ON A NEW METHOD FOR SOLVING NONLINEAR ALGEBRAIC EQUATIONS

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ABSTRACT

A new method for finding approximate solutions of nonlinear algebraic equations is proposed. Here we develop an iterative numerical scheme similar to that of Newton and free from second derivatives. It is shown illustratively by way of numerical examples that the new algorithm is an improvement over the classical Newton Raphson's method due to its faster convergence.

INTRODUCTION

It is well known that a wide class of problem which arises in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x) = 0$.Newton's method that approximates the root of a nonlinear equation in one variable using the value of the function and its derivative, in an iterative fashion, is probably the best known and most widely used algorithm, and it converges to the root quadratically. In other words, after some iterations, the process doubles the number of correct decimal places or significant digits at each iteration. It uses the idea that a continuous and differentiable function can be approximated by a straight line tangent to it.

In this study, we suggest an improvement to the iteration of Newton's method at the expense of an additional functional evaluation.
The problem we want to solve is to find solutions of the equation

$$f(x) = 0, \quad f : I \to R,\ I \subset R, \tag{1}$$

where R is the set of real numbers.

Let $\alpha \in R$ be an isolated root of (1), i.e. there exists an interval $[a,b]$ containing α. So $f(\alpha) = 0$. The numerical solutions of (1) are found using the following theorem:

Theorem 1.1(First Bolzano-Cauchy Theorem) If the function $f : [a, b] \to R$ is continuous on $[a,b]$ and $f(a)f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$.

Geometrically, this is obvious. If a continuous curve passes from one side of the Ox–axis to the other, having opposite signs at the endpoints of the interval, then it has to intersect the axis at least once.
We impose the following conditions on f:
(1) The function f and its derivatives f' and f'' are continuous on $[a,b]$;
(2) The values of the function at the endpoints of the interval have opposite signs, $f(a)f(b) < 0$;
(3) The derivatives f' and f'' have a constant sign on $[a,b]$;
(4) The function f and its second derivative f'' have the same sign at one of the endpoints of the interval $[a,b]$.

Conditions 1 and 2 guarantee the existence of the solution α of the equation (1).
By condition 3, since f' has constant positive or negative sign on the entire interval *[a,b]*, it follows that *f* either increases or decreases, and so will be 0 only once, meaning that the root α is isolated. It is easy to see that these conditions are always satisfied by algebraic equations.

Definition 1.1. A sequence $\{x_n\}$ generated by an iterative method is said to converge to a root *r* with order $p \geq 1$ if there exists $c > 0$ such that $e_{n+1} \leq ce_n^p, \forall n \geq n_o$, for some integer $n_0 \geq 0$ and $e_n = |r - x_n|$.

Theorem 1.2. (Baboliana and Biazar, 2002) Suppose that $g \in C^p[a,b]$. If $g^{(k)}(x) = 0$ for $k = 1, 2, 3, \cdots p-1$ and $g^{(p)}(x) \neq 0$, then the sequence x_n is of order p.

Iteration methods are quite useful and effective in the solution of nonlinear equations, for example, bisection method, secant method, Newton's method (Burden and Faires, 2004) and many other improved iteration methods [He (2003), He (2000), He (2004), Petkovic and Trickovic (1997), Gerlach (1994)] and the references therein. These methods have been widely used in scientific computation, engineering and technology (Wei *et al*, 2008).

Newton's Method
For a nonlinear algebraic equation (1) the most famous iteration scheme for finding its solution numerically is Newton method which has the iterative formula given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

which converges quadratically.
Equivalently, we can write

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0 \ . \tag{3}$$

The iterative process (2) is applied until $|x_{n+1} - x_n| < \varepsilon$. The error in the Newton's method is of the form

$$|x_n - \alpha| \leq \frac{|f(x_n)|}{m}, \ m = \min_{x \in (x_n, \alpha)} f'(x),$$

where α is the approximate solution of $f(x) = 0$, with precision ε. To obtain an approximate solution with error ε, it suffices that $\frac{|f(x_n)|}{m} \leq \varepsilon$.

New Iterative Method
In our scheme, we introduce the third order convergent Newton type method given by

$$x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$f(x_n) + \frac{1}{2}\left[f'(x_n)(x_{n+1} - x_n) + f'(x_{n+1}^*)(x_{n+1} - x_n) \right] = 0.$$

This implies that

$$x_{n+1} = \frac{-2f(x_n)}{\left[f'(x_n) + f'(x_{n+1}^*) \right]} + x_n \ , \quad n = 0,1,2,\cdots, \tag{4}$$

where x_{n+1}^* is predictive and x_{n+1} is corrective.
Here, x_0 is the initial guess and $f'(x_n) \neq 0$.

Numerical Simulations

In this section we consider some numerical examples to demonstrate the performance of the new proposed iterative method. In the subsequent numerical simulations, all computings are performed by using QBASIC codes with numeric data written to 13 places of decimal and the stopping criterion for the computer program being $|x_{n+1} - x_n| \le \varepsilon$, where ε is chosen as small as 10^{-13}.

Example 1. Consider the function $f(x) = x^3 - 4x^2 + 10.$ with initial guess $x_0 = -1.0$. Table 1 shows that Newton Raphson's method converges after 5 iterations while our Algorithm (4) converges after 3 iterations.

Table 1. Convergence for $f(x) = x^3 - 4x^2 + 10.$

Iteration	Newton Raphson method	Our Method
1	-1.4545454545455	-1.3450242372398
2	-1.3689004010695	-1.3652277286914
3	-1.3652366002021	-1.3652300134141
4	-1.3652300134354	
5	-1.3652300134141	

Example 2. Consider the function $f(x) = sin^2(x) - x^2 + 1$ with initial guess $x_0 = -1.0$. Table 2 shows that Newton Raphson's method converges after 6 iterations while our Algorithm (4) converges after 4 iterations.

Table 2. Convergence for $f(x) = sin^2(x) - x^2 + 1$

Iteration	Newton Raphson method	Our Method
1	-1.6491901969323	-1.3115677553482
2	-1.4390423476872	-1.4038435873578
3	-1.4053850861605	-1.4044916480360
4	-1.4044922729362	-1.4044916482153
5	-1.4044916482157	
6	-1.4044916482153	

Example 3. Consider the function $f(x) = ln(x) - sin(x)$ with initial guess $x_0 = 0.5$. Table 3 shows that Newton Raphson's method converges after 6 iterations while our algorithm (4) converges after 4 iterations.

Table 2. Convergence for $f(x) = ln(x) - sin(x)$

Iteration	Newton Raphson method	Our Method
1	1.5446850515250	1.8449323268098
2	2.4538493250134	2.2135416117177
3	2.2311226232826	2.2191071408380
4	2.2191473106998	2.2191071489138
5	2.2191071493681	
6	2.2191071489138	

Convergence Analysis

Suppose that c is a root of the equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in the neighbourhood of c, then the convergence order of our algorithm given in equation (4) is three.

Proof: Let c be a root of $f(x)$, where $f(c)=0$. Again, let

$$g(x)=x-\frac{2f(x)}{f'(x)+f'(x^*)},\text{ where } x^*(x)=x-\frac{f(x)}{f'(x)}.$$

We have calculated the derivatives of $g(x)$ generally using Mathematica and the results are shown below:

$$g(c)=c,$$

$$g'(c)=0,$$

$$g''(c)=0,$$

$$g'''(c)=\frac{2f'''(c)f''(c)[f'(c)+f'''(c)]}{5[f''(c)+f''(c)]}.$$

Therefore, according to Theorem 1.2, the convergence of our algorithm in (4) is of third order.

CONCLUSION

We have shown that algorithm (4) has third order convergence. Computed results in the examples provided amazingly support this fact. An important characteristic of the algorithm in (4) is that it is not required to compute second or higher order derivatives of the function to carry out iterations. It is observed that algorithm (4) ensures more rapid convergence than Newton Raphson's method (Tables 1, 2 and 3). Therefore, algorithm (4) improves (2) essentially.

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