) $u(t) = h(t) - \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{-u(s)}\, ds$
has a unique solution.

Solution
It is straight forward to verify that $-e^{-u}$ is near identity map. Hence by theorem 6, the integral equation (9) has unique solution.

Remark
The advantage of Riemann-Liouville definition of fractional derivative is that it yields a pointwise definition of derivative using fractional integral unlike certain definitions used in formulations of fractional Sobolev spaces defined by interpolations and global approach. The disadvantage is that it is difficult to formulate fractional Sobolev space theory with respect to Riemann-Liouville fractional derivative while the interpolation and global approach seems more suitable to application. The Riemann-Liouville derivative is used by physicists in automation, control theory and signal processing for image enhancement and texture analysis.

It is important to mention that there are many unresolved questions in the theory of fractional calculus top among which are initialization problems and the connection between fractional Sobolev spaces and classical space of functions of bounded variation. Fractional Sobolev spaces are not yet well developed with respect to Riemann-Liouville fractional calculus very much in use in engineering. Some open problems are concerned questions of properties of fractional spaces between L^1 −spaces and the Sobolev space $W^{1,1}$ and the relationship between fractional Sobolev spaces and the classical space of functions of bounded variations. Already, it is well known that the space of absolutely continuous functions $AC[a,b]$ coincide with the Sobolev space $W^{1,1}$. Given that $W^{1,1}(a,b) = \{u \in L^1(a,b): u' \in L^1(a,b)\}$ (Bergounioux *et al*, 2017).

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