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VARIATIONAL STABILITY RESULTS OF DYNAMIC EQUATIONS ON TIME-SCALES USING GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

IGOBI, D. K.[1,*], NDIYO, E.[2] AND INEH, M.P.[3]
[1, 2 & 3] *Department of Mathematics, University of Uyo, P. M. B. 1017, Nigeria.*
inehmike1@gmail.com
Correspondence: *dodiigobi@gmail.com*,

ABSTRACT

Generalized ordinary differential equations can be used to tackle the setback of the everywhere requirement of the existence of the integral of some functions of the dynamic equations on a time-scale. In this study, we investigated the variational stability and variational asymptotic stability of the zero solution of the dynamic equation on time-scale using the established results of the variational stability and variational asymptotic stability of the zero solution of the generalized ordinary differential equations as presented in this work. The regulated and rd-continuous assumptions on the integral function of the Δ-integral of the dynamic equation on the time-scale equation enhanced the compatibility of the results. An example is used to illustrate the suitability of the results

KEYWORDS: dynamic equations on time-scale, generalized ordinary differential equations, variational stability, Kurzweil integral, regulated functions.

1.0 INTRODUCTION

The occurrence of natural events is sometimes entirely continuous and at other times lying dormant until the next surge forward. Hence the domain of occurrence is either continuous or discrete and the model equations to describe such occurrences are differential equations or difference equations respectively. However, introducing of the calculus of dynamic equations on time-scales unified the differential equations and the difference equations calculus (Hilger, 1988). The has been growing research interest in this area (Bohner and Peterson, 2001; Slavik, 2012; Souahi *et al.*, 2016; Abbas, 2018; Igobi and Abasiekwere, 2019; Igobi *et al.*, 2021; Igobi and Ineh, 2024). The setback in the qualitative theory of time-scale calculus is the non-absolutely convergent integral of some functions of the dynamic equations and therefore failed the requirement of everywhere existence of the integral of the functions. This challenge is overcome by treating the dynamic equations on time-scale in the framework of the generalized ordinary differential equations (Slavik, 2012).

In this work, we developed results on variational stability and variational asymptotic stability of the dynamic equations on time-scale within the framework of the generalized ordinary differential equation using established results from previous studies (Schwabik, 1984; Igobi and Abasiekwere, 2019; Igobi *et al.*, 2021, Igobi and Ineh, 2024), by assuming that the function of the Δ-integral of the dynamic equation on time-scale is regulated and rd-continuous. This ensures that the setback of everywhere existing of the integral function is addressed

Let $A:[a,b]\times B_c \to R^n$ for $B_c = \{x \in R^n; \|x\| < c, c > 0\}$ be a regulated function with $\|A\|_\infty = \sup\{\|A(t,x(t))\|_X, t \in [a,b]\}$, which is of bounded variation on $[a,b]$ such that $\operatorname{var}_a^b A(t,x(t)) < \infty$ where $\operatorname{var}_a^b A(t,x(t)) = \sup\left\{\sum_{i=1}^{n(p)} \|A(t_i, x(t_{i-1})) - A(t_{i-1}, x(t_{i-1}))\|\right\}$. Given a unique element $I \in W$ and any value $\varepsilon > 0$, with a gauge (positive function) $\delta:[a,b] \to R_+$ defined on a finite partition $\{a = t_0, \alpha_0, t_1, \alpha_1, t_2, \ldots \alpha_{i-1}, t_i = b\}$ such that $[t_{i-1}, t_i] \subset (\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$ and $\|I - S(DA, P_\delta)\| < \varepsilon$ (1.0)
is satisfied for all $\delta - fine$ partition $P_\delta(\alpha_i, [t_{i-1}, t_i]) \subset [a,b]$,

where $$S(DA, P_\delta) = \sum_{i=1}^{n} \left(A(t_i, x(t_{i-1})) - A(t_{i-1}, x(t_{i-1}))\right) \quad , \tag{1.1}$$

is the integral sum corresponding to the function $A(t, x(t))$ and the $\delta - fine$ partition $P(\alpha_i\, [t_{i-1},\, t_i])$ Then

$$I = \int_a^b DA\,(t,\, x(t)), \tag{1.2}$$

is known as the Kurzweil integral, and the corresponding differential equation of equation (1.2) is known as the generalized ordinary differential equation, which is expressed as

$$\frac{dx}{dt} = DA(t,\ x) \tag{1.3}$$

The function $x:[a,b] \to B_c$ is a solution of the generalized ordinary differential equation (1.3) such that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} DA(t, x(t)), \tag{1.4}$$

for every $t_1, t_2 \in [a,b]$.

Remark 1.0

Consider a regulated function $A : W \to R^n$ for which $W = [a,b] \times B_c$ such that $\int_{t_1}^{t_2} DA(t, x(t))$ exists for $t_1, t_2 \in [a,b]$. Presume there exists a non-decreasing function $h : [a,b] \to R$ such that $\int_{t_1}^{t_2} Dh(t) = h(t_2) - h(t_1)$ and a continuous increasing function $w : [0, \infty) \to R$ with $w(0) = 0$, then the solution function $F : \Omega \to R^n$ for $\Omega \subset W$ satisfies the following assumptions:

a.
$$\|F(t_2, x(t_1)) - F(t_1, x(t_1))\| = \left\|\int_{t_1}^{t_2} DA(t, x(t))\right\|$$
$$= \|A(t_2, x(t_1)) - A(t_1, x(t_1))\| \le h(t_2) - h(t_1) < \infty$$

b.
$$\|F(t_2, x(t_1)) - F(t_1, x(t_1)) - F(t_2, y(t_1)) + F(t_1, y(t_1))\| = \left\|\int_{t_1}^{t_2} D[A(t, x(t)) + A(t, y(t))]\right\|$$
$$= \|A(t_2, x(t_1)) - A(t_1, x(t_1)) + A(t_2, y(t_1)) - A(t_1, y(t_1))\|$$
$$\le w\|x(t_1) - y(t_1)\| \|h(t_2) - h(t_1)$$

2.0 PRELIMINARY RESULTS

Let $f : W \to R^n$ be a regulated function, and $g : [a,b] \to R$ a strictly increasing function on $[a,b]$, then for every $\varepsilon > 0$ there exists a guage function $\delta : [a,b] \to R^+$ such that
$$\|S(f, Dg, P_\delta) - I\| < \varepsilon \tag{2.0}$$
for all $\delta -$ *fine* partitions $P_\delta(\alpha_i, [t_{i-1}, t_i]) \subset [a,b]$ on $[a,b]$, where
$$S(f, Dg, P_\delta) = \sum_{i=1}^{n(p)} f(t_i, x(t_i))[g(\alpha_{i+1}) - g(\alpha_i)] \tag{2.1}$$
and
$$I = \int_a^b f(t, x(t)) Dg(t) \tag{2.2}$$
is known as the Kurzweil-Stieltjes integral.

Let $f : W \to R^n$ be a regulated function and $g : [a,b]_{\mathrm{T}} \to R$ a strictly non-decreasing real-valued function on the dense set of the time-scale $[a,b]_{\mathrm{T}}$, then for every $\varepsilon > 0$ there exists a guage function $\delta : [a,b]_T \to R^+$ such that
$$\|S(f, \Delta g, P_\delta) - I_{\mathrm{T}}\| < \varepsilon \tag{2.3}$$
for all $\delta -$ *fine* partitions $P_\delta(\alpha_i, [t_{i-1}, t_i]) \subset [a,b]$ on $[a,b]_{\mathrm{T}}$, where
$$S(f, \Delta g, P) = \sum_{i=1}^{n(p)} f(t_i, x(t_i))[g(\alpha_{i+1}) - g(\alpha_i)] \text{ and } I_{\mathrm{T}} = \int_a^b f(t, x(t)) \Delta g(t), \tag{2.4}$$
is known as the Riemann-Stieltjes integral.

Specifically, considering that $t \in T$ is a dense point such that $t \le \sup \mathrm{T}$ and there exists $t^* = \inf\{s \in \mathrm{T}, s < t\}$, then $g(t) = t^* = t$, and equation (2.4) approximate the $\Delta -$ integral (Delta integral) $\int_a^b f(t, x(t)) \Delta t$. That is
$$\int_a^b f(t, x(t)) \Delta g(t) = \int_a^b f(t, x(t)) \Delta t. \tag{2.5}$$

Theorem 2.0 (Corresponding theorem I)

Let $t \in [a,b]_T$ be a dense point and $g : [a,b]_T \to R$ be a real-valued function that is non-decreasing on $[a,b]_T$. Let $f : W \to R^n$ a regulated function such that f_{T} and g_{T} are restriction of f and g to the time-scale T. Assume that $t \in T$ is right dense and $t \le \sup \mathrm{T}$ such that there exists $t^* = \sup\{s \in \mathrm{T}, s < t\}$. Then $g(t) = t^* = t$ and
$$\int_a^b f(t, x(t)) Dg(t) = \int_a^b f_{\mathrm{T}}(t, x(t)) \Delta(t) \tag{2.6}$$

Proof

Given that $f : W \to R^n$ is a regulated function and $g : [a,b] \to R$ be a non-decreasing real value function on $[a,b]$, then the Kurzweil-Stieltjes integral of f with respect to the function g on $[a,b]$ is well defined as in equation (2.4). Let partition $\mathrm{P}([a,b]_{\mathrm{T}})$ be defined such that $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. Then by equation (2.1)
$$\int_a^b f(t, x(t)) Dg(t) \le \sum_{i=0}^{n} M_{\Delta i} \Delta g_i(t) = U_\Delta(f_i, \Delta g_i, P), \tag{2.7}$$

where $M_{\Delta i} = \sup_{t\in[t_{i-1},t_i]} f(t,x(t))$. If we take the infimum of the right-hand side of equation (2.7) across all partition of $P([a,b]_T)$, we have $\inf_{t\in[a,b]_T} U_\Delta\left(f_i(t,x(t)),\Delta g_i(t),P\right) = \int_a^{\bar{b}} f_T(t,x(t))\Delta g_T(t)$.

Hence, $$\int_a^b f(t,x(t))Dg(t) \le \int_a^{\bar{b}} f_T(t,x(t))\Delta g_T(t). \quad (2.8)$$

Also

$$\int_a^b f(t,x(t))Dg(t) \ge \sum_{i=0}^n m_{\Delta i}\Delta g_i(t) = L_\Delta(f_i,\Delta g_i,P), \quad (2.9)$$

where $m_{\Delta i} = \inf_{t\in[t_{i-1},t_i]} f(t,x(t))$. If we take the supremum of the right hand side of equation (2.9) across all partition of $P([a,b]_T)$, we have $$\sup_{t\in[a,b]_T} L_\Delta\left(f_i(t,x(t)),\Delta g_i(t),P\right) = \int_{\underline{a}}^b f_T(t,x(t))\Delta g_T(t).$$

Hence, $$\int_a^b f(t,x(t))Dg(t) \ge \int_{\underline{a}}^b f_T(t,x(t))\Delta g_T(t). \quad (2.10)$$

Combining equations (2.8) and (2.10), we have $\int_{\underline{a}}^b f_T(t,x(t))\Delta g_T(t) \le \int_a^b f(t,x(t))Dg(t) \le \int_a^{\bar{b}} f_T(t,x(t))\Delta g_T(t)$.

Hence

$$\int_a^b f(t,x(t))Dg(t) = \int_a^b f_T(t,x(t))\Delta g_T(t),$$

for $\int_a^{\bar{b}} f(t,x(t))\Delta g = \int_{\underline{a}}^b f(t,x(t))\Delta g$ satisfied.

In particular, given that $t\in T$ is a dense point such that $t \le \sup T$ and there exists $t^* = \inf\{s\in T, s<t\}$, so that $g(t) = t^* = t$ and $$\int_a^b f(t,x(t))Dg(t) = \int_a^b f_T(t,x(t))\Delta(t). \quad (2.11$$

Hence the theorem is proved.

Definition 2.0

Let $f:[a,b]_T \times B_c \to R^n$ be an rd-continuous function which is Lebesgue integrable on $[a,b]_T$, for T being a time-scale domain and $P = \{t_0,t_1,...,t_n\} \subset [a,b]_T$ a partition, then the initial-value problem of the dynamic equation on $[a,b]_T$ is

$$\begin{aligned} x^\Delta(t) &= f(x,t) \\ x(t) &= x(t_0), \quad t_0,t\in[a,b]_T, \end{aligned} \quad (2.12)$$

which has the solution form

$$x(t) = x(t_0) + \int_{t_0}^t f(s,x)\Delta s, \quad t_0,t\in[a,b]_T \quad , \quad (2.13)$$

for $x:[a,b]_T \to B_c$.

Remark 2.0

We make the following Caratheodory assumptions on the regulated function $f:[a,b]_T \times B_c \to R^n$:

A1. We assume that $f(t,x)$ is rd-continuous and the integral function $t \to \int_a^t f(s,x)\Delta s$ is rd-continuous for all $x:[a,b]_T \to B_c$ being a continuous function.

A3. There exists a Lebesgue integrable function $m_0 : T \to R,$ such that

$$\left\|\int_{t_0}^{t_1} f(s,x)\Delta s\right\| \le \int_{t_0}^{t_1} m_0(s)\Delta s, \quad for \quad t_1,t_2\in T, \quad x\in G(T,B_c),$$

A3. there exists a Lebesgue integrable function $m_1 : T \to R,$ such that $\left\|\int_{t_0}^{t_1} f(s,x-y)\Delta s\right\| \le \int_{t_0}^{t_1} m_1(s)\|x-y\|\Delta s, \quad for \quad t_1,t_2\in T, \quad x,y\in G(T,B_c)$

Proposition 2.0

Let $f\in BV(W,R^n)$ be a Caratheodory function and $g:[a,b]\to R$ a non-decreasing function. Then, the Kurzweil integrated function $A:W\to R^n$ defined as

$$A(t,x)=\int_{t_0}^{t} f(s,x)Dg(s) \tag{2.14}$$

satisfies remark (1.0) if there exists a non-decreasing function $h(t):[a,b]\to R$ that satisfied

$$h(t)=\int_{t_0}^{t_1}(m_0(s)+m_1(s))Dg(s)$$

(2.15)

Proof

Let the non-decreasing function $g:[a,b]\to R$ be given such that $g=g_+-g_-$, and $\int_R fDg=\int_R fDg_+-\int_R fDg_-$, so that by equation (2.14) we can have that $A^+(t,x)=\int_{t_0}^{t} f(s,x)Dg_+(s)$ and $A^-(t,x)=\int_{t_0}^{t} f(s,x)Dg_-(s)$ respectively. Also, by remark (2.0), we have

$$\left\|A^-(t_2,x)-A^-(t_1,x)\right\|=\left\|\int_{t_1}^{t_2} f(s,x)Dg_-(s)\right\|=\left\|\int_{t_1}^{t_2} f(s,x)\Delta^-(s)\right\|\le\left\|\int_{t_1}^{t_2} m_0(s)ds^-\right\|,$$

and

$$\left\|A^+(t_2,x)-A^+(t_1,x)\right\|=\left\|\int_{t_1}^{t_2} f(s,x)Dg_+(s)\right\|=\left\|\int_{t_1}^{t_2} f(s,x)\Delta^+(s)\right\|\le\left\|\int_{t_1}^{t_2} m_0(s)ds^+\right\|,$$

so that

$$\begin{aligned}\left\|A(t_2,x)-A(t_1,x)\right\|&=\left\|A^+(t_2,x)-A^+(t_1,x)+A^-(t_1,x)-A^-(t_2,x)\right\|\\&\le\int_{t_1}^{t_2} m_0(s)ds^+ +\int_{t_1}^{t_2} m_0(s)ds^-\\&=\int_{t_1}^{t_2} m_0(s)\Delta s.\end{aligned}$$

We defined

$$h_1(t)=\int_{t_0}^{t}(m_0(s))\Delta s,\ \ t\in[a,b], \tag{2.16}$$

where $h_1:[a,b]\to R$ is a non-decreasing function, and m_0 a nonnegative function on $[a,b]$, then

$$\left\|A(t_2,x)-A(t_1,x)\right\|\le\int_{t_1}^{t_2} m_0(s)\Delta s\le\left|h_1(t_2)-h_1(t_1)\right|.$$

Also

$$\begin{aligned}\left\|A^+(t_2,x)-A^+(t_1,x)+A^+(t_1,y)-A^+(t_2,y)\right\|&=\left\|\int_{t_1}^{t_2} f(s,x)dg_+(s)-\int_{t_1}^{t_2} f(s,y)dg_+(s)\right\|\\&=\left\|\int_{t_1}^{t_2} f(s,x-y)\Delta^+ s\right\|\le w\left\|x-y\right\|\int_{t_1}^{t_2} m_1(s)\Delta s^+,\end{aligned}$$

and

$$\begin{aligned}\left\|A^-(t_2,x)-A^-(t_1,x)+A^-(t_1,y)-A^-(t_2,y)\right\|&=\left\|\int_{t_1}^{t_2} f(s,x)dg_-(s)-\int_{t_1}^{t_2} f(s,y)dg_-(s)\right\|\\&=\left\|\int_{t_1}^{t_2} f(s,x-y)\Delta^- s\right\|\le w\left\|x-y\right\|\int_{t_1}^{t_2} m_1(s)\Delta s^-,\end{aligned}$$

so that

$$\begin{aligned}\left\|A(t_2,x)-A(t_1,x)+A(t_1,y)-A(t_2,y)\right\|&\le w\left\|x-y\right\|\int_{t_1}^{t_2} m_1(s)\Delta^+ s+w\left\|x+y\right\|\int_{t_1}^{t_2} m_1(s)\Delta^- s\\&=w\left\|x-y\right\|\int_{t_1}^{t_2} m_1(s)\Delta s.\end{aligned}$$

We defined

$$h_2(t)=\int_{t_0}^{t} m_1(s)\Delta s,\ \ t,t_0\in[a,b], \tag{2.17}$$

where $h_2:[a,b]\to R$ is a non-decreasing function, and m_1 a nonnegative function on $[a,b]$, so that

$$\left\|A(t_2,x)-A(t_1,x)+A(t_1,y)-A(t_2,y)\right\|=w\left\|x-y\right\|\left|h_2(t_2)-h_2(t_1)\right|.$$

Then for $h(t)=h_1(t)+h_2(t)$, the theory is proved.

Definition 2.1 Let $A:W\to R^n$ be a regulated function on $[a,b]$, if the solution function $x(t):[a,b]\to B_c$ is a step function on the partition $P([a,b])$ such that $x(t)=c_i$, for $t\in(t_{i-1},t_i)$, then

$$\int_a^b DA(t,x(t))$$
$$=\sum_{i=0}^{n}\Big((A(t_i^-,c_i)-A(t_{i-1}^+,c_i))+(A(t_{i-1}^+,x(t_{i-1}))-A(t_{i-1},x(t_{i-1})))+(A(t_i\ ,x(t_i))-A(t_i^-,x(t_i)))\Big)$$

Lemma 2.0

Let $f:[a,b]_T\times B_c\to R^n$ be Kurzweil-Stieltjes integrable with a non-decreasing function $g:[a,b]\to R$, which is of bounded variation on $[a,b]$. Given any step function $x:[a,b]\to B_c$ and a regulated function $A(t,x)$ such that

$$A(t,x)=\int_{t_0}^{t} f(s,x)Dg(s),\ \text{then}\quad \int_a^b DA(t,x)=\int_a^b f(x,t)Dg(t) \tag{2.18}$$

Proof

By definition (2.1) for $A(t,x)$ being a regulated function on $[a,b]$ and $x(t):[a,b]\to B_c$ step function on the partition $P([a,b])$, then for any $\varepsilon>0$ and $x(t)=c_i$, for $t\in(t_{i-1},t_i)$, we have

$$\int_a^b DA(t,\ x(t))=\lim_{\varepsilon\to0^+}\sum_{i=0}^{n}\big((A(t_{i-1}+\varepsilon,x(t_{i-1}))-A(t_{i-1},x(t_{i-1})))\big)$$
$$+\lim_{\varepsilon\to0^+}\sum_{i=0}^{n}(A(t_i-\varepsilon,c_i)-A(t_{i-1}+\varepsilon,c_i))+\lim_{\varepsilon\to0^+}\sum_{i=0}^{n}(A(t_i,x(t_i))-A(t_i-\varepsilon,x(t_i))) \tag{2.19}$$

Also, for $f(x,t)$ being a regulated function, we have

$$\int_a^b f(t,x)Dg(t)=\sum_{i=0}^{n}\int_{t_{i-1}}^{t_i} f(s,x(s))Dg(s)$$
$$=\lim_{\varepsilon\to0^=}\sum_{i=0}^{n}\int_{t_{i-1}}^{t_{i-1}+\varepsilon} f(s,x(s))Dg(s)+\lim_{\varepsilon\to0^=}\sum_{i=0}^{n}\int_{t_{i-1}+\varepsilon}^{t_i-\varepsilon} f(s,x(s))Dg(s) \tag{2.20}$$
$$+\lim_{\varepsilon\to0^=}\sum_{i=0}^{n}\int_{t_i-\varepsilon}^{t_i} f(s,x(s))Dg(s)$$

Comparing equation (2.19) and (2.20) term by term, its observed that

$$\lim_{\varepsilon\to0^+}\sum_{i=0}^{n}\big((A(t_{i-1}+\varepsilon,x(t_{i-1}))-A(t_{i-1},x(t_{i-1})))\big)=\lim_{\varepsilon\to0^+}\int_{t_{i-1}}^{t_{i-1}+\varepsilon} f(s,x(s))Dg(s)$$
$$=f(t_{i-1},x(t_{i-1}))\Delta^+g(t_{i-1})$$

$$\lim_{\varepsilon\to0^+}\sum_{i=0}^{n}\big((A(t_i-\varepsilon,c_i)-A(t_{i-1}+\varepsilon,c_i))\big)=\lim_{\varepsilon\to0^+}\int_{t_{i-1}+\varepsilon}^{t_i-\varepsilon} f(s,x(s))Dg(s)$$
$$=\lim_{\varepsilon\to0^+}\int_{t_{i-1}+\varepsilon}^{t_i-\varepsilon} f(x(t_{i-1},t_{i-1}))\Delta^+g(t_{i-1})$$

$$\lim_{\varepsilon\to0^+}\sum_{i=0}^{n}\big((A(t,x(t_i))-A(t_i-\varepsilon,x(t_i)))\big)=\lim_{\varepsilon\to0^+}\int_{t_i-\varepsilon}^{t_i} f(s,x(s))Dg(s)$$
$$=\lim_{\varepsilon\to0^+}\int_{t_i-\varepsilon}^{t_i} f(t_i,x(t_i))\Delta^-g(t_i)$$

Hence, equation (2.18) holds and the theorem is proved.

Theorem 2.1 (Corresponding theorem II)

If $f:[a,b]_T\times B_c\to R^n$ is defined and satisfies remark (2.0) on $[a,b]_T$, and $x:[a,b]_T\to B_c$ is a solution of

$$x^\Delta(t)=f(t,x),\qquad t\in[a,b]_T \tag{2.21}$$

then $x:[a,b]_T\to B_c$ is also a solution of generalized differential equation $\dfrac{dx}{dt}=DA(t,x)$, $t\in[a,b]$, (2.22)

for t a right dense point, such that $t^*=\sup\{s\in \mathrm{T}, s<t\}$, and $g(t)=t^*=t$. Also, every solution $y:[a,b]_T\to B_c$ of (2.22) can be expressed as $y=x$.

Proof

Let $v\in[a,b]_T$. If $x:[a,b]_T\to B_c$ is a solution of equation (2.21), then $x(s)=x(v)+\int_v^s f(t,x)\Delta t,\qquad s\in[a,b]_T$

Thus, by theorem hypothesis $g(t)=t^*=t$ and by equation (2.6) we have $x(s)=x(v)+\int_v^s f(x,t)Dg(t).$

Let $s,v\in[a,b]_T$ be a compact interval and $A(t,x)$ a regulated function. Since f satisfies remark (2.0), then by lemma (2.0), we have that $x(s)=x(v)+\int_v^s DA(t,x),$

which satisfies the generalized ordinary differential equation (2.22).

To show that $y=x$, we let $y:[a,b]\to B_c$ satisfy (2.22), so that $y(s)=y(a)+\int_a^s DA(t,y(\tau)),\quad s\in[a,b]$.

If $[\alpha,\beta]_T\subset[a,b]_T$ is a time-scale interval and $\tau\in[\alpha,\beta]_T$, then

$$y(\tau)=\lim_{u\to\tau^+}\left(y(u)-A(u,y(\tau))+A(\tau,y(\tau))\right)=\lim_{u\to\tau^+}\left(y(u)-\int_\tau^u DA(s,y(\tau))\right)$$
$$=\lim_{u\to\tau^+}\left(y(u)-\int_\tau^u f(s,y(\tau))Dg(s)\right)$$
$$=\lim_{u\to\tau^+}\left(y(u)-\int_\tau^u f(\tau,y(\tau))\Delta g(\tau)\right)=\lim_{u\to\tau^+}y(u)\quad for\ \ all\ \ \tau\in[\alpha,\beta),$$

and therefore $\lim_{u\to\tau}y(u)$ exists. Likewise, for every $\tau\in(\alpha,\beta],$ we have

$$y(\tau)=\lim_{u\to\tau^-}\left(y(u)-A(u,y(\tau))+A(\tau,y(\tau))\right)=\lim_{u\to\tau^-}\left(y(u)-\int_\tau^u DA(s,y(\tau))\right)$$
$$=\lim_{u\to\tau^-}\left(y(u)-\int_\tau^u f(s,y(\tau))Dg(s)\right)$$
$$=\lim_{u\to\tau^-}\left(y(u)-\int_\tau^u f(\tau,y(\tau))\Delta^- g(\tau)\right)=\lim_{u\to\tau^-}y(u),$$

Since y is regulated and bounded on $[\alpha,\beta]$, there exists a bounded set $V\subset B_c$ such that $y(t)\in V$ for $t\in[\alpha,\beta]$. Also, the function f satisfies remark (2.0) on $[\alpha,\beta]_T\times V$ and by preposition (2.0), the function A satisfies remark (1.0) on $[\alpha,\beta]$. Then by lemma (2.0), we obtain that $y(s)=y(a)+\int_a^s f(t,y(t))Dg(t),\qquad s\in \mathrm{T}$

Thus $y=x,$ where $x:[a,b]_T\to B_c$ is the restriction of y to $[a,b]_T$.

Definition 2.2 Let $f:[t_1,t_2]_T\times B_c\to R^n$ be rd-continuous and $x:[t_1,t_2]_T\to B_c$ be of bounded variation on $[t_1,t_2]_T\subset[a,b]_T$, then

$$x(t)\equiv\int_{t_1}^{t_2}f(t,0)\Delta t=f(t_2,0)-f(t_1,0)=0 \tag{2.23}$$

defined the trivial solution of the dynamic equation (2.12).

Definition 2.3 The trivial solution ($x\equiv 0$) of equation (1.3) is variationally stable if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ and if $y:[a,b]\to B_c$ is a function of bounded variation on $[a,b]$, and for all $t\in[a,b]$, $\|y_0\|<\delta$ and

$$\mathrm{var}_a^b\left(y(t)-\int_a^t DA(s,y(s))\right)<\delta(\varepsilon), \tag{2.24}$$

then

$$\|y(t,y_0)\|<\varepsilon,\ \ for\ t\in[a,b]. \tag{2.25}$$

Definition 2.3 The trivial solution ($x\equiv 0$) of equation (1.3) is variationally attracting if there is a $\delta_0>0$, and for every $\varepsilon>0$ there exists a $\tau=\tau(\varepsilon)\geq 0$ and $\delta=\delta(\varepsilon)>0$ such that if $y:[a,b]\to B_c$ is a function of bounded variation on $[a,b]$, and for all $t\in[a,b]$ such that $\|y_0\|<\delta_0$ and $\mathrm{var}_a^b\left(y(t)-\int_a^t DA(s,y(s))\right)<\delta$, (2.26)

then

$$\|y(t,y_0)\|<\varepsilon,\quad for\ \ t\in[a,b]\cap a+\tau(\varepsilon). \tag{2.27}$$

Definition 2.4 The trivial solution ($x\equiv 0$) of equation (1.3) is variationally asymptotically stable if it is variationally stable and variationally attracting

3.0 MAIN RESULT AND DISCUSSION

Theorem 3.0 (Scwabik, 1984; Igobi *et al.*, 2021)

Let $V:[0,\infty)\times B_c\to R_+$ be such that $V(t,x)\in BV([0,\infty))$ is continuous from the left. Assume we have a $b(r)\in C([0,\infty),R)$ which is a continuous increasing function such that $b(0)=0$ and $b(r)>0$ for $r\neq 0$ satisfying

$$V(t,x) \geq b(|x|), \tag{3.0}$$
$$V(t,0) = 0. \tag{3.1}$$
$$\|V(t,x) - V(t,y)\| \leq K\|x-y\|, \quad x,y \in B_c, \quad t \in [0,\infty), \tag{3.2}$$

where $K > 0$. Also assume that

$$\limsup_{\eta \to 0+} \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \leq 0, \tag{3.3}$$

along every solution path of equation (1.3), so that for any left continuous function $y:[a,b] \to B_c$

$$V(t_1, y(t_1)) \leq V(t_0, y(t_0)) + K\,\mathrm{var}_{t_0}^{t_1}\left(y(t) - \int_{t_0}^{t} D[A(t,y(s))] \right) \tag{3.4}$$

holds. Then the trivial solution $x \equiv 0$ of equation (1.3) is variationally stable.
We formulate the theorems on the variational stability and variational asymptotic stability of the trivial solution of the dynamic equation (2.12) as a consequence of the theorem (3.0).

Theorem 3.1
Let $f:[a,b]_T \times B_c \to R^n$ satisfies remark (2.0). Assume there exist an open ball B_ρ, and function $V \in C[[0,\infty)_T \times \overline{B}_\rho, R_+]$, which is locally Lipschitzian in x, for $\overline{B}_\rho = \{x \in R^n : \|x\| \leq \rho,\ 0 < \rho < c\}$, and a Hahn class function $b(r) \in C([0,\infty]_T, R_+)$ such that $b(0) = 0$ and $b(r) > 0$ for $r \neq 0$ satisfies

$$V(t,x) \geq b(\|x\|),\ (t,x) \in [0,\infty) \times \overline{B}_\rho \tag{3.5}$$
$$V(t,0) = 0.\ t \in [0,\infty) \tag{3.6}$$
$$D^+V(t,x) \leq 0 \quad . \tag{3.7}$$

If there exists any $y:[a,b] \to \overline{B}_\rho$ such that $\|y_0\| < \delta$ which satisfies equations (2.24) and (2.25), then the trivial solution $(x \equiv 0)$ of equation (2.12) is variationally stable

Proof
We make the following assumptions:
i. $f:[a,b]_T \times B_c \to R^n$ satisfies the local existence theory,
ii. $A: W \to R^n$ satisfies remark (1.0),
iii. lemma (2.0), theorems (2.0 & 2.1), and equation (2.5) all hold.
Then we present the prove of theorem (3.1) as follow:
Let there exists $y:[t_0,t_1] \to \overline{B}_\rho$ which satisfies equation (2.12) such that $y(\sigma) = x(\sigma)$ for $a < t_0 \leq \sigma \leq t_1 < b$. If there is $\eta_1(\sigma) > 0$ such that we have $\eta \in [0, \eta_1(\sigma)]_T$, and for $x(t):[t_0,t_1]_T \to \overline{B}_\rho$ we have

$$\begin{aligned} V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma, y(\sigma)) &= V(\sigma+\eta, y(\sigma+\eta)) - V(\sigma+\eta, x(\sigma+\eta)) \\ &\quad + V(\sigma+\eta, x(\sigma+\eta)) - V(\sigma, x(\sigma)) \\ &\leq K\left\| y(\sigma+\eta) - y(\sigma) - \int_\sigma^{\sigma+\eta} f(t,y(t))\Delta t \right\| \\ &\quad + K\left\| \int_\sigma^{\sigma+\eta} [f(t, y(t) - x(t))]\Delta t \right\| , \end{aligned} \tag{3.9}$$

for $V(\sigma+\eta, x(\sigma+\eta)) - V(\sigma, x(\sigma)) = D^+V(\delta+\eta,\ x(\delta+\eta)) \leq 0$.
We expressed the first term on the right of equation (3.9) as

$$\begin{aligned} &\left\| y(\sigma+\eta) - y(\sigma) - \int_\sigma^{\sigma+\eta} f(t,y(t))\Delta t \right\| = \left\| y(\sigma+\eta) - y(\sigma) - \int_\sigma^{\sigma+\eta} f(t,y(t))Dg(t) \right\| \\ &\leq y(\sigma+\eta) - \int_{\sigma+\eta}^{t} DA(t,y(s)) - y(\sigma) - \int_\sigma^t DA(s,y(s)) \\ &= \mathrm{var}_\sigma^{\sigma+\eta}\left(y(t) - \int_\sigma^t DA(s,y(s)) \right) \end{aligned} \tag{3.10}$$

Using proposition (2.0), equation (2.11) and remark (2.0), We estimate the second term on the right of equation (3.9) as

$$\left\|\int_{\sigma}^{\sigma+\eta}[f(t,y(s)-x(s)]\Delta t\right\| \leq \int_{\sigma}^{\sigma+\eta} m_1\|y(t))-x(t)\|(t)\Delta(t)$$
$$= \lim_{\tau\to 0^+}\int_{\sigma}^{\sigma+\tau} m_1(t)\|y(t))-x(t)\|\Delta(t) + \lim_{\tau\to 0^+}\int_{\sigma+\tau}^{\sigma+\eta} m_1(t)\|y(t))-x(t)\|\Delta(t)$$
$$\leq w(\|y(\sigma))-x(\sigma)\|)\lim_{\tau\to 0^+}\int_{\sigma}^{\sigma+\tau} m_1(t)\Delta(t) + \sup_{\rho\in[\sigma,\sigma+\eta]} w(\|y(\rho))-x(\rho)\|)\lim_{\tau\to 0^+}\int_{\sigma+\tau}^{\sigma+\eta} m_1(t)\Delta(t)$$

By the assumption of $y:[t_0,t_1]\to \overline{B}_\rho$ satisfying equation (2.12) such that $y(\sigma)=x(\sigma)$

$$\left\|\int_{\sigma}^{\sigma+\eta}[f(t,y(s)-x(s)]\Delta t\right\| \leq \sup_{\rho\in[\sigma,\sigma+\eta]} w(\|y(\rho))-x(\rho)\|)(h(\sigma+\eta)-h(\sigma+)). \tag{3.11}$$

Also for $\rho\in[\sigma,\sigma+\eta]$, we have

$$\lim_{\rho\to\sigma+}(y(\rho)-x(\rho)) = \lim_{\rho\to\sigma+}\left(y(\rho)-y(\sigma)-\int_{\sigma}^{\rho} f(t,x(t))\Delta t\right)$$
$$= y(\sigma+)-y(\sigma)-\lim_{\rho\to\sigma+}\left(\int_{\sigma}^{\rho} f(t,x(t))Dg(t)\right)$$
$$= y(\sigma+)-y(\sigma)-f(\sigma,x(\sigma))(g(\sigma+)+g(\sigma))$$
$$= y(\sigma+)-y(\sigma)-\lim_{\rho\to\sigma+}\left(\int_{\sigma}^{\rho} f(t,y(t))Dg(t)\right)$$

But

$$\lim_{\rho\to\sigma+}\left(\int_{\sigma}^{\rho} f(t,y(t))dg(t)\right) = \lim_{\rho\to\sigma+}\left(\int_{\rho}^{t} f(s,y(s))Dg(s)-\int_{\sigma}^{t} f(s,y(s))Dg(s)\right)$$
$$= \lim_{\rho\to\sigma+}\left(\int_{\rho}^{t} DA(s,y(s))-\int_{\sigma}^{t} DA(s,y(s))\right),$$

so that

$$\lim_{\rho\to\sigma+}(y(\rho)-x(\rho)) = y(\sigma+)-y(\sigma)+\lim_{\rho\to\sigma+}\left(\int_{\rho}^{t} DA(s,y(s))-\int_{\sigma}^{t} DA(s,y(s))\right)$$
$$= y(\sigma+)-\lim_{\rho\to\sigma+}\left(\int_{\rho}^{t} DA(s,y(s))\right)-y(\sigma)-\int_{\sigma}^{t} DA(s,y(s))$$
$$= \mathrm{var}_{\sigma}^{\sigma+}\left(y(\rho)-\int_{\sigma}^{t} DA(s,y(s))\right) = \gamma.$$

Hence equation (3.11) is rewritten $\left\|\int_{\sigma}^{\sigma+\eta}[f(t,y(t)-x(t))]\Delta t\right\| \leq w(\gamma)(h(\sigma+\eta-h(\sigma+)))$. (3.12)

Assume $\lim_{\eta\to 0^+} h(\sigma+\eta)=h(\sigma+)$, then there exists an $\alpha>0$ such that $\eta_3(\sigma)\leq\eta_2(\sigma)$, and for $\eta\in[0,\ \eta_3(\sigma)]_T$ we have

$$h(\sigma+\eta)-h(\sigma+) < \frac{\alpha[(\delta+\eta)-\delta^+]}{\omega(\gamma)},$$

and setting

$$\alpha = \begin{cases} \dfrac{\varepsilon}{K(\delta+\eta-\delta^+)} > 0, & for\ t>\delta \\ 0 \quad , & for\ t\leq\delta \end{cases}.$$

Then equation (3.12) implies

$$\left\|\int_{\sigma}^{\sigma+\eta}[f(t,y(t)-x(t))]\Delta t\right\| \leq \frac{\varepsilon}{K}, \tag{3.13}$$

By Lemma (2.1) and for $\sigma\in[t_0,t_1]_T$, $\eta\in[0,\eta(\sigma)]_T$ and $g(t)\in BV[t_0,t_1]$, we have

$$V(t_1,y(t_1))-V(t_0,y(t_0)) \leq g(t_1)-g(t_0) = K\ \mathrm{var}_{t_0}^{t_1}\left(y(t_0)-\int_{t_0}^{t} DA(s,y(s))\right)+\varepsilon\ .$$

Choosing ε arbitrary, we have $V(t_1,y(t_1)) \leq V(t_0,y(t_0))+K\ \mathrm{var}_{t_0}^{t}\left(y(t_0)-\int_{t_0}^{t} DA(s,y(s))\right)$. (3.16)

By definition (2.2), we set $2K\delta(\varepsilon)<\gamma(\varepsilon)$ for $\gamma(\varepsilon)=\inf b(t)$ and $\lim_{\varepsilon\to 0+}\gamma(\varepsilon)=0$ for $\varepsilon>0$ we have

$$V(t_1,y(t_1)) \leq V(t_0,y(t_0))+K\ \mathrm{var}_{t_0}^{t}\left(y(t_0)-\int_{t_0}^{t} DA(s,y(s))\right)$$
$$\leq K\|y(t_0)\|+K\ \mathrm{var}_{t_0}^{t}\left(y(t_0)-\int_{t_0}^{t} DA(s,y(s))\right)$$
$$\leq 2K\delta(\varepsilon)<\gamma(\varepsilon) \tag{3.17}$$

Assume there exists $t* \in [t_0, t_1]$ such that $\|y(t*)\| \geq \varepsilon$, then we have $V(t*, y(t*)) \geq b(\|y(t*)\|) \geq \inf b(t) = \gamma(\varepsilon)$,(3.18) this contradicts equation (3.17). Hence $\|y(t)\| \leq \varepsilon$ for all $t \in [t_0, t_1]$, and the trivial solution ($x \equiv 0$) of equation (2.12) is variationally stable.

Theorem 3.2
Let $f:[a,b]_T \times B_c \to R^n$ satisfies remark (2.0). Assume there exists a function $V \in C[[0,\infty)_T \times B_c, R_+]$ which is locally Lipschitzian in x and a Hahn class function $b(r) \in C([0,\infty]_T, R_+)$ such that $b(0) = 0$, $b(r) > 0$ for $r \neq 0$ satisfying

$$V(t,x) \geq b(|x|), \tag{3.19}$$

$$V(t,0) = 0. \tag{3.20}$$

Assume for any solution $y:[t_0,t_1]_T \to \overline{B}_\rho$ there exists a strictly monotone increasing function $\phi(x(t)): W \to R_+$ for $\phi(0) = 0$ and $\phi(x) > 0$ $(x \neq 0)$ such that

$$D^+V(t,x) \leq -\phi(x(t)), \tag{3.21}$$

holds for $M = \sup_{\delta(\varepsilon)\leq\|x\|<\varepsilon}(-\phi(x(t))$. Then the trivial solution $(x \equiv 0)$ of equation (2.12) is variationally asymptotically stable for all $t > t_0 + T(\varepsilon)$.

Proof
We have shown that the trivial solution $x \equiv 0$ of equation (2.12) is variationally stable in theorem (3.1). Hence we have a $y:[t_0,t_1]_T \to \overline{B}$ which is of a bounded variation, continuous from the left on $[t_0,t_1]_T$ and for every $\varepsilon > 0$ there exists $\delta^*(\varepsilon) > 0$ such that $\|y(t_1)\| < \delta^*(\varepsilon)$ and $\operatorname{var}_{t_0}^{t_1}\left(y(t) - \int_{t_0}^{t} DA(s, y(s))\right) < \delta^*(\varepsilon)$,
then

$$\|y(t, y_0)\| < \varepsilon, \quad t \in [t_0, t_1]_T.$$

We assume the converse that $y(t) \geq \varepsilon,\ t < t_0 + T(\varepsilon)$ and defined

$$T(\varepsilon) = \begin{cases} \dfrac{K(\delta_0 + 2\delta)}{M}, & t > t_0 \\ 0, & t < t_0 \end{cases} \tag{3.22}$$

so that

$$\operatorname{var}_{t_0}^{t_1}\left(y(t) - \int_{t_0}^{t} DA(s, y(s))\right) < \delta(\varepsilon).$$

Using the hypothesis of equation (3.21), we have

$$\begin{aligned} V(t,y(t)) - V(t_0,y(t_0)) &\leq V(t,y(t)) - V(t_0+T(\varepsilon), y(t_0+T(\varepsilon))) \\ &\quad + V(t_0+T(\varepsilon), y(t_0+T(\varepsilon))) - V(t_0,y(t_0)) - MT(\varepsilon) \\ &\leq K(y(t) - y(t_0+T(\varepsilon))) + K(y(t_0+T(\varepsilon)) - y(t_0)) - MT(\varepsilon) \end{aligned} \tag{3.23}$$

But

$$\begin{aligned} y(t) - y(t_0+T(\varepsilon)) &\leq y(t) - y(t_0+T(\varepsilon)) - \int_{t_0+T(\varepsilon)}^{t} f(s.y(s))\Delta s \\ &\leq y(t) - \int_{t}^{\tau} f(s.y(s))\Delta s - y(t_0+T(\varepsilon)) - \int_{t_0+T(\varepsilon)}^{\tau} f(s.y(s))\Delta s \\ &\leq y(t) - \int_{t}^{\tau} DA(s.y(s)) - y(t_0+T(\varepsilon)) - \int_{t_0+T(\varepsilon)}^{\tau} DA(s.y(s)) \\ &\leq \operatorname{var}_{t_0+T(\varepsilon)}^{t}\left(y(t) - \int_{t_0+T(\varepsilon)}^{\tau} DA(s.y(s))\right) \end{aligned} \tag{3.24}$$

And

$$\begin{aligned} y(t_0+T(\varepsilon)) - y(t_0) &\leq y(t_0+T(\varepsilon)) - y(t_0) - \int_{t_0}^{\tau} f(s.y(s))\Delta s \\ &\leq y(t_0+T(\varepsilon)) - \int_{t_0+T(\varepsilon)}^{\tau} f(s.y(s))\Delta s - y(t_0) - \int_{t_0}^{\tau} f(s.y(s))\Delta s \\ &\leq y(t_0+T(\varepsilon)) - \int_{t_0+T(\varepsilon)}^{\tau} DA(s.y(s)) - y(t_0) - \int_{t_0}^{\tau} DA(s.y(s)) \\ &\leq \operatorname{var}_{t_0}^{t_0+T(\varepsilon)}\left(y(t_0) - \int_{t_0}^{\tau} DA(s.y(s))\right) \end{aligned} \tag{3.25}$$

Putting equation (3.24) and (3.25) into equation (3.23) we obtain

$$\begin{aligned} V(t,y(t)) - V(t_0,y(t_0)) &\leq K \operatorname{var}_{t_0+T(\varepsilon)}^{t}\left(y(t) - \int_{t_0+T(\varepsilon)}^{\tau} DA(s.y(s))\right) \\ &\quad + K \operatorname{var}_{t_0}^{t_0+T(\varepsilon)}\left(y(t_0) - \int_{t_0}^{\tau} DA(s.y(s))\right) - MT(\varepsilon) \\ &\leq -K\delta_0 \end{aligned}$$

So that

$$V(t,y(t)) \le V(t_0, y(t_0)) - K\delta_0$$
$$\le K\|y(t_0)\| - K\delta_0 < K\delta_0 - K\delta_0 = 0.$$

This is a contradiction of equation (3.18), hence we have $t \geq, t_0 + T(\varepsilon)$ such that $y(t_0) < \delta_0$ so that

$$\text{var}_{t_0}^{t_1}\left(y(t) - \int_{t_0}^{t} DA(s, y(s))\right) < \delta(\varepsilon), \text{ and } \quad \|y(t)\| < \varepsilon, \quad t \ge t_0 + T(\varepsilon),$$

and the trivial solution $x \equiv 0$ of equation (2.12) is variationally attracting. Thus, system solution is variationally asymptotically stable.

3.1 Illustration

We consider the Lyapunov function for the dynamic equation (2.12) of the form $V(t, x(t)) = \int_{t_0}^{t} \| f(s, x(s))\| \Delta s$, (3.26)

which is locally Lipschitz. That is

$$V(t, y(t)) - V(t, x(t)) = \left\| \int_{t_0}^{t} \| f(s, y(s))\| \Delta s - \int_{t_0}^{t} \| f(s, x(s))\| \Delta s \right\|$$
$$\le \left\| \int_{t_0}^{t} \| f(s, y(s) - x(s))\| \Delta s \right\|$$
$$\le w\|y(t) - x(t))\| \int_{t_0}^{t} m(s) \Delta s$$
$$\le w\|y(t) - x(t))\| (h(t) - h(t_0)), \; t \in [a,b]$$

Also, for any $a < t_0 \le \sigma \le t_1 < b$, there is an $\eta(\sigma) > 0$, $t_0 + \eta \in [t_0, t_1]_T$, and by theorem (3.1) we have

$$V(t_0 + \eta, y(t_0 + \eta)) - V(t_0, y(t_0)) = \int_{t_0+\eta}^{t} \| f(s, y(s))\| \Delta s - \int_{t_0}^{t} \| f(s, y(s))\| \Delta s$$
$$\le \int_{t_0}^{t_0+\eta} \| f(s, y(s)))\| \Delta s - \|y(t) - y(t_0)\|$$
$$\le \int_{t_0}^{t_0+\eta} m(s) \Delta s - \|y(t) - y(t_0)\|$$
$$\le \|h(t_0 + \eta) - h(t_0)\| - \|y(t) - y(t_0)\| \qquad (3.27)$$

Assume $\lim_{\eta \to 0} h(t_0 + \eta) = h(t_0)$, then there exists an $\varepsilon > 0$ such that $\|h(t_0 + \eta) - h(t_0)\| < \varepsilon$.

Also considering the second term on the right of the equation, we have

$$\lim_{t \to t_+} (y(t) - y(t_0)) = \lim_{t \to t_+} \left(y(t) - y(t_0) - \int_{t_0}^{t} f(s, y(s)) \Delta s \right)$$
$$= y(t_+) - y(t_0) - \lim_{t \to t_+} \int_{t_0}^{t} f(s, y(s)) \Delta s$$
$$= y(t_+) - \int_{t_+}^{t} f(s, y(s)) \Delta s - y(t_0) - \int_{t_+}^{t} f(s, y(s)) \Delta s$$
$$= y(t_+) - \int_{t_+}^{t} DA(s, x(s)) - y(t_0) - \int_{t_0}^{t} DA(s, y(s))$$
$$\text{var}_{t_0}^{t_+} \left(y(t) - \int_{t_0}^{t} DA(s, y(s)) \right)$$

So that equation (3.27) implies

$$V(t_0 + \eta), x(t_0 + \eta)) \le V(t_0, x(t_0)) + \text{var}_{t_0}^{t_0+\eta} \left(x(t) - \int_{t_0}^{t} \|DA(s, x(s))\| \right) + \varepsilon$$
$$\le \|x(t_0)\| + \delta(\varepsilon) + \varepsilon$$
$$\le 2\delta(\varepsilon) + \varepsilon$$

Also following the contradictory argument of equation (3.18), and choosing ε arbitrary, the system solution is stable in the Lyapunov sense.

Conclusion

In this work, we established the results on variational stability and variational asymptotic stability of the dynamic equation on time-scale in the framework of the generalized ordinary equations. Theorems and proof of the concept were presented. An illustration was used to confirm the suitability of the idea.

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