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NOVEL CHARACTERIZATION OF ABSOLUTE CONTINUITY

JOHNSON, U, UDO-UTUN, X AND EDET, U
Department of Mathematics,
University of Uyo, Nigeria
xavierudoutun@gmail.com
***Correspondence:** xavierudoutun@uniuyo.edu.ng*

ABSTRACT

We establish newer characterizations for absolutely continuous functions and obtain tremendous interesting enrichment and consequences in its theory and applications. The significance of our studies include independence, of our characterization, on differentiability and integrability properties which turns out a very promising lucrative departure from traditional characterizations.

KEYWORDS: Absolute Continuity, characterizations, differentiability and integrability, rapidly Cauchy sequences

INTRODUCTION / BACKGROUND THEORY

We recall (Niculescu, 2008, Stanislav, 2002, Royden and Fitzpatrick, 2010, Sremr, 2010, Stanislav (2002), and Zhou, 2019) a real valued function $f:[a,b]\to\mathbb{R}$ on a closed bounded interval $[a,b]$ is called absolutely continuous provided for each $\epsilon > 0$ there exists a $\delta > 0$ such that for each finite collection $\{(a_k,b_k)\}_{k=1}^{n}$ of open sub-intervals $(a_k,b_k)\subseteq(a,b)$ we have

(1) $\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\epsilon$ whenever $\sum_{k=1}^{n}|b_k-a_k|<\delta.$

The space of absolutely continuous functions on $[a,b]$ often denoted by $AC[a,b]$, is a well known subspace of continuous functions $C[a,b]$ introduced by Vitali in 1905 (Marly, 1998) in order as a naturally enlargement to the class of uniformly continuous functions which contains the important class of Lipschitz functions as proper subclass. The concept of absolute continuity is both important and natural generalization of the class of monotone functions for enabling formulation of a version of fundamental theorem of calculus which is compatible with Lebesgue theory of integration. This explains why earlier characterizations of absolute continuity by Vitali's (Porter, 1915) and many followers were based on either differentiability requirement $f'\in L'[a,b]$ or integrability in sense of Lebesgue. To date, it still persists that most characterizations of the space $AC[a,b]$ are dependent upon differentiability or integrability properties (Welland and Devito, 1967 and Zhou, 2019).

The aim of this article is to establish newer characterizations of $AC[a,b]$ which together with their consequences are independent of differentiability and integrability properties. Our main result leads natural connections of the space of absolutely continuous functions with the concept of rapidly Cauchy sequences.

We recall a real valued function $f:[a,b]\to\mathbb{R}$ on a closed bounded interval $[a,b]$ is called absolutely continuous provided for each $\epsilon>0$ there exists a $\delta>0$ such that for each finite collection $\{(a_k,b_k)\}_{k=1}^{n}$ of open sub-intervals $(a_k,b_k)\subseteq(a,b)$ we have

(2) $\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\epsilon$ whenever $\sum_{k=1}^{n}|b_k-a_k|<\delta$

The space of absolutely continuous functions on $[a,b]$ denoted by $AC[a,b]$, is a well known subspace of continuous functions $C[a,b]$ introduced by Vitali in 1905 (Marly, 1998) as a natural generalization of the class of continuous monotone functions. The space $AC[a,b]$ of absolutely continuous functions plays invaluable roles both in theoretical, applied, and industrial directions. The most profound usefulness salient features in the theory of absolute continuity are the following properties of absolute continuous functions:

Lemma 1:
Let X be an arbitrary subset of $\mathbb{R}$. Suppose $f:X\to\mathbb{R}$ is absolutely continuous on X. Let E be a bounded subset of X. Then f is of bounded variation on E.

Lusin Condition
A function $f : X\to\mathbb{R}$ is said to satisfy the *Lusin condition*, if f maps set of measure zero to sets of measure zero. The function with this condition is said to be a *Lusin function*. We also call a set of measure zero, a *null set*.

Lemma 2:
Let X be an arbitrary subset of $\mathbb{R}$. Suppose: $f:X\to\mathbb{R}$ is absolutely continuous on X. Then f is a Lusin function on X, i.e., f maps set of measure zero to sets of measure zero.

Theorem 3:
Suppose X is a closed and bounded subset of $\mathbb{R}$. Suppose $f:X-\mathbb{R}$ is a finite-valued function, continuous of bounded variation on X and is a Lusin function. Then f is absolutely continuous on X.

Theorem 3 usually known as the Banach-Zarecki theorem is a sufficient condition for absolute continuity, therefore the combination of Lemma 1, Lemma 2 and Theorem 3 yields the important early characterization of absolute continuity below:

Theorem 4: (Characterization of Absolute Continuity)

Suppose X is a closed and bounded subset of and $f: X \to \mathbb{R}$ is a finite valued continuous function. Then f is absolutely continuous on X if, and only if, f is of bounded variation on X and is a Lusin function.

Other invaluable features of the theory are the following properties:

Theorem 5:
Suppose I is an interval and $f: I \to \mathbb{R}$ is an absolutely continuous function on I. Suppose $f'(x) = 0$ almost everywhere on I. Then f is a constant function.

Theorem 6:
Suppose $f: [a, b] \to \mathbb{R}$ is an absolutely continuous function on $[a, b]$. Then $\int_a^b f'(x)dx = f(b) - f(a)$.

Theorem 7:
Suppose X is a non-empty subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a finite-valued continuous function and also a Lusin function. Then f maps measurable subsets of X to measurable sets.

Corollary 8:
Suppose X is a non-empty subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is an absolutely continuous function. Then f maps measurable subsets of X to measurable sets.

Theorem 9:
Let X be an arbitrary subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a finite-valued function. Suppose f has finite derivative at every point of a measurable set D in X, then f is a Lusin function on D.
These properties lead to the following classical characterizations of absolute continuity:

Theorem 10:
Suppose I is an interval not necessarily bounded. I may be open, half open, or closed. Suppose $f: I \to \mathbb{R}$ is a continuous function of bounded variation. Then f is absolutely continuous if, and only if, f is a Lusin function.

Theorem 11:
Suppose X is a non-empty closed and bounded subset of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a continuous function of bounded variation. Then f is absolutely continuous if, and only if, $m\big(f(\{x \in X: Df(x) = \pm\infty\})\big) = 0$.

Theorem 12:
Suppose $f: [a, b] \to \mathbb{R}$ is a continuous Lusin function. Suppose $f'(x)$ exists and is finite almost everywhere on $[a, b]$. Then f is absolutely continuous on $[a, b]$ if, and only if, f' is Lebesgue integrable on $[a, b]$.

The characterizations above lead to the classical consequences below:

Corollary 13:
Suppose $f: [a, b] \to \mathbb{R}$ is a continuous function. Suppose $f'(x)$ exists and is finite except for x in a denumerable set in $[a, b]$. Then f is absolutely continuous on $[a, b]$ if, and only if, f' is Lebesgue integrable on $[a, b]$.

Corollary 14:
Suppose $f: [a, b] \to \mathbb{R}$ is a continuous function. Suppose $f'(x)$exists and is finite for every x in $[a, b]$. Suppose f' is Lebesgue integrable or $f'(x)$ is bounded for all x in $[a, b]$.

Then f is absolutely continuous on $[a, b]$ and $f(x) = f(a) + \int_a^x f'(x)dx$ for x in $[a, b]$.

A sequence $\{u_n\}_{n=1}^{\infty} \subset X$ in a normed linear space $(X, \|.\|)$ is said to be rapidly Cauchy if there exists a convergent series $\sum_{k=1}^{\infty} \epsilon_k$ such that

(3) $||u_{n+1} - u_n|| \le \epsilon_n^2$ or $||u_{n+1} - u_n|| \le \frac{\eta}{n^4}$ for some $\eta > 0$.

It is important to remark that the property $||u_{n+1} - u_n|| \le \frac{\eta}{n^4}$ (the second alternate defining property of absolutely continuous functions) in equation number (3) can be found in works of (Lax, 2009)

MAIN RESULTS

Theorem 15:
A function $f: [a, b] \to \mathbb{R}$ is absolutely continuous if and only if for each finite partition $\mathcal{P}_n = \{z_k\}_{k=1}^n$ of $[a, b]$ of the form $\mathcal{P}_n = \{z_k\}_{k=1}^n = \left\{\frac{(n-k)a+kb}{n}\right\}_{k=1}^n$ the following property holds:

(4) $\sup_{1\le k\le n} |f(z_{k+1}) - f(z_k)| = \mathcal{O}\left(\frac{1}{n}\right)$ for all $z_k \in \mathcal{P}$;

where $\mathcal{O}$ is the Bachmann-Landau asymptotic notation often called the big $\mathcal{O}$ defined by $\sup_{1\le k\le n} |f(z_{k+1}) - f(z_k)| \le \frac{n_k}{n}$ for some $\eta_k \ge 0$..

PROOF
For any $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a finite partition, of the bounded interval $[a, b]$, of the form $\mathcal{P}_n = \{z_k\}_{k=1}^n = \left\{\frac{(n-k)a+kb}{n}\right\}_{k=1}^n$ then $|z_{k+1} - z_k| = \frac{b-a}{n}$. Assuming $f: [a, b] \to \mathbb{R}$ is absolutely continuous; then for any $\epsilon > 0$, there is $\delta > 0$ such that $\sum_{k=1}^n |f(z_{k+1}) - f(z_k)| < \epsilon$ whenever $\sum_{k=1}^n |z_{k+1} - z_k| < \delta \le b - a$. Also, it follows that there exists, for each k, a constant η_k such that $|f(z_{k+1}) - f(z_k)| \le \eta_k |z_{k+1} - z_k| = \frac{\eta_k(b-a)}{n}$, yielding $\sup_{1\le k\le n} |f(z_{k+1}) - f(z_k)| \le \frac{\eta_k(b-a)}{n}$ where $\eta = \sup_{1\le k\le n} \eta k$

Conversely, for each subinterval (x_k, y_k) of $[a, b]$, assuming $\sup_{1\le k\le n} |f(z_{k+1}) - f(z_k)| = \mathcal{O}\left(\frac{1}{n}\right)$ for all finite partitions $\mathcal{P}_{kn}$ of (x_k, y_k), in the form of $\mathcal{P}_n$ above, then $nsup_{1\le k\le n} |f(z_k + 1) - f(z_k)| \le \eta_k |z_{k+1} - z_k|$ for some

constant $\eta_k > 0$. Let $\{(x_k, y_k)\}_{k=1}^{n}$ be a family of such subintervals of $[a,b]$; then given any $\epsilon > 0$ we shall compute $\delta > 0$ such that $\sum_{k=1}^{n}|f(y_k) - f(x_k)| < \epsilon$ whenever $\sum_{k=1}^{n}|y_k - x_k| < \delta$
δ.

Without loss of generalization we may assume that $|y_k - x_k| \le \beta_k|z_{k(j+1)} - z_{kj}|$ while $|f(y_k) - f(x_k)| \le \theta_k|f(z_{k(j+1)}) - f(z_{kj})|$ for some $\beta_k, \theta_k > 0$. Here, the partition $\mathcal{P}_{kn} = \{z_{kj}\}_{j=1}^{n}$ for the subinterval (x_k, y_k) should be understood to satisfy$z_{kj} = \frac{(n-j)x_k + jy_k}{n}$ so that $|z_{k(j+1)} - z_{kj}| = \frac{y_k - x_k}{n}$; $j, k = 1,2,\ldots,n$.

This yields $|f(y_k) - f(x_k)| \le \sum_{k=1}^{n}\theta_k|f(z_{k(j+1)}) - f(z_{kj})| \le n\sup\theta_k|f(z_{k(j+1)}) - f(z_{kj})| \le n\sup\theta_k|f(z_{k(j+1)}) - f(z_{kj})| \le \eta_k$ for some $\eta_k > 0$ Hence $\sum_{k=1}^{n}|f(y_k) - f(x_k)| \le \sum_{k=1}^{n}\eta_k < \epsilon$ whenever $\sum_{k=1}^{n}|y_k - x_k| < \delta$

where $\epsilon > \sum_{j=1}^{n}\theta_k|f(z_{k(j+1)}) - f(z_{kj})|$ and $\delta > \sum_{j=1}^{n}\theta_k|z_{k(j+1)} - z_{kj}\beta_k| \le b - a$ End of proof. ■

Corollary 16:
$f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is rapidly Cauchy whenever $\{x_n\}_{n=1}^{\infty}$ is rapidly Cauchy.

PROOF
Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous then for any finite partition $\mathcal{P}_n = \{z_j\}_{j-1}^{n}$ of $[a,b]$ we have that, by Theorem 15, there exists $\eta_j > 0$ such that $\frac{|f(z_j+1) - f(z_j)|}{|z_j+1 - z_j|} \le \eta_j$. Let $\{x_n\}_{k=1}^{\infty}$ be a rapidly Cauchy sequence in $[a,b]$ then its *nth* term x_n satisfies $|x_{n+1} - x_n| \le \frac{1}{n^4}$ and by Archimedean property we also have that there exist $\beta_j, \theta_j > 0$ such that $|z_{j+1} - z_j| \le \beta_j|x_{n+1} - x_n|$ and $|f(x_{n+1}) - f(x_n)| \le \theta_j|f(z_{j+1}) - f(z_j)|$.

From above we obtain $|f(x_{n+1}) - f(x_n)| \le \theta_j|f(z_{j+1}) - f(z_j)| \le \theta_j\eta_j|z_{j+1} - z_j| \le \theta_j\eta_j\beta_j|x_{n+1} - x_n| \le \frac{\theta_j\eta_j\beta_j}{n^4}$. This proves that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is rapidly Cauchy.

On the other hand, suppose the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is rapidly Cauchy for each rapidly Cauchy sequence $\{x_n\}_{n=1}^{\infty}$. For any $\mathrm{N} \in \mathbb{N}$ let $\mathcal{P}_{\mathrm{N}} = \{u_j\}_{j=1}^{\mathrm{N}}$ of $[a,b]$, defined above, we shall show that $\sup_{1 \le k \le \mathrm{N}} |f(u_{j+1}) - f(u_j)| = \mathcal{O}\left(\frac{1}{\mathrm{N}}\right)$.

For the rapidly Cauchy sequence $\{x_n\}_{n=1}^{\infty}$, there exists $n(k)$ and a partition $\{z_k\}_{k=1}^{N} \subset (x_{n(k)}, x_{n(k)+1})$ of $(x_{n(k)}, x_{n(k)+1})$ such that $|f(z_{k+1}) - f(z_k)|\eta_k|f(x_{n(k)+1}) - f(x_{n(k)})|$. Also, since the partitions $\{u_k\}_{k=1}^{N}$ and $\{z_k\}_{k=1}^{N}$ are finite, there exists θ_k such that $|f(u_{k+1}) - f(u_k)| = \theta_k|f(z_{k+1}) - f(z_k)|$. These yield $|f(u_{k+1}) - f(u_k)| = \theta_k|f(z_{k+1}) - f(z_k)| = \theta_k\eta_k|f(x_{n(k)+1}) - f(x_{n(k)})| = \frac{N\theta_k\eta_k}{n(k)^4}\frac{1}{N} \le \frac{\eta}{N}$ where $\eta \le \sup_{1 \le k \le N}\frac{N\theta_k\eta_k}{n(k)^4}$. End of proof. ■

APPLICATIONS.
Theorem 17:
An absolutely continuous function $f: [a,b] \to \mathbb{R}$ is Lipschitz along any rapidly Cauchy sequence, that is; if $\{x_n\}_{n=1}^{\infty} \subset [a,b]$ is a rapidly Cauchy sequence in $[a,b]$ then there exists a constant $L > 0$ such that $|f(x_m) - f(x_n)| \le L|x_m - x_n|$ for all $m, n \in \mathbb{N}$.

PROOF
Given an absolutely continuous function $f: [a,b] \to \mathbb{R}$, let $\{x_n\}_{n=1}^{\infty}$ be a rapidly Cauchy sequence in $[a,b]$ and $\mathcal{P}_N = \{z_k\}_{k=1}^{N} = \left\{\frac{(n-j)c + kd}{n}\right\}_{j=1}^{n}$ a finite partition of $[c,d]$ of the form above where $c = inf\{x_n, x_m\}$ and $d = max\{x_n, z_m\}$. Mimicking the procedure above: by Theorem 15 we have $|f(z_{j+1}) - f(z_j)| \le \eta_j|z_{j+1} - z_j|$ and by Archimedean property, there exist $\beta_j, \theta_j > 0$ such that $|z_{j+1} - z_j \le \beta_j|x_n - x_m|$ and $|f(x_{n+1}) - f(x_n)| \le \theta_j|f(z_{j+1}) - f(z_j)|$ for some j such that $1 \le j \le N$. Hence $|f(x_n) - f(x_m)| \le \theta_j|f(z_{j+1}) - f(z_j)| \le \theta_j\eta_j|z_{j+1} - z_j| \le \theta_j\eta_j\beta_j|x_n - x_m$. End of proof. ■

Example:
If g is absolutely continuous and $g(u(x_n))$ is rapidly Cauchy sequence whenever $\{x_n\}_{n=1}^{\infty}$ is rapidly Cauchy, then $u(x_n)$ is rapidly Cauchy and $u(x)$ is absolutely continuous.

Concluding Remark
The famous Riesz characterization of absolute continuity below makes use of one of our hypotheses and is therefore closely related to our main result:

Theorem 19: (Riesz)
A function $F: [a,b] \to \mathbb{R}$ can be represented in the form of 1:

$$F(x) = c + \int_a^x f(t)d\mu$$

if and only if for every partition $\mathcal{P}_n$ of $[a,b]$ by points $a = x_0 < x_1 < \ldots x_n < b$ the following inequality holds:

$$\sum_{j=1}^{n}\frac{|F(x_{k+1}) - F(x_k)|^p}{|x_{k+1} - x_k|^{p-1}} \le k$$

where k is a constant independent of the choice partition points in $\mathcal{P}_n$.

But is limited to $L_p[a,b]$ spaces with $p > 1$ thereby excluding the rich theory of the invaluable Sobolev space $W^{1,1}$ which is identified with the space $AC[a,b]$ of absolutely continuous functions; i.e. when $p > 1$. In a recent submission, we have demonstrated that the Riesz characterization of absolute continuity implies our main result not only for $p > 1$, but also for $p \geq 1$. On passing, it is also of importance to remark that the Riesz L_p characterization of absolute continuity was later studied by Medvedev (Appel et al, 2014) under the subject area of nonclassical bounded variations spaces and generalized using the concepts of functions of bounded Riesz variation and Riesz–Medvedev variations (Appel *et al*, 2014) which we have generalized in a recent submission. Such nonclassical generalizations prove to be very versatile techniques in signal processing.

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