On the Probability Density Functions of Forster-Greer-Thorbecke (FGT) Poverty Indices

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Abstract

Distributional properties of poverty indices are generally unknown due to the fact that statistical inference for poverty measures are mostly ignored in the field of poverty analysis where attention is usually based on identification and aggregation problems. This study considers the possibility of using Pearson system of distributions to approximate the probability density functions of Forster-Greer-Thorbecke (FGT) poverty indices. The application of the Pearson system reveals the potentials of normal and four parameter distributions in poverty analysis.

Keywords: Distributional properties, Pearson system of distributions, FGT poverty indices, Normal distribution, Four parameter beta distribution.

1.0 Introduction

The poverty situation in Nigeria presents a paradox, because despite the fact that the nation is rich in natural resources, the people are poor. [1] referred to this situation as poverty in the midst of plenty. In 1992, for instance, 34.7 million Nigerians (one-third of the population) were reported to be poor, while 13.9 million people were extremely poor [1]. The incidence of poverty increased from 28.1 percent in 1980 to 46.3 percent in 1985. The poverty problem grew so worse in the 1990s that in 1996, about 65.6 percent of the population was poor, while the rural areas accounted for 69.3 percent [2]. Recent data showed that in 2004, 54.4 percent of Nigerians were poor [3]. Also, more than 70 percent of the people are poor, living on less than \$1 a day. Similarly, Nigeria's Human Development Index (HDI) of 0.448 ranks 159^{th} among 177 nations in 2006, portraying the country as one of the poorest in the world [4-5]. This paradox was further highlighted in (Soludo, 2006). He noted that Nigeria is a country abundantly blessed with natural and human resources but the potential remain largely untapped and even mismanaged. With a population estimated at about 140 million, Nigeria is the largest country in Africa and one sixth of the black population in the world. It is the eight largest oil producers and has the sixth largest deposit of natural gas in the world. The growth in per capita income in the 1990s was zero while the incidence of poverty in 1999 was 70% [6].

Traditional approaches to measurement usually start with the specification of poverty line and the value of basic needs considered adequate for meeting minimum levels of decent living in the affected society. Poverty can also be measured using the head count ratio which is based on the ratio or percentage of the number of individuals or households having incomes not equal to the poverty line to the total number of individuals or households [7-9]. Another method of measuring intensity of poverty is the "income-gap" ratio. Here the deviation of the incomes of the poor from the poverty line is averaged and divided by the poverty line [10]. These are the convectional approaches to poverty analysis where the population is classified into two dichotomous groups of poor and non-poor, defined in relation to some chosen poverty line based on household income/expenditure [11]. In the last few years, poverty analyses made substantial improvements by gradually moving from the conventional one-dimensional approach to multidimensional approach [12-14].

Statistical inference for poverty and inequality measures are widely ignored in the field of poverty analysis where attention is usually based on identification and aggregation problems [15] . The implication of this is that distributional properties of poverty and inequality indices are generally unknown. This study therefore intends to demonstrate how moments and cumulants of Forster-Greer-Thorbecke (FGT) poverty indices could be obtained from knowledge of their probability density functions from the Pearson system of distributions.

2.0 FORSTER-GREER THORBECKE (FGT) POVERTY INDICES

In analyzing poverty, it has become customary to use the so called FGT P-Alpha poverty measures proposed by [11]. These FGT P-Alpha measures are usually used to measure the poverty level. This is a family of poverty indexes, based on a single formula, capable of incorporating any degree of concern about poverty through the "poverty aversion" parameter, α. This measure is given as

$$P_\alpha = \frac{1}{n}\sum_1^n \left(\frac{z-y_i}{z}\right)^\alpha I(z, y_i) \qquad (1)$$

where z is the poverty line; n is the total number of individuals in the reference population; y_i is the expenditure/income of the household in which individual lives, α takes on values, 0, 1, and 2. The quantity in parentheses is the proportionate shortfall of expenditure/income below the poverty line. This quantity is raised to a power α. By increasing the value of α, the aversion to poverty as measured by the index is also increased [16]. The P-alpha measure of poverty becomes head count, poverty gap and square poverty gap indices respectively when α = 0, 1, and 2 in that order.

3.0 The Pearson System of Distributions

Several well known distributions like Gaussian, Gamma, Beta and Student's t-distributions belong to the Pearson family. The system was introduced by [17] who worked out a set of four-parameter probability density functions as solutions to the differential equation

$$\frac{f'(x)}{f(x)} = \frac{P(x)}{Q(x)} = \frac{x-a}{b_0 + b_1 x + b_2 x^2} \qquad .(2)$$

where f is a density function and a, b_0, b_1 and b_2 are the parameters of the distribution. What makes the Pearson's four-parameter system particularly appealing is the direct correspondence between the parameters and the central moments $(\mu_1, \mu_2, \mu_3\ and\, \mu_4)$ of the distribution [18]. The parameters are defined as

$$\left.\begin{aligned} b_1 &= a = -\frac{\mu_3(\mu_4 + 3\mu_2^2)}{A} \\ b_0 &= -\frac{\mu_2(4\mu_2\mu_4 - 3\mu_3^2)}{A} \\ b_2 &= -\frac{(2\mu_2\mu_4 - 3\mu_3^2 - 6\mu_2^3)}{A} \end{aligned}\right\} \qquad (3)$$

The scaling parameter A is obtained from

$$A = 10\mu_4\mu_2 - 18\mu_2^3 - 12\mu_3^2 \qquad (4)$$

When the theoretical central moments are replaced by their sample estimates, the above equations define the moment estimators for the Pearson parameters a, b_0, b_1 and b_2. As alternatives to the basic four-parameter systems, various extensions have been proposed with the use of higher-order polynomials or restrictions on the parameters. Typical extension modifies (2) by setting $P(x) = a_0 + a_1 x$ so that

$$\frac{f'(x)}{f(x)} = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \qquad (5)$$

This parameterization characterizes the same distributions but has the advantage that a_1 can be zero and the values of the parameters are bound when the fourth cumulant exists [19]. Several attempts to parameterize the model using cubic and quadratic curves have been made already by Pearson and others, but these systems proved too cumbersome for general use. Instead the simpler scheme with linear numerator and quadratic denominator are more acceptable.

3.1 Classification and Selection of Distributions in the Pearson System

There are different ways to classify the distributions generated by the roots of the polynomials in (2) and (5). Pearson himself organized the solution to his equation in a system of twelve classes identified by a number. The numbering criterion has no systematic basis and it has varied depending on the source. An alternative approach suggested by [20] for distribution selection based on two statistics that are functions of the four Pearson parameters will be adopted. The scheme is presented in Tables 1 and 2 where D and λ denote the selection criteria. D and λ are defined as

$$\left.\begin{array}{l} D = b_0 b_2 - b_1^2 \\ \lambda = \dfrac{b_1^2}{b_0 b_2} \end{array}\right\} \qquad (6)$$

Table 1: Pearson Distributions

The table provides a classification of the Pearson Distributions, f(x) satisfying the differential equation $(\frac{1}{f})\frac{df}{dx} = \frac{P(x)}{Q(x)} = \frac{(a_0+a_1x)}{(b_0+b_1x+b_2x^2)}$. The signs and values for selection criteria, $D = b_0b_2 - b_1^2$ and $\lambda = \frac{b_1^2}{b_0b_2}$, are given in columns three and four.

Table 1: Person Distributions

P(x) = a_0 , Q(x) = 1					
	Restrictions	**D**	**λ**	**Support**	**Density**
1.	$a_0 < 0$	0	0/0	$\mathbf{R}^+$	$\gamma e^{-\gamma x}, \gamma > 0$
P(x) = a_0 , $Q(x) = b_2 x(x+\alpha)$					
	Restrictions	**D**	**λ**	**Support**	**Density**
2(a).	$\alpha > 0$	< 0	∞	[- α , 0]	$\frac{m+1}{\alpha^{m+1}}(x+\alpha)^m, m < -1$
2(b).	$\alpha > 0$	< 0	∞	[- α , 0]	$\frac{m+1}{\alpha^{m+1}}(x+\alpha)^m, -1 < m < 0$
P(x) = a_0 , $Q(x) = b_0 + 2b_1 x + x^2 = (x-\alpha)(x-\beta), \alpha < \beta$					
	Restrictions	**D**	**λ**	**Support**	**Density**
3(a).	$a_0 \neq 0$ $0 < \alpha < \beta$	< 0	>1	[β, ∞]	$\frac{(\beta-\alpha)^{-(m+n+1)}}{B(-m-n-1, n+1)}(x-\alpha)^m(x-\beta)^n$ $m > -1, n > -1, m \neq 0, n \neq 0, m = -n$
3(b).	$a_0 \neq 0$ $\alpha < \beta < 0$	< 0	>1	[-∞,α]	$\frac{(\beta-\alpha)^{-(m+n+1)}}{B(-m-n-1, m+1)}(x-\alpha)^m(x-\beta)^n$ $m > -1, n > -1, m \neq 0, n \neq 0, m = -n$

4.	$a_0 \neq 0$ $\alpha < 0 < \beta$	< 0	< 0	[α, β]	$\frac{\alpha^{2m}\beta^{2n}}{(\alpha+\beta)^{m+n+1}B(m+1,n+1)}(x-\alpha)^m(x-\beta)^n$ $m>-1, n>-1, m\neq 0, n\neq 0, m=-n$
	$P(x) = a_0 + a_1x, Q(x) = 1$				
	Restrictions	**D**	**λ**	**Support**	**Density**
5.	$a_1 \neq 0$	0	0/0	**R**	$\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Table 2: Pearson Distributions (Continued)

$P(x) = a_0 + a_1x, Q(x) = x-\alpha$					
	Restrictions	**D**	**λ**	**Support**	**Density**
6.	$a_1 \neq 0$	< 0	∞	[α,∞]	$\frac{k^{m+1}}{\Gamma(m+1)}(x-\alpha)^{-m}e^{-k(x-\alpha)}, k>0$
$P(x) = a_0 + a_1x, Q(x) = b_0 + 2b_1x + x^2 = (x-\alpha)(x-\beta), \alpha \neq \beta$					
	Restrictions	**D**	**λ**	**Support**	**Density**
7(a).	$a_1 \neq 0$ $0 < \alpha < \beta$	< 0	>1	[β, ∞]	$\frac{(\beta-\alpha)^{-(m+n+1)}}{B(-m-n-1,n+1)}(x-\alpha)^m(x-\beta)^n$ $m>-1, n>-1, m\neq 0, n\neq 0, m\neq -n$
7(b).	$a_1 \neq 0$ $\alpha < \beta < 0$	< 0	>1	[-∞,α]	$\frac{(\beta-\alpha)^{-(m+n+1)}}{B(-m-n-1,m+1)}(x-\alpha)^m(x-\beta)^n$ $m>-1, n>-1, m\neq 0, n\neq 0, m\neq -n$
8.	$a_1 \neq 0$ $\alpha < 0 < \beta$	< 0	< 0	[α, β]	$\frac{\alpha^{2m}\beta^{2n}}{(\alpha+\beta)^{m+n+1}B(m+1,n+1)}(x-\alpha)^m(x-\beta)$ $m>-1, n>-1, m\neq 0, n\neq 0, m\neq -n$
	$P(x) = a_0 + a_1x, Q(x) = b_0 + 2b_1x + x^2 = (x-\alpha)(x-\beta), \alpha = \beta$				
9.	$a_1 > 0$ $\alpha = \beta$	0	1	[α, ∞]	$\frac{\gamma^{m-1}}{\Gamma(m-1)}(x-\alpha)^{-m}e^{-\gamma/x}, \gamma>0, m>1$
	$P(x) = a_0 + a_1x, Q(x) = b_0 + 2b_1x + x^2$, complex roots				
	Restrictions	**D**	**λ**	**Support**	**Density**
10.	$a_0 = 0, a_1 < 0$ $b_1 = 0, b_0 = \beta^2$ $\beta \neq 0$	>0	0	**R**	$\frac{\alpha^{2m-1}}{B(m-\frac{1}{2},\frac{1}{2})}(x^2+\beta^2)^{-m}, m>\frac{1}{2}$
11.	$a_0 \neq 0, a_1 < 0$ $b_1 \neq \frac{a_0}{a_1}$	>0	0> <1	**R**	$c(b_0+2b_1x+x^2)^{-m}e^{-\mathrm{var}c\tan((x+b_1)/\beta)}$ $m>\frac{1}{2}, \beta=\sqrt{b_0-b_1^2}$

The advantage of this approach in statistical modeling in the Pearson framework is its simplicity. Implementation is done in accordance with the following steps:

(1) Estimate moments from data.
(2) Calculate the Pearson parameters a, b_0, b_1 and b_2 using (3) and (4).
(3) Use the estimates of the parameters to compute the selection criteria D and λ as given in (6).
(4) Select an appropriate distribution from Tables (1) and (2) based on the signs of the values of the selection criteria.

4.0 Bootstrapping

Poverty indices are complex in nature and this makes direct analytic solutions very tedious and complex. Alternative numerical solutions are possible through simulation Bootstrapping. Bootstrapping is essentially a re-sampling method. That is, re-sampling is a Monte-Carlo method of simulating data set from an existing data set, without any assumption on the underlying population. Bootstrapping was invented by [21-22] and further developed by [23]. It is based on re-sampling with replacement from the original sample. Thus each bootstrap sample is an independent random sample of size n from the empirical distribution. The elements of bootstrap samples are the same as those of the original data set. Bootstrapping, like other asymptotic methods, is an approximate method, which attempts to get results for small samples (unlike other asymptotic methods). The estimates of the parameters of the selection criteria for the purpose of selecting appropriate probability distributions from the Pearson system for head count, poverty gap and square poverty gap indices were obtained through bootstrap simulation method. The bootstrap sample size was 10,000 and the number of iterations was 5,000.

5.0 Results and Discussion

The methods presented are applied to The Nigerian Living Standard Survey (NLSS, 2004) data. The survey was designed to give estimates at National, Zonal and State levels. The first stage was a cluster of housing units called numeration Area (EA), while the second stage was the housing units. One hundred and twenty EAs were selected and sensitized in each state, while sixty were selected in the Federal Capital Territory. Ten EAs with five housing units were studied per month. Thus a total of fifty housing units were canvassed per month in each state and twenty-five in Abuja. Data were collected on the following key elements: demographic characteristics, educational skill and training, employment and time use, housing and housing conditions, social capital, agriculture, income, consumption expenditure and non-farm enterprise. The total number of households in the survey was 19,158.

The estimates of the selection criteria for the selection of probability distributions from the Pearson system were obtained as shown in Table 3. Based on the values and signs of these criteria, the normal and four parameter beta distributions were selected for the poverty indices based on the classifications in Tables 1 and 2. The normal distribution was selected for the head count index while, the four parameter beta distribution was selected for both poverty gap and square poverty gap indices respectively. The estimates of the parameters of these selected distributions were equally estimated as shown in Tables 4, 5 and 6.

Table 3: Estimates of Selection Criteria ($D = b_0b_2 - b_1^2$ and $\lambda = \dfrac{b_1^2}{b_0b_2}$) for FGT Poverty Indices

	P_0 Poverty Head Count Index	P_1 Poverty Gap Index	P_2 Square Poverty Gap Index
b_0	-1.13687×10^{-5}	-3.40804×10^{-6}	-1.62081×10^{-6}
$b_1 =$ a	-4.91928×10^{-5}	-3.84213×10^{-5}	-1.33289×10^{-5}
b_2	-1.45081×10^{-2}	6.80230×10^{-3}	6.40434×10^{-3}
D	1.62518×10^{-7}	-2.46587×10^{-8}	-1.05579×10^{-8}
λ	3.08817 X	6.36771 X	-1.71151 X

	10^{-6}	10^{-2}		10^{-2}	
A	2.17318×10^{-14}	4.32546×10^{-16}	X	4.67674×10^{-17}	X

Table 4: Parameter Estimates of the Normal Distribution for Head Count Poverty Index

Parameter	**Estimate**
μ	0.52096
σ	0.00345

Table 5: Parameter Estimates of the Four Parameter Beta Distribution for Poverty Gap Index

Parameter	**Estimate**
α_1	224.73
α_2	388.02
a	0.17752
b	0.27147

Table 6: Parameter Estimates of the Four Parameter Beta Distribution for Square Poverty Gap Index

Parameter	**Estimate**
α_1	47.953
α_2	48.085
a	0.10164
b	0.12648

6.0 Conclusion

The probability distributions of head count, poverty gap and square poverty gap indices have been determined. The distributions appropriate for the indices obtained using the procedure given by Andreev for the selection of probability distributions from the Pearson system of distributions were the normal distribution for head count index and the four parameter beta distribution for both poverty gap and square poverty gap indices. The normality confirms the applicability of laws of large numbers and the consequent validity of the central limit theorem. Hence, study on poverty indices should involve large samples. The selection of the beta distribution for the two indices may be due to the fact that the Beta distribution is often used to mimic other distributions when a vector of random variables is suitably transformed and normalized.

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References

[1]. World Bank (1996) “Poverty in the Midst of Plenty: The challenge of growth with inclusion in Nigeria” A World Bank Poverty Assessment, May 31, World Bank, Washington, D.C.

[2]. Federal Office of Statistic (FOS) (1999), Poverty and Agricultural sector in Nigeria FOS, Abuja, Nigeria.

[3]. Federal Republic of Nigeria (FRN) (2006). Poverty Profile for Nigeria. National Bureau of Statistics (NBS) FRN.

[4]. United Nations Development Program (UNDP) (2006). Beyond scarcity: Power, poverty and the global water crisis. Human Development Report 2006.

[5]. IMF (2005). Nigeria: Poverty Reduction Strategy Paper— National Economic Empowerment and Development Strategy. IMF Country Report No. 05/433.

[6]. Soludo (2006) ‘’Potential Impacts of the New Global Financial Architecture on Poor Countries’’: Edited by Charles Soludo, Monsuru Rao, ISBN 9782869781580, 2006, CODERSIA, Senegal, paperback. 80 pgs.

[7]. Bardhan, P. K. (1973) “On the Incidence of Poverty in Rural India”. Economic and Political Weekly,. March

[8]. Ahluwalia, M.S. (1978) “*Inequality, Poverty and Development*”. Macmillan Press U.K.

[9]. Ginneken, W. V.(1980), "Some Methods of Poverty Analysis: An Application to Iranian Date," *World Development, Vol. 8*
[10]. World Bank (1980), Poverty and Basic Needs Development Policy Staff Paper, Washington D.C.
[11]. Foster, James, J. Greer and Eric Thorbecke.(1984) . "A Class of Decomposable Poverty Measures,"*Econometrica*, 52(3): 761-765.
[12]. Hagenaars A.J.M. (1986), *The Perception of Poverty,* North Holland, Amsterdam.
[13]. Dagum C. (1989), "Poverty as Perceived by the Leyden Evaluation Project. A Survey of Hagenaars' Contribution on the Perception of Poverty", *Economic Notes*, 1, 99-110.
[14]. Sen A.K. (1992), *Inequality Reexamined,* Harvard University Press, Cambridge (MA).
[15]. Sen, A. (1976) "An Ordinal Approach to Measurement", *Econometrica, 44, 219- 232.*
[16]. Boateng, E.O., Ewusi, K., Kanbur, R., and McKay, A. 1990. *A Poverty Profile for Ghana*, 1987-1988 in Social Dimensions of Adjustment in Sub-Saharan Africa, Working Paper 5. The World Bank: Washington D.C.
[17]. Pearson, K.1895. Memoir on Skew Variation in Homogeneous Material. *Philosophical Transactions of the Royal Society. A186: 323-414*
[18]. Stuart, A. and Ord. J.1994. *Kendall's Advanced Theory of Statistics, Vol. I: Distribution Theory.* London: Edward Arnold.
[19]. Karvanen, J.2002. Adaptive Methods for Score Function Modeling in Blind Source Separation. Unpublished Ph.D. Thesis, Helsinki University of Technology.
[20]. Andreev, A., Kanto, A., and Malo, P.2005. Simple Approach for Distribution Selection in the Pearson System. Helsinki School of Economics Working Papers: W-388.
[21]. Efron, B.1982. *The Jacknife, the Bootstrap, and Other Resampling Plans*. Philadelphia: SIAM.
[22]. Efron, B. 1983. Bootstrap Methods; Another Look at the Jacknife. *The Annals of Statistics. 7: 1-26.*
[23]. Efron, B. and R.J. Tibshirani.1993. *An Introduction to the Bootstrap*. London: Chapman and Hall.