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APPLICATION OF EXTENDED CORNELL POTENTIAL WITH SPIN-SPIN INTERACTION TO DIATOMIC MOLECULES AND MASS SPECTRA OF HEAVY MESONS

ABRAHAM, I. S., OKON, I. B., ANTIA, A. D.,
AKANKPO, A. O., ISONGUYO, C. N. AND ITUEN, E. E*.
Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria
*Correspondence: ituenokon@uniuyo.edu.ng

ABSTRACT

In this work, the study of mass spectra of mesons with extended Cornell potential coupled with spin-spin interaction has been carried out. This involves approximating the Gaussian function to harmonic term in radial Schrodinger wave equation with the help of Greene-Aldrich approximation to centrifugal term using parametric Nikiforov-Uvarov (NU) method. The addition of spin-spin interaction to the extended Cornell potential helped in determining the singlet ($s = 0$) and triplet ($s = 1$) state of mesons which cannot be determined with the absence of the spin. Using the minimization approach implemented using the Mathematica package, the potential free parameters (a, b, c, σ and α_s) were obtained by fitting the mesons mass spectra states with experimentally observed values of the Particle Data Group (PDG). The potential model was reduced to Pseudoharmonic, Kratzer and extended Cornell potential as special cases. The numerical energies of the Kratzer potential for selected diatomic molecules and for extended Cornell potential as applied to mass spectra of charmonium and bottomonium were in excellent agreement when compared to the work of existing literature. The total wave function for the proposed potential model was expressed in term of Jacobi polynomial. The wave function and probability density plots for charmonium were obtained using Mathematica package.

KEYWORDS: Extended Cornell potential; Schrödinger equation; Greene-Aldrich approximation.

INTRODUCTION

Elementary particles are sub-atomic particles which are not composed of any other particle. They are classified according to families, sub-families, and spin. Elementary particles are classified either as bosons or fermions. Bosons are particles with zero or integer spin while fermions are particles with half-integer spin. The study of elementary particles is significant with enormous applications in different field of study such as; Medicine, diagnosing, security, synchrotron light source, nuclear weapons and Statistical Physics. Quarks are elementary particles and fundamental constituents of matter which are classified into up, down, top, bottom, strange and charm quarks. The theory of quantum chromodynamics (QCD) has been successfully used in describing the properties of these particles with some considerable potential models. Meanwhile, one of the interesting significant research questions is whether to consider hadrons as non-relativistic bound states of quarks (Omugbe *et al.*, 2023a).

Cornell and extended Cornell potential is the main focus potential used for the description of quarks, mesons and other elementary particles (Omugbe *et al.*, 2022). The study of quark and antiquark which constitutes mesons and other subatomic particles held together by strong interaction is described by the theory of quantum chromodynamics (Omugbe *et al.*, 2020). The investigation of mass spectra of mesons under the spinless-Salpeter equation couple with Cornell potential has been carried out using semi-classical WKB approximation approach (Omugbe *et al.*, 2021). The proposed potential was used to obtain the mass spectra of heavy and heavy-light mesons. Also, the non-relativistic energy of the Coulomb potential, mass spectrum free parameters of the meson systems were obtained by fitting

analytical solution with experimental data and solve the resulting nonlinear equations simultaneously via numerical approach. The masses of the mesons systems with the Salpeter equation for different quantum states which was in excellent agreement with available experimental data as reported in the existing literature. The electromagnetic spectrum transition of bottomonium as well as the new bottomonium state was determined by Wei-Jun *et al.*, 2017. The author also calculated the mass spectra and adopted a nonrelativistic screened potential model. A lot of work has been done in determining bound state of heavy mesons. The radial Schrödinger equation (Schrodinger, 1926) was solved with a three-point difference central method, where the spin-dependent potentials were solved using a non-perturbative method. With this treatment, the corrections of the spin dependent potentials to the wave functions were included successfully. Hulthén plus Hellmann potentials was adopted by (Akpan *et al.*, 2021) using quark-antiquark interaction potential to study the mass spectra of heavy mesons and their present potential provides satisfying results when compared with experimental data. The mass spectra of charmonium using a Coulomb plus linear potential were studied by (Kher and Rai, 2018). The trajectories of the charmonium were basically nonlinear and exhibit nonlinear behaviour in the lower mass region. Edurado *et al.*, 2003, studied the hadron spectra and other properties of quark systems in the framework of a non-relativistic spin independent phenomenological model. The chosen confining potential was harmonic, which allowed them to obtain analytical solutions for both meson and baryon (of equal constituent quarks) spectra.

Many quarkonium potential is solved within the nonrelativistic regime, using the Nikiforov-Uvarov method. Inyang *et al.*, 2022 studied the thermal properties and masses

of heavy mesons with quarkonium potential where they obtained the energy spectrum and the wave function in terms of Laguerre polynomials. The results were used in predicting the masses of heavy mesons which is (charmonium $c\bar{c}$) and (bottomonium $b\bar{b}$) as well as studying thermal properties of the system such as mean energy, entropy, free energy, and specific heat capacity obtained from the partition function by setting Debye mass to zero. Correspondingly, using the nonrelativistic quark-antiquark Cornell potential model, Sonia *et al.*, 2018, employed the numerical solution of Schrödinger equation of mass spectra and decay properties of heavy quarkonia only using four mass spectra parameters. Inyang *et al.*, 2020 investigated the class of Yukawa potential as the quark-antiquark interaction potential for studying the mass spectra of heavy mesons. The potential was made to be temperature dependent by replacing the screening parameters with Debye mass. Kumar and Singh, 2022 investigated analytical solutions to the Schrodinger equation for a generalized Cornell potential and its applications to diatomic molecules and heavy mesons where the analytical expressions of energy eigenvalues and eigenfunctions for a generalized or standard Cornell potential were obtained by solving the non-relativistic Schrodinger equation using the Nikiforov–Uvarov functional analysis method (Nikiforov and Uvarov ,1988) along with Greene–Aldrich approximation. Using the energy eigenvalue spectra, diatomic molecules like CO, NO, O₂, LiH and HCl were computed. The results obtained were in good agreement with the results of others researchers

Omugbe *et al.*, 2023a proposed a global spin-dependent potential energy function and described a nonrelativistic quark model. (Omugbe *et al.*, 2023b) carried out similar research where they used the non-relativistic Schrödinger equation with a spin-spin, spin-orbit and tensor term to calculate root mean square radii of heavy mesons. The Cornell Potential, which is the sum of the Coulomb and linear potentials, is one of the most well-known QCD-inspired potential models as exemplified by (Godfrey and Isgur, 1985). The Cornell Potential takes the form.

$$\psi''(s) + \frac{c_1 - c_2 s}{s(1 - c_3 s)} \psi'(s) + \frac{1}{s^2(1 - c_3 s)} [-\xi_1 s^2 + \xi_2 s - \xi_3] \psi(s) = 0 \quad (5)$$

The parametric constants are given as;

$$c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \quad c_6 = c_5^2 + \xi_1, \quad c_7 = 2c_4 c_5 - \xi_2,$$

$$c_8 = c_4^2 + \xi_3, \quad c_9 = c_3 c_7 + c_3^2 c_8 + c_6, \quad c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \quad c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}), \quad c_{12} = c_4 + \sqrt{c_8}, \quad c_{13} = c_5 - \sqrt{c_9} + c_3\sqrt{c_8}$$

$$c_1, \quad c_2, \quad c_3 \quad \text{and} \quad c_4 = 0 \quad (6)$$

The parametric energy-eigen equation is presented as;

$$C_2 n - (2n + 1)C_5 + (2n + 1)(\sqrt{C_9} + C_3\sqrt{C_8}) + n(n - 1)C_3 + C_7 + 2C_3 C_8 + 2\sqrt{C_8 C_9} = 0 \quad (7)$$

The wave function is given as;

$$\psi(s) = N_{nl} s^{c_{12}} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n \left[\frac{c_{10} - 1; c_{11} - c_{10} - 1}{c_3} \right] (1 - 2C_3 s) \quad (8)$$

$$V(r) = \frac{-a}{r} + br \quad (1)$$

Here, the first term is the Coulomb term which is responsible for the short-range gluon exchange interaction between a quark and its antiquark while the second term is responsible for the large-scale quark confinement. According to Kumar and Singh, 2022, the extended or generalized Cornell potential with radial distance-dependent functions is given as

$$V_C(r) = ar^2 + br - \frac{c}{r} + \frac{d}{r^2} + e, \quad (2)$$

Also, Cornell potential with spin-spin interaction as studied by Omugbe *et al.*, 2023a is given as

$$V(r) = \frac{-4\alpha_s}{3r} + br + \frac{16\alpha_s\pi(\frac{\sigma}{\sqrt{\pi}})^3 e^{-\sigma^2 r^2} (s(s+1) - \frac{3}{2})}{9m_q m_{\bar{q}}} \quad (3)$$

In this work, we study energy spectra of Charmonium and bottomonium using extended Cornell potential coupled with spin -spin interaction which to the best of our knowledge has not been reported in any existing literature. We also obtain energy equation for special cases and applies it to study bound states of diatomic molecules.

The Generalized Parametric Nikiforov-Uvarov (NU) Method

The NU method was presented by Nikiforov and Uvarov in 1988 and has been employed in solving second order linear differential equations to a generalized equation of hypergeometric-type. This method provides exact solutions in terms of special orthogonal functions as well as corresponding energy eigenvalues. This NU method applies to both relativistic and non-relativistic equations.

With the appropriate coordinated transformation $S = s(x)$ the equation can be written as;

$$\psi''(s) + \frac{\check{\tau}(s)}{\sigma(s)} \psi'(s) + \frac{\check{\sigma}(s)}{\sigma^2(s)} \psi(s) = 0 \quad (4)$$

Where $\check{\tau}(s)$ is a polynomial of degree one while $\sigma(s)$ and $\check{\sigma}(s)$ are polynomial of at most degree of two.

The parametric NU differential equation according to Tezcan and Sever in 2009 the wave function is given as;

Analytical Solutions of the Extended Cornell Potential with Spin-spin interaction

In this section, we shall present the analytical solution of this system, which is necessary for obtaining the wave function and probability density needed for the analysis. The radial Schrodinger equation is given as (Greiner, 2000; Galindo, *et al.*, 1978 and Ita, *et al.*, 2015)

$$\frac{d^2R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left\{ (E - V(r)) - \frac{\hbar^2 l(l+1)}{2\mu r^2} \right\} R(r) = 0, \quad (9)$$

The extended Cornell potential couple with Spin-spin is the combination of equation (2) and (3) and it is given as

$$V(r) = ar^2 + br - \frac{c}{r} + \frac{d}{r^2} + e + f \left\{ \frac{16\pi\alpha_s}{9mqmq} \left(\frac{\delta}{\sqrt{\pi}} \right)^3 e^{-\delta^2 r^2} \left(s(s+1) - \frac{3}{2} \right) \right\} \quad (10)$$

Where a, b, c, d and e are constant potential parameters for a system which should be fix approximately to achieve results comparable to experimental and theoretical values. α_s is the coupling constant, m_q is the mass of quark, $m_{\bar{q}}$ is the mass of antiquark, s is the spin number. f takes the value between zero and one.

By substituting equation (10) into equation (9) and with the help of coordinate transformation reduces to equation (11)

$$\begin{aligned} \frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dR(s)}{ds} + \frac{1}{s^2(1-s)^2} \left\{ \left(\frac{2\mu E_{nl}}{\hbar^2 \alpha^2} - \frac{12\mu a}{\hbar^2 \alpha^4} - \frac{6\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} - \frac{2\mu D_1 D_2 f}{\hbar^2 \alpha^2} + \frac{12\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) s^2 + \left(\frac{-4\mu E_{nl}}{\hbar^2 \alpha^2} + \frac{8\mu a}{\hbar^2 \alpha^4} + \frac{6\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{4\mu e}{\hbar^2 \alpha^2} + \frac{4\mu D_1 D_2 f}{\hbar^2 \alpha^2} - \frac{8\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) s + \left(\frac{2\mu E_{nl}}{\hbar^2 \alpha^2} - \frac{2\mu a}{\hbar^2 \alpha^4} - \frac{2\mu b}{\hbar^2 \alpha^3} + \frac{2\mu c}{\hbar^2 \alpha^2} - \frac{2\mu d}{\hbar^2} - \frac{2\mu e}{\hbar^2 \alpha^2} - \frac{2\mu D_1 D_2 f}{\hbar^2 \alpha^2} + \frac{2\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} - l(l+1) \right) + \left(\frac{-2\mu a}{\hbar^2 \alpha^4} + \frac{2\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) s^4 + \left(\frac{8\mu a}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{8\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) s^3 \right\} \end{aligned} \quad (11)$$

By neglecting higher powers of s, equation (11) reduces to

$$\begin{aligned} \frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dR(s)}{ds} + \frac{1}{s^2(1-s)^2} \left\{ - \left(\frac{12\mu a}{\hbar^2 \alpha^4} + \frac{6\mu b}{\hbar^2 \alpha^3} + \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{2\mu D_1 D_2 f}{\hbar^2 \alpha^2} - \frac{2\mu E_{nl}}{\hbar^2 \alpha^2} - \frac{12\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) s^2 + \left(\frac{8\mu a}{\hbar^2 \alpha^4} - \frac{4\mu E_{nl}}{\hbar^2 \alpha^2} + \frac{6\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{4\mu e}{\hbar^2 \alpha^2} + \frac{4\mu D_1 D_2 f}{\hbar^2 \alpha^2} - \frac{8\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) s + \left(\frac{2\mu a}{\hbar^2 \alpha^4} - \frac{2\mu E_{nl}}{\hbar^2 \alpha^2} + \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{2\mu d}{\hbar^2} + \frac{2\mu e}{\hbar^2 \alpha^2} + \frac{2\mu D_1 D_2 f}{\hbar^2 \alpha^2} - \frac{2\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} + l(l+1) \right) \right\} R(s) = 0 \end{aligned} \quad (12)$$

Equation (12) is comparable to NU differential equation. Using equation (7), the energy eigen equation for extended Cornell potential coupled with spin-spin interaction is given as

$$\begin{aligned} E = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} + \frac{\hbar^2 \alpha^2}{2\mu} \left(\frac{2\mu a}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{4\mu e}{\hbar^2 \alpha^2} + \frac{4\mu d}{\hbar^2} + \frac{2\mu D_1 D_2 f}{\hbar^2 \alpha^2} - \frac{2\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} \right) - \hbar^2 \alpha^2 \left\{ \frac{\left((n^2 + n + \frac{1}{2}) + (n + \frac{1}{2}) \sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} - \frac{24\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} + \frac{8\mu b}{\hbar^2 \alpha^3} + \frac{8\mu d}{\hbar^2} + 4l(l+1) + 1} - \frac{4\mu a}{\hbar^2 \alpha^4} - \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{4\mu D_1 D_2 f}{\hbar^2 \alpha^2} + \frac{4\mu d}{\hbar^2} + 2l(l+1)} \right)^2}{(2n+1) + \sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} - \frac{24\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} + \frac{8\mu b}{\hbar^2 \alpha^3} + \frac{8\mu d}{\hbar^2} + 4l(l+1) + 1}} \right\} \end{aligned} \quad (13)$$

Using equation (8), the total un-normalized wave function is given as

$$\psi_n(s) = N_{nl} s^\beta (1-s)^{\frac{1}{2} + \eta} P_n^{[2\beta, 2\eta]}(1-2s) \quad (14)$$

Where

$$\beta = \sqrt{\frac{-2\mu E_{nl}}{\hbar^2 \alpha^2} + l(l+1) + \chi_1}, \quad \eta = \frac{1}{2} \sqrt{4\chi_2 + 4l(l+1) + 1} \quad (15)$$

Normalisation of the wave function

The Normalization constant can be obtained using the expression

$$\int_0^\infty |R_{nl}(r)|^2 dr = 1 \Rightarrow \int_0^\infty \left[N_{nl} s^{\beta_1} (1-s)^{\eta_1} P_n^{(2\beta_1, \eta_1-1)}(1-2s) \right]^2 ds = 1 \quad (16)$$

$$\beta_1 = \sqrt{\frac{-2\mu E_{nl}}{\hbar^2 \alpha^2} + l(l+1) + \chi_1}, \quad \eta_1 = \frac{1}{2} + \frac{1}{2} \sqrt{4\chi_2 + 4l(l+1)},$$

$$\chi_1 = \frac{2\mu a}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha^2} + \frac{2\mu d}{\hbar^2} + \frac{2\mu e}{\hbar^2 \alpha^2} + \frac{2\mu D_1 D_2 f}{\hbar^2 \alpha^2} - \frac{2\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4}, \quad \chi_2 = \frac{6\mu a}{\hbar^2 \alpha^4} - \frac{6\mu D_1 D_2 f \delta^2}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} + \frac{2\mu d}{\hbar^2} \quad (17)$$

The wave function is assumed to be in bounded at $r \in (0, \infty)$ and $se^{-\alpha r} \in (1, 0)$.

Using Mathematica package, the total normalized wave function for ground state (n = 0), first excited state (n = 1) and second excited state (n = 2) is showing in Equations (18), (19) and (20)

$$R_{nl}(s) = \sqrt{\frac{\alpha\Gamma(2\beta_1+2\eta_1+2)}{\Gamma(2\beta_1)\Gamma(2+2\eta_1)}} (1 - e^{-\alpha r})^{\eta_1+\frac{1}{2}} (e^{-\alpha r})^{\beta_1} \quad (18)$$

$$R_{nl}(s) = \sqrt{\frac{2\alpha\beta_1(3+2\beta_1+2\eta_1)\Gamma(2\beta_1+2\eta_1+2)}{(3+2\eta_1)\Gamma(2+2\beta_1)\Gamma(2+2\eta_1)}} (1 - e^{-\alpha r})^{\eta_1+\frac{1}{2}} (e^{-\alpha r})^{\beta_1} P_1^{(2\beta_1, \eta_1)} (1 - 2e^{-\alpha r}) \quad (19)$$

$$R_{nl}(s) = \sqrt{\frac{4\alpha\beta_1(5+2\beta_1+2\eta_1)\Gamma(2\beta_1+2\eta_1+3)}{(3+2\eta_1)\Gamma(2+2\beta_1)\Gamma(2+2\eta_1)}} (1 - e^{-\alpha r})^{\eta_1+\frac{1}{2}} (e^{-\alpha r})^{\beta_1} P_1^{(2, 2\beta_1, 2\eta_1)} (1 - 2e^{-\alpha r}) \quad (20)$$

Special Cases from the Proposed Potential

The special cases of the potential were obtained after making a suitable adjustment of the parameters of the general potential in Equation (10). Some of the important forms of these potentials such are: Kratzer, the Pseudoharmonic and the Coulomb perturbed potentials. These potentials are very useful in describing interactions of diatomic molecules and had been successfully use by many authors in different areas of physical and chemical sciences.

(a) Pseudoharmonic Potential

This form of potential had been successfully employed in the past to study the eigenvalue spectra of various diatomic molecules. By adjusting the potential parameters.

Substituting $a = \frac{D_e}{r_e^2}, b = c = 0, d = D_e r_e^2, e = -2D_e, f = 0$ in Equation (10) gives the Pseudoharmonic potential as;

$$V(r) = ar^2 + \frac{d}{r^2} + e = \frac{D_e r^2}{r_e^2} + \frac{D_e r_e^2}{r^2} - 2D_e \quad (21)$$

Substituting the same boundary condition into Equation (13) gives the energy equation for Pseudoharmonic potential as;

$$E = \frac{\hbar^2 \alpha^2}{2\mu} \left(\frac{2\mu D_e}{\hbar^2 \alpha^4 r_e^2} - \frac{4D_e \mu}{\hbar^2 \alpha^2} + \frac{2\mu D_e r_e^2}{\hbar^2} + l(l+1) \right) - \frac{\hbar^2 \alpha^2}{2\mu} \left\{ \frac{(n^2+n+\frac{1}{2})+(n+\frac{1}{2})\sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} + \frac{8\mu d}{\hbar^2} + 4l(l+1)+1} - \frac{4\mu a}{\hbar^2 \alpha^4} + \frac{4\mu D_e r_e^2}{\hbar^2} + 2l(l+1)}}{(2n+1)+\sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} + \frac{8\mu b}{\hbar^2 \alpha^3} + 4l(l+1)+1}} \right\}^2 \quad (22)$$

(b) Standard Kratzer Potential

This is one of the most important potentials of the molecular spectroscopy.

Substituting $a = b = e = f = 0, \alpha = 0, d = D_e r_e^2, e = D_e, c = 2D_e r_e$ into Equation (10) and carry out some simplifications gives the energy equation of standard Kratzer potential as;

$$V(r) = \frac{-c}{r} + \frac{d}{r^2} = \frac{D_e r_e}{r} + \frac{D_e r_e^2}{r^2} \quad (23)$$

The energy for standard Kratzer potential is given as;

$$E_{nl} = \frac{-\hbar^2}{2\mu} \left\{ \frac{\frac{4\mu D_e r_e}{\hbar^2}}{(2n+1)+\sqrt{\frac{8\mu d}{\hbar^2} + 4l(l+1)+1}} \right\}^2 \quad (24)$$

(c) Perturb Cornell Potential

This form of potential has a very useful applications in calculating the mass spectra of quark and antiquark pairs. By substituting $d = e = f = 0$ in Equation (10) gives the perturb Cornell potential as;

$$V(r) = ar^2 + br - \frac{c}{r} \quad (25)$$

The energy of perturbed Cornell potential is given below using the same condition

$$\frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} + \frac{\hbar^2 \alpha^2}{2\mu} \left(\frac{2\mu a}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{2\mu c}{\hbar^2 \alpha} \right) - \frac{\hbar^2 \alpha^2}{2\mu} \left\{ \frac{(n^2+n+\frac{1}{2})+(n+\frac{1}{2})\sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} + \frac{8\mu d}{\hbar^2} + 4l(l+1)+1} - \frac{4\mu a}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} + \frac{4\mu c}{\hbar^2 \alpha} + 2l(l+1)}}{(2n+1)+\sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} + \frac{8\mu b}{\hbar^2 \alpha^3} + 4l(l+1)+1}} \right\}^2 \quad (26)$$

Perturbed Cornell potential with Spin-spin interaction

Substituting $d = e = 0, f = 1$ into Equation (10) gives the perturbed Cornell potential with spin-spin interaction as;

$$V(r) = ar^2 + br - \frac{c}{r} + D_1 D_2 (1 - \delta^2 r^2) \quad (27)$$

The energy corresponding energy is given as;

$$E = \frac{\hbar^2 \alpha^2 l(l+1)}{2\mu} + \frac{\hbar^2 \alpha^2}{2\mu} \left(\frac{2\mu a}{\hbar^2 \alpha^4} + \frac{2\mu b}{\hbar^2 \alpha^3} - \frac{2\mu D_1 D_2}{\hbar^2 \alpha^2} - \frac{2\mu D_1 D_2 \delta^2 f}{\hbar^2 \alpha^4} \right) - \frac{\hbar^2 \alpha^2}{2\mu} \left\{ \frac{(n^2+n+\frac{1}{2})+(n+\frac{1}{2})\sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} + \frac{24\mu D_1 D_2 \delta^2}{\hbar^2 \alpha^4} + \frac{8\mu b}{\hbar^2 \alpha^3} + 4l(l+1)+1} - \frac{4\mu a}{\hbar^2 \alpha^4} - \frac{2\mu b}{\hbar^2 \alpha^3} + \frac{4\mu c}{\hbar^2 \alpha} + \frac{4\mu D_1 D_2 \delta^2}{\hbar^2 \alpha^4} + 2l(l+1)}}{(2n+1)+\sqrt{\frac{24\mu a}{\hbar^2 \alpha^4} + \frac{24\mu D_1 D_2 \delta^2}{\hbar^2 \alpha^4} + \frac{8\mu b}{\hbar^2 \alpha^3} + 4l(l+1)+1}} \right\}^2 \quad (28)$$

NUMERICAL RESULTS

Table 1: Spectroscopic parameters for various diatomic molecules

| Molecules | D_e (eV) | μ (a. m. u) | r_e (\AA) |
|-----------|-------------|-----------------|------------------------|
| LiH | 2.155283695 | 0.8801221 | 1.5956 |
| HCl | 4.619061175 | 0.9801045 | 1.2746 |

| | | | |
|----------------|-------------|-----------|--------|
| O ₂ | 5.156658828 | 7.9974575 | 1.2080 |
|----------------|-------------|-----------|--------|

(Vinod and Ram, 2022)

D_e is the disassociation Energy

μ is the reduce mass measured in atomic mass unit (a.m.u)

r_e is the equilibrium bond length

Now using the data of Table 1 in Equation (21), the energy eigenvalue spectra of the three diatomic molecules are computed and presented in Tables 2 to Table 6. A close observation of the data of Tables 2 to Table 6 reveals that the present results are in excellent agreement to work of an existing literature.

Table 2: Numerical Solutions of Mass spectra of Charmonium in GeV ($a = 0.0297 \text{ GeV}^3$, $b = 0.7609 \text{ GeV}^2$, $c = 60.586$ and $m_c = 1.317 \text{ GeV}$, $\alpha = 0.6585$, $\alpha = 0.0959$).

| State | Present Work | Kumar and Singh, 2022 | Sonia <i>et al.</i> , (2018) | Richa <i>et al.</i> , (2018) | Wei-Jun <i>et al.</i> , (2017) | Ramesh and Fakir (2012) | Bai-Qing, and Kuang-Ta. (2009) | Beringer <i>et al.</i> , (2012) | Ebert <i>et al.</i> , (2003) |
|-------|--------------|-----------------------|------------------------------|------------------------------|--------------------------------|-------------------------|--------------------------------|---------------------------------|------------------------------|
| 1S | 3.09129 | 3.091 | 3.094 | 3.096 | 3.097 | 3.078 | 3.097 | 3.096 | 3.096 |
| 2S | 3.62773 | 3.627 | 3.681 | 3.265 | 3.679 | 3.581 | 3.673 | 3.686 | 3.686 |
| 3S | 4.12091 | 4.121 | 4.129 | 3.328 | 4.078 | 4.085 | 4.022 | 4.040 | 4.088 |
| 4S | 4.57373 | 4.573 | 4.514 | 3.737 | 4.412 | 4.589 | 4.273 | 4.236 | - |
| 5S | 4.98880 | 4.988 | 4.863 | - | 4.711 | - | 4.463 | - | - |
| 6S | 5.36852 | 5.368 | 5.185 | - | - | - | 4.608 | - | - |
| 1P | 3.11638 | 3.123 | 3.468 | 3.216 | 3.516 | 3.415 | 4.608 | - | 3.510 |
| 2P | 3.65192 | 3.658 | 3.938 | 3.362 | 3.937 | 3.917 | 3.510 | 3.773 | 3.686 |
| 3P | 4.14428 | 4.150 | 4.338 | - | 4.284 | - | 3.901 | - | - |
| 4P | 4.59632 | 4.602 | 4.696 | - | - | - | 4.178 | - | - |
| 5P | 5.01066 | 5.016 | 5.026 | - | - | - | 3.787 | - | - |
| 1D | 3.15936 | 3.187 | 3.775 | 3.440 | 3.787 | 3.749 | - | - | 3.798 |
| 2D | 3.69365 | 3.819 | 4.188 | - | 4.144 | - | 4.089 | - | - |
| 3D | 4.18485 | 4.209 | 4.555 | - | 4.456 | - | 4.137 | - | - |
| 4D | 4.63582 | 4.658 | 4.891 | - | - | - | - | - | - |

Table 3: Numerical Solutions of Mass spectra of Bottomonium in GeV ($a = 0.0297 \text{ GeV}^3$, $b = 1.378 \text{ GeV}^2$, $c = 10.473$ and $m_b = 4.584 \text{ GeV}$).

| State | Present Work | Kumar and Singh, 2022 | Sonia <i>et al.</i> , (2018) | Richa <i>et al.</i> , (2018) | Stephen and Kenneth, 2015 | Ikhdaire and Sever, 2009 | Bai-Qing and Kuang-Ta, 2009 | Beringer <i>et al.</i> , 2012 | Ebert <i>et al.</i> , 2003 |
|-------|--------------|-----------------------|------------------------------|------------------------------|---------------------------|--------------------------|-----------------------------|-------------------------------|----------------------------|
| 1S | 9.46093 | 9.460 | 9.463 | 9.460 | 9.0465 | 9.510 | 9.463 | 9.460 | 9.460 |
| 2S | 9.98182 | 9.981 | 9.979 | 10.023 | 10.003 | 10.038 | 9.979 | 10.023 | 10.023 |
| 3S | 10.40181 | 10.401 | 10.359 | 10.585 | 10.354 | 10.566 | 10.359 | 10.355 | 10.355 |
| 4S | 10.73647 | 10.736 | 10.683 | 11.148 | 10.635 | 11.094 | 10.683 | 10.580 | - |
| 5S | 10.99819 | 10.998 | 10.975 | - | 10.878 | - | 10.975 | - | - |
| 6S | 11.19697 | 11.196 | 11.243 | - | 11.102 | - | 11.243 | - | - |
| 1P | 9.52601 | 9.544 | 9.819 | 9.492 | 9.876 | 9.862 | 9.819 | - | 9.982 |
| 2P | 10.04106 | 10.056 | 10.217 | 10.038 | 10.246 | 10.390 | 10.217 | - | 10.255 |
| 3P | 10.45615 | 10.468 | 10.553 | - | 10.538 | - | 10.553 | - | - |
| 4P | 10.78662 | 10.796 | 10.853 | - | 10.788 | - | 10.853 | - | - |
| 5P | 11.04475 | 11.052 | 11.127 | - | 11.014 | - | 11.127 | - | - |
| 1D | 9.63687 | 9.808 | 10.074 | 9.551 | 10.138 | 10.214 | - | - | 10.153 |
| 2D | 10.14368 | 10.202 | 10.423 | - | 10.441 | - | 4.089 | - | - |
| 3D | 10.55191 | 10.599 | 10.731 | - | 10.698 | - | 4.137 | - | - |
| 4D | 10.87662 | 10.915 | 11.013 | - | 10.928 | - | - | - | - |

Energy Spectra of Some Diatomic Molecules

In this section, we shall obtain the energy eigenvalue spectra of three (3) diatomic molecules, which shall be LiH, HCl and O₂ by using the Kratzer potential which is given in Equation (24). For this purpose, we fixed various spectroscopic parameters for the three diatomic molecules showing below with the use of Maple Mathematical application as listed in Table 1.

Table 4: Energy spectra of LiH

| No. | <i>L</i> | LiH (Present Study) | Kumar and Singh, 2022 | Purohit <i>et al.</i> , 2020 | Ikot <i>et al.</i> , 2019 | Ikhdaire and Sever, 2009 |
|-----|----------|---------------------|-----------------------|------------------------------|---------------------------|--------------------------|
| 1 | 0 | -2.375802636 | -2.375958478 | -2.375802634 | -2.375802636 | -2.375819921 |
| | 1 | -2.374091378 | -2.374249269 | -2.374091378 | -2.374091378 | -2.374107972 |
| 2 | 0 | -2.289307673 | -2.291120943 | -2.289307672 | -2.289307674 | -2.289324266 |
| | 1 | -2.287688995 | -2.289500771 | -2.287689000 | -2.287688996 | -2.287705602 |
| | 2 | -2.284458584 | -2.286267374 | -2.284458585 | -2.284458584 | -2.284475515 |
| 3 | 0 | -2.207451626 | -2.210834101 | -2.207451625 | -2.207451626 | -2.207468200 |
| | 1 | -2.205918968 | -2.209299462 | -2.205918969 | -2.205918968 | -2.205935555 |
| | 2 | -2.202860140 | -2.206236677 | -2.202860142 | -2.202860140 | -2.202876749 |
| 4 | 3 | -2.198288038 | -2.201658657 | -2.198288039 | -2.198288040 | -2.198304679 |
| | 0 | -2.129908603 | -2.134778267 | -2.129908601 | -2.129908602 | -2.129925128 |
| | 1 | -2.128455978 | -2.133323227 | -2.128455977 | -2.128455976 | -2.128472514 |
| | 2 | -2.125556793 | -2.130419223 | -2.125556794 | -2.125556792 | -2.125733505 |
| | 3 | -2.121223115 | -2.126078336 | -2.121223116 | -2.121223116 | -2.121239701 |
| 5 | 4 | -2.115472884 | -2.120318526 | -2.115472882 | -2.115472884 | -2.115489505 |
| | 0 | -2.056380833 | -2.062661341 | -2.056380832 | -2.056380834 | -2.056397256 |
| | 1 | -2.055002762 | -2.061280475 | -2.055002760 | -2.055002762 | -2.055019226 |
| | 2 | -2.052252304 | -2.058524429 | -2.052252305 | -2.052252304 | -2.052268786 |
| | 3 | -2.048140758 | -2.054404523 | -2.048140758 | -2.048140758 | -2.048157264 |
| 5 | 4 | -2.042684927 | -2.048937588 | -2.042684926 | -2.042684928 | -2.042701466 |
| | 5 | -2.035906942 | -2.042175782 | -2.035906942 | -2.035906942 | -2.035923525 |

Table 5: Energy spectra of HCl for different state of quantum number *l*

| No. | <i>l</i> | HCl (Present Study) | Kumar and Singh, 2022 | Purohit <i>et al.</i> , 2020 | Okorie <i>et al.</i> , 2021 | Ikhdaire and Sever, 2009 |
|-----|--------------|---------------------|-----------------------|------------------------------|-----------------------------|--------------------------|
| 1 | 0 | -4.393727022 | -4.393713794 | -4.393727026 | -4.393727024 | -4.393727956 |
| | 1 | -4.391292902 | -4.393713794 | -4.393727026 | -4.393727024 | -4.393727956 |
| 2 | 0 | -4.252735635 | -4.260091578 | -4.252735268 | -4.252735636 | -4.252737112 |
| | 1 | -4.250417718 | -4.257768847 | -4.250417710 | -4.250417718 | -4.250419208 |
| | 2 | -4.245789525 | -4.253131060 | -4.245789523 | -4.245789526 | -4.245791052 |
| 3 | 0 | -4.118423404 | -4.132810777 | -4.118423405 | -4.118423404 | -4.118425371 |
| | 1 | -4.116214408 | -4.130595364 | -4.116214406 | -4.116214408 | -4.116216289 |
| | 2 | -4.111803615 | -4.126171764 | -4.111803613 | -4.111803616 | -4.111805631 |
| | 3 | -4.105205380 | -4.119554377 | -4.105205379 | -4.105205380 | -4.105207449 |
| 4 | 0 | -3.990375014 | -4.011476907 | -3.990375014 | -3.990375014 | -3.990377425 |
| | 1 | -3.988268221 | -4.009362246 | -3.988268219 | -3.988268222 | -3.998270645 |
| | 2 | -3.984061423 | -4.005139724 | -3.984061421 | -3.984061424 | -3.984063879 |
| | 3 | -3.977768153 | -3.998822947 | -3.977768151 | -3.977765152 | -3.977770657 |
| | 4 | -3.969408570 | -3.990432148 | -3.969408569 | -3.969408570 | -3.969411138 |
| 5 | 0 | -3.868206936 | -3.895725694 | -3.868206938 | -3.868206938 | -3.868209749 |
| | 1 | -3.866196139 | -3.893705709 | -3.866196137 | -3.866196140 | -3.866198963 |
| | 2 | -3.8621826949 | -3.889672170 | -3.862180946 | -3.862180950 | -3.862183802 |
| | 3 | -3.856174133 | -3.883637909 | -3.856174132 | -3.856174134 | -3.856177032 |
| | 4 | -3.848194719 | -3.875622042 | -3.848194720 | -3.848194720 | -3.848197679 |
| 5 | -3.838267838 | -3.865649819 | -3.83827839 | -3.838267840 | -3.838270870 | |

Table 6: Energy spectra of O₂

| No. | <i>l</i> | O ₂ (Present Study) | Kumar and Singh, 2022 | Purohit et al., 2020 | Okorie et al., 2021 | Ikhdaïr and Sever, 2009 |
|-----|--------------|--------------------------------|-----------------------|----------------------|---------------------|-------------------------|
| 1 | 0 | -5.066640766 | -5.066607135 | -5.0666407663 | -5.0666411467 | -5.066641151 |
| | 1 | -5.066291938 | -5.066258133 | -5.0662919381 | -5.0662923214 | -5.066292323 |
| 2 | 0 | -5.007960489 | -5.011267857 | -5.0079604870 | -5.0079611102 | -5.007961116 |
| | 1 | -5.007617701 | -5.010924782 | -5.0076177016 | -5.0076183271 | -5.007618329 |
| | 2 | -5.006932271 | -5.010238777 | -5.0069322711 | -5.0069329023 | -5.006932904 |
| 3 | 0 | -4.950293764 | -4.956885574 | -4.9502937621 | -4.9502946186 | -4.950294624 |
| | 1 | -4.949956878 | -4.956548293 | -4.9499568798 | -4.9499577391 | -4.949957740 |
| | 2 | -4.949283256 | -4.955873878 | -4.9492832534 | -4.9492841183 | -4.949284119 |
| | 3 | -4.948273160 | -4.984862598 | -4.9482731590 | -4.9482740326 | -4.948274034 |
| 4 | 0 | -4.893617381 | -4.903438377 | -4.8936173815 | -4.8936184638 | -4.893618469 |
| | 1 | -4.893286266 | -4.903106760 | -4.8932862680 | -4.8932873530 | -4.893287355 |
| | 2 | -4.892624177 | -4.902443673 | -4.8926241763 | -4.8926252668 | -4.892625268 |
| | 3 | -4.891631377 | -4.901449374 | -4.8916313768 | -4.8916324755 | -4.891632476 |
| | 4 | -4.890308274 | -4.901242074 | -4.8903082749 | -4.8903093844 | -4.890309388 |
| 5 | 0 | -4.837908798 | -4.850904987 | -4.8379087980 | -4.8379100982 | -4.837910103 |
| | 1 | -4.837583322 | -4.8505718911 | -4.8375833223 | -4.8375846252 | -4.837584627 |
| | 2 | -4.836932502 | -4.849926891 | -4.8369325034 | -4.8369338116 | -4.836933812 |
| | 3 | -4.833956605 | -4.848949193 | -4.8339566060 | -4.8339579221 | -4.833957923 |
| | 4 | -4.834656026 | -4.847646214 | -4.8346560268 | -4.8346573535 | -4.834657357 |
| 5 | -4.833031293 | -4.846018483 | -4.8330312942 | -4.8330326341 | -4.833032637 | |

Table 7: Mass Spectrum of Charmonium meson in GeV under spin-spin interaction potential

| S/N | State $n^{2s+1}L_j$ | Present | Omugbe et al, 2023a | Sonia et al., 2018 | Wei-Jun et al, 2017 LP(SP) | Kher and Rai, 2018 | Ebert et al., 2011 | Mansour et al, 2020 | EXP. (Workman et al., 2022) |
|-----|-------------------------------|---------|---------------------|--------------------|----------------------------|--------------------|--------------------|---------------------|-----------------------------------|
| 1 | 1 ¹ S ₀ | 3.43401 | 3.081 (3.097) | 3.094 | 3.097 (3.097) | 3.094 | 3.096 | 3.126 | 2.984 ± 0.0004 |
| 2 | 1 ³ S ₀ | 3.51148 | 3.061 (2.979) | 2.989 | 2.983 (2.984) | 2.995 | 2.981 | 3.033 | 3.097 ± (6 x 10 ⁻⁶) |
| 3 | 2 ¹ S ₀ | 3.43429 | 3.717 (3.736) | 3.681 | 3.679 (3.679) | 3.649 | 3.685 | 3.701 | 3.638 ± (1.1 x 10 ⁻³) |
| 4 | 2 ³ S ₀ | 3.56593 | 3.696 (3.640) | 3.602 | 3.635 (3.637) | 3.606 | 3.635 | 3.666 | 3.686 ± (6 x 10 ⁻⁵) |
| 5 | 3 ¹ S ₀ | 3.51123 | 4.003 (4.039) | 4.129 | 4.078 (4.030) | 4.036 | 4.039 | 4.055 | - |
| 6 | 3 ³ S ₀ | 3.58363 | 3.983 (3.937) | 4.058 | 4.048 (4.004) | 4.000 | 3.989 | 4.158 | 4.039 ± 10 ⁻³ |

Table 8: Theoretically Calculated Parameter with Spin-spin interaction

| Parameters | Charmonium |
|-----------------------|-----------------------------|
| mass (GeV) | $M_c = M_{\bar{c}} = 1.317$ |
| a (GeV ³) | 0.00909316 |
| b (GeV ²) | 2.55864 x 10 ⁻⁷ |
| C | 0.799991 |
| σ (GeV) | 0.0833493 |
| α _s | 0.998458 |

Discussion

Table 1 is spectroscopic parameters for various diatomic molecules used in calculating the bound state energies for the diatomic molecules. Table 2 shows the numerical solutions of mass spectra of charmonium. This table composes the present calculation with work reported in existing literature. It can be observed that the numerical calculation is in excellent agreement to authenticate the Mathematical accuracy of our present analytical calculation. Table 3 shows the mass spectra of bottomonium using extended Cornell potential. Our result is also in excellent agreement to work of existing literature. Tables 4, 5 and 6 are numerical bound state solutions of Kratzer potential for LiTH, HCL, and O₂ molecules for some quantum state. Our

numerical calculation is also consistent when compared to work of existing literature for the selected diatomic molecules. Table 7 is the numerical bound state solution of extended Cornell potential coupled with spin-spin interaction for singlet and triplet state of charmonium in s-orbital. The numerical bound state energies obtained for singlet and triplet are in consistent with experimental values to ascertain the high level of Mathematical accuracy of analytical and computational calculations.

Table 8 is the theoretical calculated values of free potential parameters using minimization approach. The parameters of Table 8 are used to obtained numerical solutions in Table 7

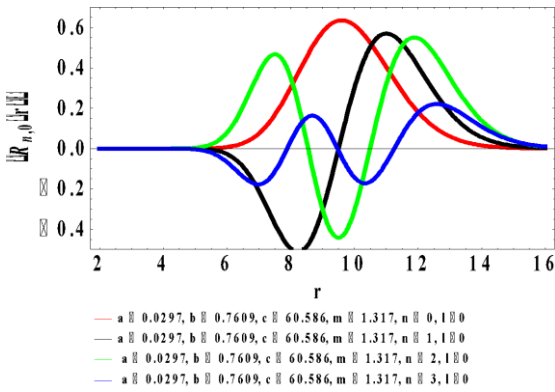


Figure 1: Wave Function Plot for Extended Cornell Potential of Charmonium without Spin-spin interaction for

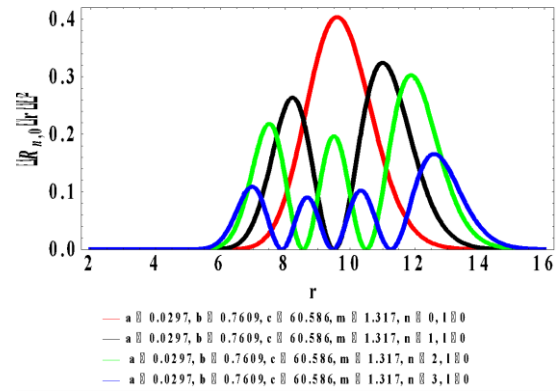


Figure 2: Probability Density Wave Function Plot for Extended Cornell Potential of Charmonium without Spin-spin interaction for

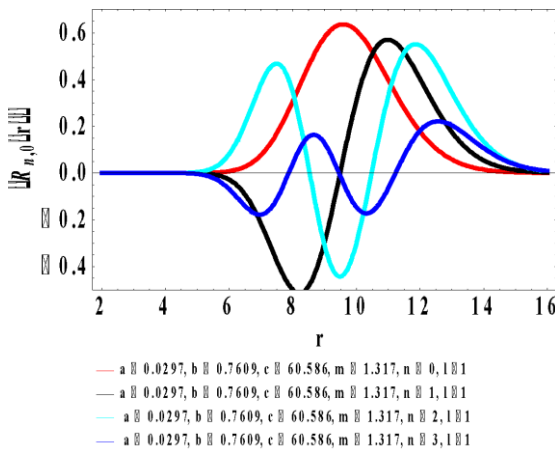


Figure 3: Wave Function Plot for Extended Cornell Potential of Charmonium without Spin-spin interaction for

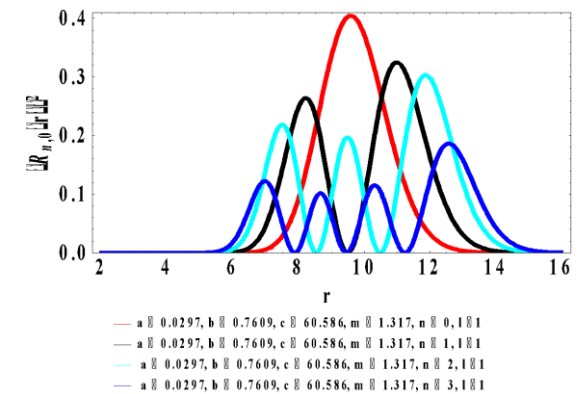


Figure 4: Probability Density Wave Function Plot for Extended Cornell Potential of Charmonium without Spin-spin interaction for

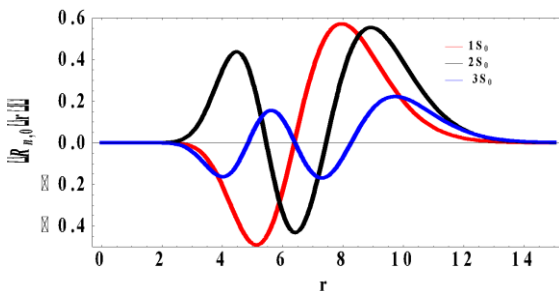


Figure 5: Wave Function Plot for Extended Cornell Potential of Charmonium Singlet state with Spin-spin interaction for

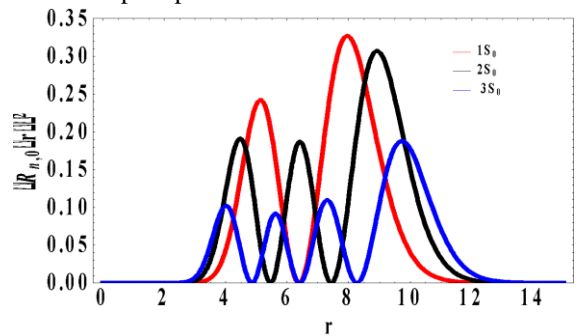


Figure 6: Probability Density Wave Function Plot for Extended Cornell Potential of Charmonium Singlet state with Spin-spin interaction for

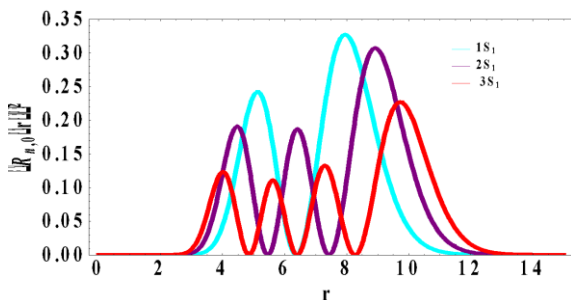


Figure 7: Wave Function Plot for Extended Cornell Potential of Charmonium Triplet state with Spin – spin interaction for ($s = 1$)

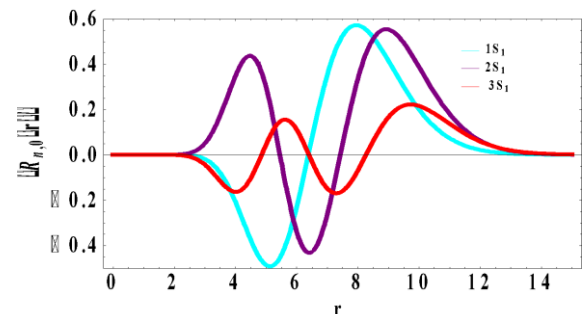


Figure 8: Probability Density Function Plot for Extended Cornell Potential of Charmonium Triplet state with Spin – spin interaction for ($s = 1$)

Figure 1 is a wave function plot for extended Cornell potential of Charmonium without spin-spin interaction for orbital angular quantum number $l = 0$. Here the wave function is periodic with different maximum and minimum turning point which increases periodically with a radial distance. The probability density for $l = 0$ which follows normal distribution is presented in figure 2. Figures 3 and 4 are respectively wave functions and probability density plots for extended Cornell potential without spin-spin interactions for orbital angular quantum number ($l = 1$). Hence, the wave function is also periodic while the probability density follows normal distribution curve. In Figure 5, we considered wave function plot for Charmonium singlet state ($S = 0$) using extended Cornell potential with spin-spin interaction. Here, the periodicity of the wave function is also established. That is the addition of the spin does not change the periodic nature of that wave function. The probability density for the singlet state follows Gaussian distribution function which makes the particle that is the singlet state more localized as presented in figure 6. The same explanation is applicable to Charmonium wave function triplet state ($S = 1$) and Chormonium probability density triplet state ($S = 1$) as shown in Figures 7 and 8 respectively.

CONCLUSION

In this research work, we have determined bound state solution of charmonium mesons with extended Cornell potential coupled with spin-spin interaction through parametric Nikiforov-Uvarov method. The addition of the spin-spin interaction helps in determining the singlet and triplet state of charmonium. Numerical bound state solutions for Charmonium and Bottomonium using extended Cornell potential are in excellent agreement with work of existing literature as well as Kratzer potential as applied to diatomic molecules, which form special cases of the main potential. This study has application in particle and high energy physics.

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