https://dx.doi.org/10.4314/WOJAST.v14i1b.44

ENTROPIC UNCERTAINTY RELATIONS AND THE FISHER INFORMATION FOR THE GENERALIZED RADIAL YUKAWA POTENTIAL

Supported by tetfund TERTIARY EDUCATION TRUST FUND

ISSN: 2141 – 3290
www.wojast.org

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ABSTRACT

The Heisenberg uncertainty relation as well as the Fisher information are presented analytically and numerically for the Generalized radial Yukawa potential. The probability density for the ground and first excited state has been analyzed via the Fisher information for this potential model. Some numerical results are obtained. From the numerical results obtained, we observed that, for $n = 0, 1$, the position-space Fisher information I_r increases with increasing potential parameter *a,* while the momentum-space Fisher information I_ρ initially increases, and later decreases with increasing potential parameter *a*. The Fisher-information-based uncertainty relation and the Heisenberg uncertainty relation have been verified to hold for this atomic model. In addition, we observed a squeezed phenomenon in some of the results in position r and momentum ρ for the ground and first excited states.

KEYWORDS: Schrődinger Equation; Nikiforov-Uvarov method; Manning Rosen plus exponential Yukawa Potential.

INTRODUCTION

As stated by the density functional theory (DFT), the physical and chemical properties of fermionic systems (atoms, molecules, nuclei, solids) can be completely described by means of the single-particle probability densities in the position and momentum spaces (Hohenberg, *et al.*, 1964). The spread of the probability densities which characterizes their allowed quantum states are quantified by the information-theoretic measures in a more appropriate manner than the celebrated variance or other measures of dispersion (Sen, 1995 and Dehesa, *et al.*, 2005). This is due to the fact that, these information-theoretic tools do not make any reference to some specific point of the corresponding Hilbert space. These measures play important roles in the uncertainty and other quantum parameters.

Fisher information is one of the main information-theoretic measures that was first introduced by Fisher in 1925, as a measure of intrinsic accuracy in statistical estimation theory (Fisher, 1925 and Sears, *et al.*, 1980). Since then, it has found many applications in different areas of sciences, communication and quantum computation (Hohenberg, 1964 and Dehesa, *et al.,* 2005). Several laws and fundamental equations in physics have been obtained through the principle of minimum/maximum Fisher information, examples include: the equation of non-relativistic quantum mechanics, the time-dependent Kohn-Sham equations and the time-dependent Euler equation of density functional theory, amongst others (Frieden, 1998; Reginatto, 1998 and Nalewajski, 2003). Fisher information of the electronic distribution functions is closely related to the vonWeizsäcker kinetic energy functional of atomic and molecular systems and the kinetic energy (Parr, *et al.* 1989; Romera, *et al.*, 1994 and Sears, 1980). As reported by Nalewajski, Fisher information have been used in numerous ways in quantum chemistry and more generally, molecular electronic structure theory (Nalewajski, 2008). In addition, it is used as descriptors of chemical reactivity of molecules (Montgomery, *et al.,* 2008; Esquivel, *et al.,* 2010 and Grassi, 2011).

However, the study of information theory of quantum-mechanical systems have been extensively used in recent years to study a variety of quantum mechanical systems (Majernik, *et al*., 1996; Yanez, *et al.*, 1994; Dehesa, *et al.,* 1997; Dehesa, *et al.*, 2006; Yahya, e*t al.,* 2014a; Yahya, *et al.*, 2014b; Yahya, *et al.,* 2013; Osobonye, *et al.,* 2020; Patil, *et al.,* 2007; Isonguyo, *et al.*, 2018; Okon, *et al.*, 2018 and Antia, *et al.,* 2018). This is because, it provides a deeper knowledge into the internal structure of the quantum systems and it is closely related with modern quantum computation and information, which is a basic theory for numerous technological developments (Gadre, *et al.*, 1991 and Nielson, 2001). The Fisher information in the position-space I_r is defined by (Fisher, 1925, Isonguyo, *et al.*, 2018; Okon, *et al.*, 2018),

$$I_r = \int \frac{[\rho\prime(r)]^2}{\rho(r)} dr = 4\int [\psi'(r)]^2 dr = \langle \rho^2 \rangle, \qquad (1)$$

the corresponding quantity for the momentum space Fisher information I_ρ is defined as,

$$I_p = \int \frac{[\rho\prime(p)]^2}{\rho(p)} dr = 4\int [\phi'(p)]^2 dr = \langle r^2 \rangle. \qquad (2)$$

The Fisher information product is expresses as $I_{rp} = I_r I_p$, where $\psi(r)$ is a normalized eigenfunction in the spatial coordinate and $\phi(p)$ is its normalized eigenfunction in the momentum coordinate which is obtained by the Fourier transform of $\psi(r)$. Fisher information is the gradient functional of probability density, as such, it is a local measure of the extent and concentration of the probability density of the system in the spatial localization of the electron cloud. The higher this quantity, the more concentrated the single particle density, the smaller the uncertainty and the higher the accuracy in predicting the localization of the particles (Yahya, *et al.*, 2014a and Falaye, *et al.,* 2014).

In this paper, the aim is to investigate the Fisher information for the generalized radial Yukawa Potential (GYP) and its uncertainty principle. The Yukawa potential is very useful in describing the nuclear interaction between protons and neutrons due to the creation and exchange of pion (Yukawa, 1935; Oluwadare, *et al*., 2016 and Adamowski, 1985). This model has found extensive applications in the various branches of Physics such as atomic and nuclear physics, solid state physics, Plasma physics, and alot more (Greiner, 2000 and Preston, *et al.*, 1975). GYP is a combination of a long range Yukawa interaction and a short range repulsive inversely quadratic Yukawa interaction (Oluwadare, *et al.,* 2016 and Isonguyo*, et al.,* 2018).

Furthermore, we shall study the Uncertainty Relation (Heisenberg uncertainty relation) which is an important aspect of quantum mechanics. This Principle was introduced by Heisenberg in 1927. The Heisenberg uncertainty principle for the product of the uncertainties in the position and momentum spaces, expressed in terms of Planck's constant is given as (Heisenberg, 1927)

$$\Delta r \Delta p \geq \frac{\hbar}{2}, \quad (3)$$

where,

$$\Delta r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} \text{ and } \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \ . \quad (4)$$

Δr and Δp represent the position and momentum uncertainties, respectively. This relation implies that it is physically impossible to measure exactly both position and momentum simultaneously (Heisenberg, 1927 and Chen, *et al.*, 2013). In addition, we shall study the squeezing behaviour of the ground and first excited states for some of the potential parameter a.

This work is structured as follows: In the next Section, we obtained the eigensolution for the Generalized Yukawa potential, the normalized wave function and the probability density of the system. Furthermore, the Fisher information and the Heisenberg uncertainty relation is presented for GYP, some numerical results are also given. Finally, the discussion and conclusion follows.

GENERALIZED YUKAWA POTENTIAL AND ITS ANALYTICAL SOLUTION

In this section, we shall present the analytical solution of this system, which is necessary for obtaining the wave function and probability density needed for the analysis. The radial Schrodinger equation is given as (Greiner, 2000; Galindo, *et al.,* 1978 and Ita, *et al.,* 2015)

$$\frac{d^2\psi_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left\{\left(E_{nl} - V(r)\right) - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right\}\psi_{nl}(r) = 0, \quad (5)$$

with the Generalized radial Yukawa potential (GYP) defined as (Oluwadare, *et al.,* 2016)

$$V(r) = -\left(\frac{V_0 e^{-ar}}{r}\right) - \frac{V_1 e^{-2ar}}{r^2}, \quad (6)$$

where E is the exact bound state energy eigenvalue, $\psi(r)$ is the eigenfunction, μ represents the reduced mass,($\hbar$=μ=1). n denotes the principal quantum number (n and l are known as the vibration-rotation quantum numbers), r is the inter-nuclear separation. V_0 and V_1 are constants which determine the potential strength, a is the screening parameter which characterize the range of the interaction. The plot of this potential model as a function of r is shown below.

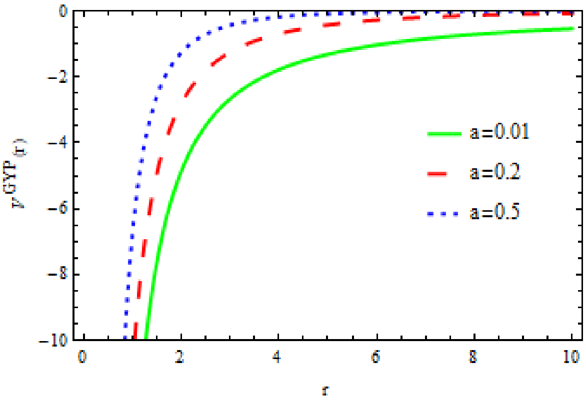

Figure 1: Shape of the Generalized Yukawa Potential (GYP) as a function of r, for some values of a, with V_0 = 5.0, V_1 = 10.0

On substituting equation (6) into equation (5) and then solve, the wave function and energy eigenvalue equation for the generalized Yukawa potential are obtained respectively as:

$$\psi_{nl}^{GYP}(r) = N_{nl}(e^{-ar})^{\varepsilon}(1-e^{-2ar})^{\eta}P_n^{(2\varepsilon,\ 2\eta-1)}(1-2e^{-2ar})$$
$$= N_{nl}s^{\varepsilon}(1-s)^{\eta}P_n^{(2\varepsilon,\ 2\eta-1)}(1-2s)\,,\quad s = e^{-2ar} \quad (7)$$

and

$$E = -\frac{2\hbar^2 a^2}{\mu}\left[\frac{n^2+n+\frac{1}{2}-\frac{\mu V_0}{\hbar^2 a}+l(l+1)+(2n+1)\delta}{(2n+1)+2\delta}\right]^2, \quad (8)$$

where

$$\varepsilon = \sqrt{-\frac{\mu E}{2a^2\hbar^2}}\ ,\ \delta = \sqrt{\frac{1}{4}+l(l+1)-\frac{2\mu V_1}{\hbar^2}}\ ,\ A = \sqrt{\frac{1}{4}+2V_1+l(l+1)}\ . \quad (9)$$

The normalization constant can be calculated using $\int_{-\infty}^{\infty}\left|\psi_{nl}^{GYP}(r)\right|^2 dr = 1$ (Greiner, 2000 and Galindo, *et al.,* 1978). For n = 0, 1 we have

$$N_0^{GYP} = 2\sqrt{\frac{2a\Gamma(2\varepsilon+2\eta+1)}{\Gamma(2\eta+1)\Gamma(2\varepsilon)}}\ , \quad (10)$$

$$N_1^{GYP} = \sqrt{\frac{a\varepsilon\Gamma(2\varepsilon+2\eta+3)}{\eta(\eta+1)\Gamma(2\eta)(2\eta+2\varepsilon+1)\Gamma(2\varepsilon+2)}}\ . \quad (11)$$

Hence, the normalized wave function in position space for two low lying states $n = 0, 1$, are then given by:

$$\psi_0^{GYP}(r) = 2\sqrt{\frac{2a\Gamma(2\varepsilon+2\eta+1)}{\Gamma(2\eta+1)\Gamma(2\varepsilon)}}\,(e^{-ar})^{\varepsilon}(1-e^{-2ar})^{\eta} \qquad (12)$$

and

$$\psi_1^{GYP}(r) = \frac{N_1^{GYP}}{2}(e^{-ar})^{\varepsilon}(1-e^{-2ar})^{\eta}\left((1-2e^{-2ar})(2\varepsilon+2\eta+1)-2\eta+2\varepsilon+1\right), \qquad (13)$$

while the corresponding normalized wave function in momentum-space is obtained by finding the Fourier transform of $\psi_{nl}^{GYP}(r)$ as (Greiner, 2000):

$$\psi_0^{GYP}(p) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ipr}\,\psi_0^{GYP}(r)dp = \frac{N_0^{GYP}}{\sqrt{2\pi}}\frac{\Gamma(\eta+1)\Gamma\left(\frac{ip}{2a}+\varepsilon\right)}{2a\Gamma\left(\frac{ip}{2a}+\varepsilon+\eta+1\right)}, \qquad (14)$$

$$\psi_1^{GYP}(p) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ipr}\,\psi_1^{GYP}(r)dp = \frac{N_1^{GYP}}{\sqrt{2\pi}}\frac{\Gamma(\eta+1)(a(n+2\varepsilon+1)-i\eta p)\Gamma\left(\frac{ip}{2a}+\varepsilon\right)}{2a^2\Gamma\left(\frac{ip}{2a}+\varepsilon+\eta+2\right)}, \qquad (15)$$

The probability density for GYP is obtained by squaring equation (7) which gives

$$\rho(r) = \left|\psi_{nl}^{GYP}(r)\right|^2 = N_{nl}^2 s^{2\varepsilon}(1-s)^{2\eta}\left[P_n^{(2\varepsilon,\ 2\eta-1)}(1-2s)\right]^2 . \qquad (16)$$

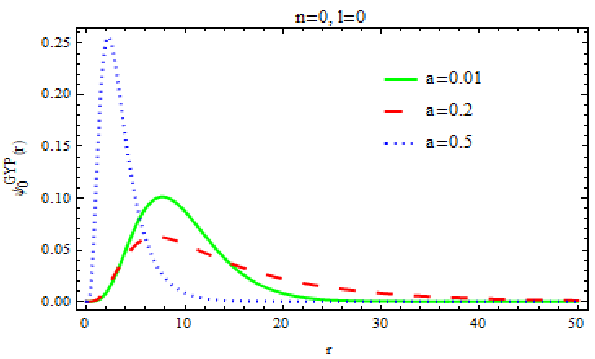

Figure 2: Wave function plot in the ground state for GYP

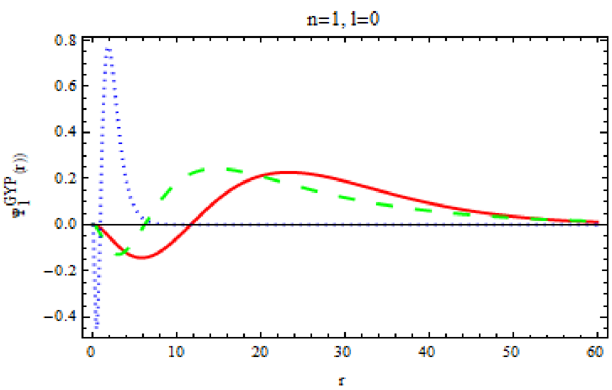

Figure 3: Wave function plot in the first excited state for GYP

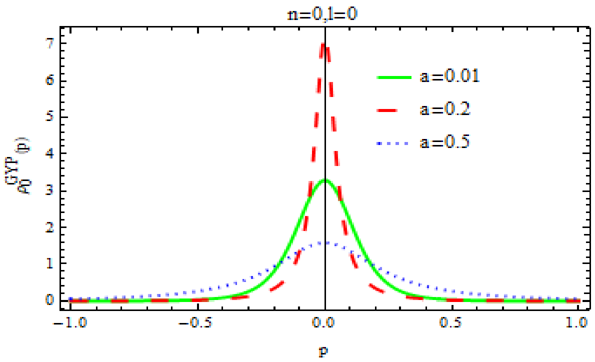

Figure 4: Probability density plot in the ground state for GYP

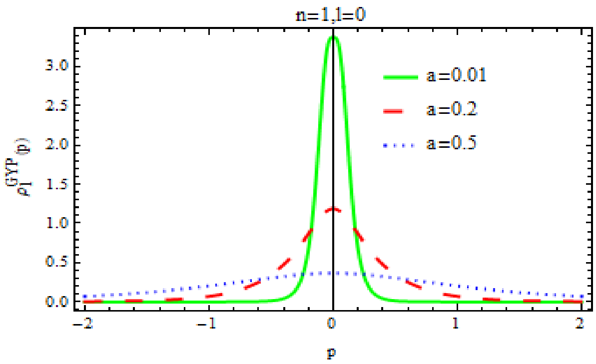

Figure 5: Probability density plot in the first excited state for GYP

UNCERTAINTY PRINCIPLE AND FISHER INFORMATION

The expectation values for the GYP are evaluated using the following equations (Greiner, 2000):

$$\langle r\rangle_n = \int_0^{\infty}\psi_{nl}^{GYP}(r)\,r\,\psi_{nl}^{GYP}(r)dr\ ,\langle r^2\rangle_n = \int_0^{\infty}\psi_{nl}^{GYP}(r)\,r^2\,\psi_{nl}^{GYP}(r)dr \qquad (17)$$

$$\langle p^2\rangle_n = \int_0^{\infty}\psi_{nl}^{GYP}(r)\,p^2\,\psi_{nl}^{GYP}(r)dr\ ,\langle p\rangle_n = 0. \qquad (18)$$

On substituting equations (14), (15)into (17) and (18), then simplify the integrals above for the ground and first excited state,
($n = 0, 1$) yields

$$\langle r\rangle_0 = \frac{\Gamma(2\eta+1)\Gamma(2\varepsilon)(H_{2(\varepsilon+\eta)}-H_{2\varepsilon-1})}{4a^2\Gamma(2\varepsilon+2\eta+1)}, \qquad (19)$$

$$\langle r\rangle_1 = \frac{1}{8a^2(\varepsilon+\eta+1)^2\Gamma(2\varepsilon+2\eta+2)}\times\begin{pmatrix}\Gamma(2\eta+1)\Gamma(2\varepsilon)(2(\eta+1)(2\varepsilon+1)(\eta+\varepsilon+1)(2\varepsilon+2\eta+1)H_{2(\varepsilon+\eta)}\\ +2(\eta+1)((\eta+1)((2\eta+3)+8\varepsilon^2+10(\eta+1)\varepsilon)\\ -2(\eta+1)(2\varepsilon+1)(\eta+\varepsilon+1)(2\eta+2\varepsilon+1)(\psi^{(0)}(2\varepsilon)+\gamma))\end{pmatrix}, \qquad (20)$$

$$\langle r^2\rangle_0 = \frac{\Gamma(2\eta+1)\Gamma(2\varepsilon)((-H_{2(\varepsilon+\eta)}+\psi^{(0)}(2\varepsilon)+\gamma)^2-\psi^{(1)}(2\varepsilon+2\eta+1)+\psi^{(1)}(2\varepsilon))}{8a^3\Gamma(2\varepsilon+2\eta+1)} \qquad (21)$$

where H denote the Harmonic number, $\psi^{(0)}$ is digamma function, $\psi^{(1)}$ denote polygamma function and γ is the Euler-Mascheroni constant (Adamowski, 1965).

The Fisher information for the generalized radial Yukawa potential is evaluated by using the probability density obtained for this system in equation (16) and then, analyze via the Fisher information formula in equations (1) and (2). The Fisher information for GYP in the spatial coordinate is expressed as:

$$I_n^{GYP} = 4\int_0^{\infty}[\psi'(r)]^2 r^2 dr = 16a\int_{-1}^{1}\left(\frac{2}{1-\nu}\right)[\psi'(\nu)]^2\ d\nu\,,\quad \nu = 1-2s,\ s = e^{-2ar}. \tag{22}$$

Therefore, substituting equation (7) into equation (22) gives the following expression

$$I_n^{GYP} = 32a^2 N_{nl}^2 \int_{-1}^{1}\left\{\begin{array}{c} -\varepsilon\left(\frac{1+\nu}{2}\right)^{\eta}\left(\frac{1-\nu}{2}\right)^{\varepsilon}P_n^{(2\varepsilon,2\eta-1)}(\nu) \\ +\left(\frac{1+\nu}{2}\right)^{\eta}\left(\frac{1-\nu}{2}\right)^{\varepsilon+1}(2\varepsilon+\eta+2\eta)\,P_n^{(2\varepsilon,2\eta-1)}(\nu) \\ +\eta\left(\frac{1+\nu}{2}\right)^{\eta-1}\left(\frac{1-\nu}{2}\right)^{\varepsilon+1}P_n^{(2\varepsilon,2\eta-1)}(\nu)\end{array}\right\}\left(\frac{2}{1-\nu}\right)d\nu\ . \tag{23}$$

Due to the complicated form of the integral in equation (23) for n-state, we shall be limited to studying few low-lying state, that is, the ground and first excited state. The Fisher information in spatial coordinate for GYP is obtained as follows (Yanez, *et al.*, 2008; Sánchez-Moreno, *et al.*, 2011 and Guerrero, *et al*., 2010):

For the ground state, equation (23) becomes

$$I_0^{GYP}(r) = \frac{4a\eta\Gamma(2\eta-1)\Gamma(2\varepsilon+1)}{\Gamma(2\varepsilon+2\eta)}(N_0^{GYP})^2\,, \tag{24}$$

also, for the first excited state, we have

$$I_1^{GYP}(r) = \frac{8a\eta\Gamma(3\eta-1)\Gamma(2\varepsilon+2)}{\Gamma(2\varepsilon+2\eta+1)}(N_1^{GYP})^2. \tag{25}$$

It is very complicated to obtain the Fisher information in momentum coordinate $I_1^{GYP}(p)$ analytically. Hence, we will compute it numerically by finding the Fourier transform of the position coordinate wave function in equation (7) , then, substitute into equation (22) and simplify.

NUMERICAL RESULTS

Table 1: Numerical Results For Fisher Information in the ground eigenstates for various values of a, with l=0, $V_0 = 0.5$, $V_1 = 1.0$, min = minimum value for the Generalized Yukawa Potential (GYP)

a	I_r	I_p	$I_rI_p \geq 36.0$	$\min(I_rI_p)$
0.01	0.08960	446.652	40.02002	36.0
0.02	0.09507	421.566	40.07689	36.0
0.03	0.09973	402.772	40.16978	36.0
0.04	0.10360	389.012	40.30164	36.0
0.06	0.10667	379.477	40.47767	36.0
0.07	0.10893	373.674	40.70543	36.0
0.08	0.11040	371.337	40.99560	36.0
0.09	0.11107	372.398	41.36113	36.0
0.10	0.11093	376.967	41.81808	36.0
0.20	0.11000	385.353	42.38883	36.0
0.30	0.05667	1262.05	71.51621	36.0
0.40	0.08333	1101.25	91.77080	36.0
0.50	0.35000	167.448	58.60680	36.0
0.60	0.75000	70.0530	52.53975	36.0
0.70	1.28333	39.0202	50.07579	36.0
0.80	1.95000	25.0028	48.75546	36.0
0.90	3.68333	12.8635	47.38052	36.0

Table 2: Numerical Results For Uncertainty Relation in the ground eigenstates for various values of a, with l=0, $V_0 = 0.5$, $V_1 = 1.0$, min = minimum value for the Generalized Yukawa Potential (GYP)

a	$\langle r^2\rangle$	$\langle r\rangle$	$\Delta(r)$	$(\Delta r)^2$	$\langle p^2\rangle$	$\Delta(p)$	$\Delta(r)\Delta(p) \geq \hbar/2$	$(\Delta r)^2(\Delta p)^2$	$\min\{(\Delta r)^2(\Delta p)^2\}$
0.01	111.663	9.64398	4.319334	18.6566	0.02240	0.149666	0.646459	0.41790	0.250000
0.02	105.391	9.36281	4.210616	17.7294	0.02377	0.154164	0.649127	0.42141	0.250000
0.03	100.693	9.14193	4.137404	17.1182	0.02493	0.157903	0.653308	0.43418	0.250000
0.04	97.2530	8.97159	4.094334	16.7634	0.02590	0.160935	0.658921	0.43403	0.250000
0.06	93.4185	8.75862	4.087184	16.7049	0.02723	0.165024	0.674488	0.44488	0.250000
0.07	92.8343	8.70929	4.120985	16.9826	0.02760	0.166132	0.684629	0.46872	0.250000
0.08	93.0995	8.69625	4.180279	17.4746	0.02777	0.166634	0.696575	0.48527	0.250000
0.09	94.2418	8.71992	4.266702	18.2049	0.02773	0.166533	0.710547	0.50482	0.250000
0.10	96.3383	8.78211	4.383240	19.2129	0.02750	0.165831	0.726878	0.52835	0.250000
0.20	315.513	14.4686	10.30399	106.172	0.01417	0.119024	1.226419	1.50447	0.250000
0.30	275.312	13.1241	10.15236	103.070	0.02083	0.144338	1.465367	2.14695	0.250000
0.40	41.8620	5.42482	3.526092	12.4334	0.08750	0.295804	1.043032	1.08792	0.250000
0.50	17.5133	3.57460	2.176117	4.73546	0.18750	0.433013	0.942286	0.97892	0.250000
0.60	9.75505	2.69184	1.583997	2.50902	0.32083	0.566421	0.897209	0.80498	0.250000
0.70	6.25070	2.16609	1.248501	1.55877	0.48750	0.698212	0.871718	1.94611	0.250000
0.80	4.35788	1.81484	1.031600	1.06421	0.68750	0.829157	0.855370	1.09787	0.250000
0.90	3.21588	1.56279	0.879524	0.77354	0.92083	0.959600	0.843991	0.71232	0.250000

https://dx.doi.org/10.4314/WOJAST.v14i1b.44

Table 3: Numerical Results For Fisher Information in the first excited eigenstates for various values of a, with l=0, $V_0 = 0.5$, $V_1 = 1.0$, min = minimum value for the Generalized Yukawa Potential (GYP)

a	I_r	I_p	$I_r I_p \geq 36.0$	$\min(I_r I_p)$
0.01	0.064765	2736.54	177.2331	36.0
0.02	0.064000	2745.94	175.7402	36.0
0.03	0.059432	2919.82	173.5310	36.0
0.04	0.051062	3352.11	171.1644	36.0
0.05	0.038889	4393.56	170.8608	36.0
0.06	0.022914	8105.23	185.7200	36.0
0.07	0.003136	2093930	656.6146	36.0
0.08	0.022222	9944.06	220.9789	36.0
0.09	0.056173	3255.44	182.8672	36.0
0.10	0.098765	1770.79	174.8928	36.0
0.20	1.000000	170.590	170.5900	36.0
0.30	2.765430	61.8560	171.0584	36.0
0.40	5.395060	31.7561	171.3261	36.0
0.50	8.888890	19.2925	171.4889	36.0
0.60	13.24690	12.9538	171.5977	36.0
0.70	18.46910	9.29522	171.6743	36.0
0.80	24.55560	6.99362	171.7325	36.0
0.90	31.50620	5.45217	171.7772	36.0

Table 4: Numerical Results For Uncertainty Relation in the first excited eigenstates for various values of a, with l=0, $V_0 = 0.5$, $V_1 = 1.0$, min = minimum value for the Generalized Yukawa Potential (GYP)

a	$\langle r^2 \rangle$	$\langle r \rangle$	$\Delta(r)$	$(\Delta r)^2$	$\langle p^2 \rangle$	$\Delta(p)$	$\Delta(r)\Delta(p) \geq \hbar/2$	$(\Delta r)^2(\Delta p)^2$	$\min\{(\Delta r)^2(\Delta p)^2\}$
0.01	684.135	24.3656	9.51065	90.4501	0.01619	0.127245	1.210185	1.46443	0.250000
0.02	686.485	24.3930	9.56382	91.4639	0.01600	0.126491	1.209737	1.46346	0.250000
0.03	729.955	25.1147	9.96026	99.2068	0.01486	0.121893	1.214091	1.42963	0.250000
0.04	838.028	26.8056	10.9310	119.488	0.01277	0.112984	1.235033	1.52586	0.250000
0.05	1098.39	30.3730	13.2616	175.871	0.00972	0.098601	1.307615	1.79465	0.250000
0.06	2026.31	39.9130	20.8149	433.265	0.00573	0.075686	1.575400	2.48259	0.250000
0.07	52348.3	172.500	150.314	22592.0	0.00078	0.027999	4.208444	17.6218	0.250000
0.08	2486.02	42.4889	26.0904	680.708	0.00556	0.074536	1.944662	3.78477	0.250000
0.09	813.860	25.4090	12.9708	168.241	0.01404	0.118504	1.537097	2.36213	0.250000
0.10	442.698	19.0307	8.97385	80.5328	0.02469	0.157135	1.410104	1.98829	0.250000
0.20	42.6475	6.03208	2.50230	6.26149	0.25000	0.500000	1.251151	1.56528	0.250000
0.30	15.4640	3.64010	1.48784	2.21369	0.69136	0.831479	1.237109	1.53044	0.250000
0.40	7.93903	2.61021	1.06105	1.12581	1.34877	1.161363	1.232266	1.51849	0.250000
0.50	4.82313	2.03532	0.82498	0.68061	2.22222	1.490712	1.229812	1.51243	0.250000
0.60	3.23845	1.66818	0.67500	0.45562	3.31173	1.819815	1.228375	1.50891	0.250000
0.70	2.32381	1.41334	0.57121	0.32628	4.61728	2.148785	1.227396	1.50653	0.250000
0.80	1.74841	1.22608	0.49511	0.24513	6.13890	2.477680	1.226721	1.50488	0.250000
0.90	1.36304	1.08266	0.43691	0.19089	7.87655	2.806519	1.226194	1.50353	0.250000

DISCUSSION

We report the numerical values of the Fisher information and uncertainty relations both in the position and momentum spaces I_r , I_p, $\langle r^2 \rangle$, $\langle r \rangle$, $\langle p^2 \rangle$, $\Delta(r)\Delta(p)$ and their associated uncertainty relations for generalized radial Yukawa potential. The results in Tables (1) and (3) show that, the position-space Fisher information I_r increases with increasing potential parameter a, in the ground and first excited states (n = 0,1) while the momentum space Fisher information I_p first increases and later decreases with increasing potential parameter a. This implies that, the Fisher information with large values have high accuracy in predicting the localization of the particles in the atomic system. Also, we have been able to verify that the Fisher-information-based-uncertainty relation ($I_r I_p \geq 36.0$) holds for the potential model under study. In addition, we observed from Tables (2) and (4) a squeezed phenomenon for some of the screening parameter a in the momentum space *p* for the ground state when a ≤ 0.7, while in the first excited state, the squeezed phenomenon (squeezed states) occurs in both position *r* and momentum *p* when a ≥ 0.6 and a ≤ 0.2, respectively. A state is defined to be squeezed if $(\Delta r)^2 < 0.5$ or $(\Delta p)^2 < 0.5$, where $(\Delta y)^2 = \langle y^2 \rangle - \langle y \rangle^2$, $y = r$ or p (Esquivel *et al.*, 2000, Grassi, 2011). Our results also obey the Heisenberg uncertainty relation $\Delta(r)\Delta(p) \geq \hbar/2$ for GYP. Figures 2-4 demonstrate the variation of the wave function and the probability densities with position r and momentum p respectively, for the ground and first excited states for some values of the screening parameter a.

CONCLUSION

In this article, we have presented the Fisher information and uncertainty relations for the generalized radial Yukawa potential in both the position and momentum spaces. The Fisher information was calculated by utilizing the probability density, which is the square of the wave function, obtained through the exact solution of this system. The validity of the Fisher-information based uncertainty relation which is stronger version of Heisenberg uncertainty principle have been verified to hold for this atomic model. We found from our results that, for *n* = 0, 1, the position-space Fisher information I_r increases with increasing potential parameter a, while the momentum-space Fisher information I_p first increases, and then decreases with increasing potential parameter a. We have also observed that, there exist a squeezed phenomenon in both position and momentum space in the ground and first excited state for some values of the potential parameter a. The squeezed in position r is compensated for by an increase in the momentum p and otherwise, such that, the Heisenberg Uncertainty principle is satisfied for the atomic model as displayed in the numerical results.

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