https://dx.doi.org/10.4314/wojast.v15i2.30

ISSN: 2141 – 3290
www.wojast.com

A GENERALIZATION OF RIESZ CHARACTERIZATION OF ABSOLUTE CONTINUITY IN $Lp[a,b]$−SPACES

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ABSTRACT

We initiate studies of a new classical characterization of absolute continuity which promises to be invaluable in studies and further generalizations of nonclassical characterizations of absolute continuity in diversely more general function spaces. Our main objective is showing that our newer characterization for absolute continuity includes Riesz characterization in Lp−spaces ($p > 1$) by allowing the situation $p \geq 1$. Our method also enriches studies of connection between absolute continuity and rapidly Cauchy sequences.

KEYWORDS: Absolute Continuity, Riesz characterizations, rapidly Cauchy sequences

INTRODUCTION

In his studies of absolute continuity Riesz was led to formulate the first L^p characterization of absolute continuity for $p > 1$. It is the aim of this work to modify the Riesz characterization from $p > 1$ to a more general characterization which admits $p \geq 1$. It is important to observe that the condition $p > 1$ of Riesz characterization excludes the fundamentally important space of Lebesgue integrable functions $L_1[a,b]$ which our newer characterization seeks to admit.

Let $f \in L_p[a,b]$ and $F(x)$ be real-valued functions, it is well known that if $F(x)$ is absolutely continuous on $[a,b]$, then F can be represented in terms some function $f \in L_p[a,b]$ in the form

(1) $\quad F(x) = c + \int_a^x f(t)d\mu$

where μ denotes Lebesgue measure and $c \in R$ is a constant of integration. The famous Riesz theorem which characterizes absolute continuity is stated below:

Theorem 1. (Riesz) see (Natanson, 1964)
A function $F:[a,b] \to \mathbb{R}$ can be represented in the form of 1:

$$F(x) = c + \int_a^x f(t)d\mu$$

if and only if for every partition $\mathcal{P}_n$ of $[a,b]$ by points $a = x_0 < x_1 < \ldots x_n < b$ the following
inequality holds:

(2) $\quad \sum_{j=1}^{n} \frac{|F(x_{k+1}) - F(x_k)|^p}{|x_{k+1} - x_k|^{p-1}} \leq k$

where k is a constant independent of the choice partition points in $\mathcal{P}_n$.

It is of importance to remark that the Riesz L_p characterization of absolute continuity was later studied and generalized by Medvedev using the concepts of functions of bounded Riesz variation and Riesz–Medvedev variations (Appel, 2014) which we have generalized in a recent submission.

We recall a real valued function $f:[a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is called absolutely continuous provided for each $\epsilon > 0$ there exists a $\delta > 0$ such that for each finite collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open sub-intervals $(a_k, b_k) \subseteq (a,b)$ we have

(3) $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$ whenever $\sum_{k=1}^{n} |b_k - a_k| < \delta$

The space of absolutely continuous functions on $[a,b]$ denoted by $AC[a,b]$, is a well-known subspace of continuous functions $C[a,b]$ introduced by Vitali in 1905 (Marly, 1998) as a natural generalization of the class of continuous monotone functions. The space $AC[a,b]$ of absolutely continuous functions plays invaluable roles both in theoretical, applied, and industrial directions. The most profound usefulness
Salient features in the theory of absolute continuity are the following properties of absolute continuous functions:

Lemma 2: Let A be an arbitrary subset of $\mathbb{R}$. Suppose $f: A \to \mathbb{R}$ is absolutely continuous on A. Let E be a bounded subset of A. Then f is of bounded variation on E.

Lusin Condition
A function $f : A \to \mathbb{R}$ is said to satisfy the *Lusin condition*, if f maps set of measure zero to sets of measure zero. The function with this condition is said to be a *Lusin function*. We also call a set of measure zero, a *null set*.

Lemma 3
Let A be an arbitrary subset of $\mathbb{R}$. Suppose: $f: A \to \mathbb{R}$ is absolutely continuous on A. Then f is a Lusin function on A, i.e., f maps set of measure zero to sets of measure zero.

Theorem 4: (Banach-Zarecki)
Suppose X is a closed and bounded subset of $\mathbb{R}$. Suppose $f: X - \mathbb{R}$ is a finite-valued function, continuous of bounded variation on X and is a Lusin function. Then f is absolutely continuous on X.

Theorem 4 usually known as the Banach-Zarecki theorem is a sufficient condition for absolute continuity, therefore the combination of Lemma 2, Lemma 3 and Theorem 4 yields the important early characterization of absolute continuity below:

Theorem 5: (Characterization of Absolute Continuity)
Suppose X is a closed and bounded subset of and $f: X \to \mathbb{R}$ is a finite valued continuous function. Then f is absolutely continuous on X if, and only if, f is of bounded variation on X and is a Lusin function.
Other invaluable features of the theory are the following properties:

Theorem 6:
Suppose I is an interval and $f: I \to \mathbb{R}$ is an absolutely continuous function on I . Suppose $f'(x) = 0$ almost everywhere on I. Then f is a constant function.

Theorem 7:
Suppose $f: [a, b] \to \mathbb{R}$ is an absolutely continuous function on $[a, b]$. Then $\int_a^b f'(x)dx = f(b) - f(a)$.

Theorem 8:
Suppose X is a non-empty subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a finite-valued continuous function and also a Lusin function. Then f maps measurable subsets of X to measurable sets.

Corollary 9:
Suppose X is a non-empty subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is an absolutely continuous function. Then f maps measurable subsets of X to measurable sets.

Theorem 10:
Let X be an arbitrary subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a finite-valued function. Suppose f has finite derivative at every point of a measurable set D in X, then f is a Lusin function on D.

These properties lead to the following classical characterizations of absolute continuity:

Theorem 11:
Suppose I is an interval not necessarily bounded. I may be open, half open, or closed. Suppose $f: I \to \mathbb{R}$ is a continuous function of bounded variation. Then f is absolutely continuous if, and only if, f is a Lusin function.

Theorem 12:
Suppose X is a non-empty closed and bounded subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a continuous function of bounded variation. Then f is absolutely continuous if, and only if, $m\big(f(\{x \in X: Df(x) = \pm\infty\})\big) = 0$.

Theorem 13:
Suppose $f : [a, b] \to \mathbb{R}$ is a continuous Lusin function. Suppose $f'(x)$ exists and is finite almost everywhere on $[a, b]$. Then f is absolutely continuous on $[a, b]$ if, and only if, f' is Lebesgue integrable on $[a, b]$.
The characterizations above lead to the classical consequences below:

Corollary 14:
Suppose $f: [a, b] \to \mathbb{R}$ is a continuous function. Suppose $f'(x)$ exists and is finite except for x in a denumerable set in $[a, b]$. Then f is absolutely continuous on $[a, b]$ if, and only if, f' is Lebesgue integrable on $[a, b]$.

Corollary 15:
Suppose $f: [a, b] \to \mathbb{R}$ is a continuous function. Suppose $f'(x)$exists and is finite for every x in $[a, b]$. Suppose f' is Lebesgue integrable or $f'(x)$ is bounded for all x in $[a, b]$. Then f is absolutely continuous on $[a, b]$ and $f(x) = f(a) + \int_a^x f'(x)dx$ for x in $[a, b]$.

Modern studies of space $AC[a, b]$ of absolutely continuous function have been tremendously generalized in various directions via introduction of nonclassical space $BV[a, b]$ of bounded variations functions by N. Wiener and L. C Young based on guage functions and pth powers ; (Apell, 2014) and references there-in. But their contributions follow from the classical Reisz characterization of absolute continuity in L_p -spaces. The Reisz characterization constitute the framework for nonclassical setting of guage functions employed by Medvedev to obtain elegant nonclassical characterizations of absolutely continuous functions we are investigating currently. In the present study we obtain a classical characterization which extends the famous Riesz characterization. We also enrich the connection between concept of rapidly Cauchy sequences and absolute continuity initiated by (Lax, 2009).

Our method is based on relating the space of absolutely continuous functions with the concept of rapidly Cauchy sequences (Lax, 2009). We recall a sequence $\{u_n\}_{n=1}^{\infty} \subset X$ in a normed linear space $(X, \|.\|)$ is said to be rapidly Cauchy if there exists a convergent series $\sum_{k=1}^{\infty} \epsilon_k$ such that

(4) $\left||u_{n+1} - u_n|\right| \leq \epsilon_n^2$ or $\left||u_{n+1} - u_n|\right| \leq \frac{\eta}{n^4}$ for some $\eta > 0$.

MAIN RESULTS
Theorem 16:
A function $F: [a, b] \to \mathbb{R}$ can be represented in the form of 1:

$$F(x) = c + \int_a^x f(t)d\mu$$

if and only if the following property holds:

(5) $\sup\limits_{1\leq j\leq n} |F(z_j) - F(z_{j-1})| = \mathcal{O}\left(\frac{1}{n}\right)$ for all $z_j \in \mathcal{P}_n$;

where $\mathcal{O}$ is the Bachmann-Landau asymptotic notation often called the big O defined by $\sup\limits_{1\leq j\leq n} |f(z_j) - f(x_{j-1})| \leq \frac{\eta_j}{n}$ for some $\eta_j \geq 0$.

PROOF
The proof is an application of Theorem3 below which asserts that a function $G(x)$ is absolutely

continuous if and only if:
$\sup_{1\le j\le n} |G(z_{j+1}) - G(z_j)| = \mathcal{O}\left(\frac{1}{n}\right)$ for all $z_j \in \mathcal{P}_n$; where $\mathcal{P}_n = \{z_j\}_{j=1}^n = \left\{\frac{(n-j)a+kb}{n}\right\}_{j=1}^n$.

The proof is equivalent to showing that if Riesz property (2) holds then the property (6) in Theorem 17 below also holds as a special case. In this direction we set $G(x) = \int_a^x |f(t)|^p dt$. Clearly, $G(x)$ is absolutely continuous since $|f(t)|^p \in L_1[a,b]$ and by Theorem 3, we have $\sup|G(x_{j+1}) - G(x_j)| = \sup_{1\le j\le n} \int_{z_j}^{x_{j+1}} |f(t)|^p dt \le \frac{\eta}{n}$ for some $\eta > 0$.
To complete the proof, we rewrite (2) in the form below:

$$|F(z_j) - F(z_{j-1})|^p \le \frac{k_j}{|G(z_j)-G(z_{j-1})|} |G(z_k) - G(z_{k-1})|(z_k - z_{k-1})^{p-1}$$

$$\le \frac{k_j}{|G(z_j)-G(z_{j-1})|} |G(z_k) - G(z_{k-1})| \frac{(b-a)^{p-1}}{n^{p-1}}$$

$$\le \frac{k_j}{|G(z_j) - G(z_{j-1})|(b-a)} \frac{\eta(b-a)^p}{n^p}$$

$$\sup_{1\le j\le n} |F(z_j) - F(z_{j-1})| \le \sup_{1\le j\le n} \frac{k_j}{|G(z_j)-G(z_{j-1})|(b-a)} \frac{\eta_1(b-a)}{n} \le \frac{\eta}{n}$$

The conclusion follows from the fact that the constants $k_j's$ are linear functions of $|G(z_j) - G(z_{j-1})|$ as demonstrated in (Natanson, (1964) - p.257). End of proof. □

As mentioned above the proof of Theorem 2 is an application Theorem 3 which we prove next. Theorem 3 is a very important characterization of absolute continuity in $L_1[s,b]$−space which is excluded in Riesz characterization.

Theorem 17:
A function $f: [a,b] \to \mathbb{R}$ is absolutely continuous if and only if for each finite partition $\mathcal{P}_n = \{z_k\}_{k=1}^n$of $[a,b]$ of the form $\mathcal{P}_n = \{z_k\}_{k=1}^n = \left\{\frac{(n-k)a+kb}{n}\right\}_{k=1}^n$ the following property holds:

(6) $\sup_{1\le j\le n} |F(z_{k+1}) - F(z_k)| = \mathcal{O}\left(\frac{1}{n}\right)$ for all $z_k \in \mathcal{P}_n$;

where $\mathcal{O}$ is the Bachmann-Landau asymptotic notation often called the big O defined by $\sup_{1\le k\le n} |f(z_{k+1}) - f(z_k)| \le \frac{\eta_k}{n}$ for some $\eta_k \ge 0$.

PROOF
For any $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a finite partition, of the bounded interval $[a,b]$, of the form $\mathcal{P}_n = \{z_k\}_{k=1}^n = \left\{\frac{(n-k)a+kb}{n}\right\}_{k=1}^n$, then $|z_{k+1} - z_k| = \frac{b-a}{n}$. Assuming $f: [a,b] \to \mathbb{R}$ is absolutely continuous; then for
any $\epsilon > 0$ there is $\delta > 0$ such that $\sum_{k=1}^n |f(z_{k+1}) - f(z_k)| < \epsilon$ whenever $\sum_{k=1}^n |z_{k+1} - z_k| < \delta \le b-a$.

Also, it follows that there exists, for each k, a constant η_k such that $|f(z_{k+1}) - f(z_k)| \le \eta_k|z_{k+1} - z_k| = \frac{\eta_k(b-a)}{n}$, yielding $\sup_{1\le k\le n} |f(z_{k+1}) - f(z_k)| \le \frac{\eta(b-a)}{n}$ where $\eta = \sup_{1\le k\le n} \eta_k$.

Conversely, for each subinterval (x_k, y_k) of $[a,b]$, assuming $\sup_{1\le k\le n} |f(z_{k(j+1)} - f(z_{kj})| = \mathcal{O}\left(\frac{1}{n}\right)$ for all finite partitions $\mathcal{P}_{kn}$ of (x_k, y_k), in the form of $\mathcal{P}_n$ above, then $n \sup_{1\le k\le n} |f(z_k + 1) - f(z_k)| \le$
$\eta_k|z_{k+1} - z_k|$ for some constant $\eta_k > 0$. Let $\{(x_k, y_k)\}_{k=1}^n$ be a family of such subintervals of $[a,b]$; then given any $\epsilon > 0$ we shall compute $\delta > 0$ such that $\sum_{k=1}^n |f(y_k) - f(x_k)| < \epsilon$ whenever $\sum_{k=1}^n |y_k - x_k| < \delta$.

Without loss of generalization we may assume that $|y_k - x_k| \le \beta_k|z_{k(j+1)} - z_{kj}|$ while $|f(y_k) - f(x_k)| \le \theta_k|f(z_{k(j+1)}) - f(z_{kj})|$ for some $\beta_k, \theta_k > 0$. Here, the partition $\mathcal{P}_n = \{z_{kj}\}_{j=1}^n$ for the subinterval (x_k, y_k) should be understood to satisfy $z_{kj} = \frac{(n-j)x_k+jy_k}{n}$

so that $|z_{k(j+1)} - z_{kj}| = \frac{y_k-x_k}{n}$; $j,k = 1,2,\ldots,n$. This yields $|f(y_k) - f(x_k)| \le \sum_{j=1}^n \theta_k|f(z_{k(j+1)}) - f(z_{kj})| \le n\sup\theta_k|f(z_{k(j+1)}) - f(z_{kj})| \le \eta_k$ for some $\eta k > 0$.

Hence $\sum_{k=1}^n |f(y_k) - f(x_k)| \le \sum_{k=1}^n \eta_k| < \epsilon$ whenever $\sum_{k=1}^n |y_k - x_k| < \delta$.

Where $\epsilon > \sum_{j=1}^n \theta_k|f(z_{k(j+1)}) - f(z_{kj})|$ and $\delta > \sum_{j=1}^n \theta_k|z_{k(j+1)} - z_{kj} \le b-a|$. End of proof. □

CONCLUSION
In concluding, it is important to mention that the space of absolutely continuous functions is invaluable in theory and applications of fractional dynamic problems where it is identified with the Sobolev space $W^{1,1}$ which happens to be the very space excluded in Riesz characterization which have been resolved in our main result. In deed it is properties of absolute continuity that makes fractional calculus invaluable in signal processing for image enhancement and restoration. It has also been established that fractional calculus is currently the best techniques for revelation of faint objects from astronomical image processing.

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