A survey of Strong Convergent Schemes for the Simulation of Stochastic Differential Equations

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Abstract

We considered strong convergent stochastic schemes for the simulation of stochastic differential equations. The stochastic Taylor's expansion, which is the main tool used for the derivation of strong convergent schemes; the Euler Maruyama, Milstein scheme, stochastic multistep schemes, Implicit and Explicit schemes were considered. A simple SDE, which is known to have analytic solution, was used to illustrate the simulation technicalities. A MatLab script file was written to implement the Euler Maruyama and the Milstein schemes. The result showed graphically the closeness of the Milstein and the Euler Maruyama scheme as well as the error between the analytic result and the numerical approximation. The error appeared dispersed as t increases and tens to T.

1.0 Introduction

The concept of Stochastic Differential Equations though not new, is not as popular as the deterministic ordinary differential equations. Consequently, the method of solutions is much more unpopular. More so, the fact that most stochastic differential equations of practical applications are not analytically solvable makes it even more unpopular. Despite these facts, there are wide range of stochastic numerical schemes in literature that can be used for the simulation of the trajectories of SDEs. The focus of this work is to survey of applicable strong convergent numerical schemes for the simulation of stochastic differential equations, [6].

Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. It is used to model systems that behave randomly the best known of which is the Wiener process (named in honour of Norbert Wiener), which is used for modelling Brownian motion as described by Albert Einstein and other physical diffusion processes in space of particles subject to random forces. Since the 1970s, the Wiener process has been widely applied in financial mathematics and economics to model the evolution in time of stock prices and bond interest rates, [4].

The Itō calculus is named after *Kiyoshi Itō* and it extends the methods of calculus to stochastic processes such as Wiener process. It has important applications in mathematical finance and stochastic differential equations. The central concept is the Itō stochastic integral which is a generalization of the ordinary concept of a Riemann–Stieltjes integral. The generalization is in two respects. Firstly, if deals with random variables (more precisely, stochastic processes). Secondly, the method allows integration with respect a non-differentiable function (technically, stochastic process), [3]. An alternative method to the Ito calculus is the Stratonovich calculus which was introduced by Ruslan L. Stratonovich and D. L. Fisk is the preferred method for modelling stochastic processes in applied mathematics, [5].

1.1 Itó Integral

The stochastic calculus of Itô originated with the investigation of conditions under which the drift and the diffusion coefficient of the diffusion of Markov process could be used to characterize this process. Kolmogorov had made similar attempt but Itó focused on the functional form of the process which resulted to some meaningful mathematical formulation for stochastic differential equations, [2].

Following the Einstein's explanation of Brownian motion in the first decade of the 19th century, there were rigorous efforts by Langevin and others to formulate the dynamics of the motion in terms of stochastic differential equation. The resulting equation were written in the form

$$dX_t = a(t, X_t) + b(t, X_t)dW_t \tag{1.1}$$

This symbolic differential form can be written in integral form as

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_t \tag{1.2}$$

Since the Brownian motion B_t is a derivative of the Wiener process W_t, the second integral in equation (1.1) above cannot be interpreted as the Riemann or Lebesgue integral because Brownian motion is nowhere differentiable. Furthermore, because the continuous sample path of Brownian motion is not of a bounded variation in any interval, the integral cannot also be interpreted as Riemann-Stieljes integral. The integral of the form $\int_a^b X(t)dB(t)$ where $X(t)$ is adapted to the filtration $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$ is called a stochastic or Itó integral because the concept was originally introduced by Itó and was used to construct the diffusion process which is a subclass of the Markov process. This type of integral arises naturally as a solution of stochastic differential equations or martingale.

1.2 Stochastic Differential Equations

Differential equations are used generally to describe the evolution of a system over time. Stochastic differential equation arises when a differential equation is subject to some random perturbation called the White noise. For example, if x(t) is a differential equation defined for t $\geq$0, u(x,t) is a function of x and t and the following relation is satisfied for all t, 0$\leq$ t $\leq$T

$$\frac{dx(t)}{dt} = x'(t) = u(x(t), t), \quad x(0) = x_0 \tag{1.4}$$

Then $x(t)$ is said to be the solution of the ordinary differential equation (1) with the initial condition $x(0) = x_0$. Equation (1) can be written as $dx(t) = u(x(t), t)\,dt$ and by the continuity of $x'(t)$, we can write:

$$x(t) = x(0) + \int_0^t U(x(s), s)ds \qquad (1.5)$$

As mentioned above stochastic differential equation arises when a differential equation is put to a random perturbation by a White noise. The White noise is defined as a derivative of a Brownian motion i.e. $\xi(t) = \frac{dB(t)}{dt} = B'(t)$.

Now, the White noise does not exist as a usual function of t since a Brownian motion is now where differentiable. If we denote by $\sigma(t)$ the intensity of the noise at a point x at time t, then we can write

$$\int_0^T \sigma(t)(x(t), t)\xi(t) = \int_0^T \sigma(t)(x(t), t)dB'(t)d(t) = \int_0^T \sigma(t)(x(t), t)dB(t) \qquad (1.6)$$

The integral in equation (1.4.2) above is an Ito integral; hence, we can say that stochastic differential equation arises when the coefficients of ordinary differential equation are perturbed by white Noise.

2.0 Stochastic Taylor's Formula

In order to appreciate the stochastic Taylor's formula, it is appropriate to start from the deterministic Taylor's formula before proceeding to the stochastic counterpart

.

Let X_t be a 1-dimensional ordinary differential equation so that

$$\frac{d}{dt}X_t = a(X_t) \qquad (2.1)$$

with initial value X_{t_0} for $t \in [t_0, T]$where $0 \le t \le T$.This can be written in integral form as:

$$X_t = X_{t_0} + \int_{t_0}^t a(X_s)ds \qquad (2.2)$$

Suppose (2.2) above is sufficiently smooth and has a linear growth bound and let $f: \mathcal{R} \to \mathcal{R}$ be a continuously differentiable function, then by the chain rule,

$$\frac{d}{dt}f(X_t) = a(X_t)\frac{\partial}{\partial x}f(X_t) \qquad (2.3)$$

If we use the operator

$$L = a\frac{\partial}{\partial x} \qquad (2.4)$$

we can then express the integral in (2) above as

$$X_t = f(X_{t_0}) + \int_{t_0}^t Lf(X_s)ds \qquad (2.5)$$

for all $t \in [t_0, T]$. When $f(x) = x$, we have $Lf = a, L^2f = La$ and (2.5) reduces to (2.2) i.e.

$$X_t = X_{t_0} + \int_{t_0}^t a(X_s)ds \qquad (2.6)$$

Applying the relation (5) to $f = a$ in the integral in (2.6) we obtain

$$= X_{t_0} + a(X_{t_0})\int_{t_0}^t ds + \int_{t_0}^t \int_{t_0}^s La(X_s)dzds \quad X_t = X_{t_0} + \left(\int_{t_0}^t a(X_{t_0}) + \int_{t_0}^t La(X_s)\right)ds \qquad (.2.7)$$

and this is the simplest nontrivial Taylor's expansion for X_t. We can apply (5) again to the function $f = La$ in the double integral to have

$$X_t = X_{t_0} + a(X_{t_0})\int_{t_0}^{t} ds + La(X_{t_0})\int_{t_0}^{t}\int_{t_0}^{s} La(X_s)dzds + R_3 \tag{2.8}$$

where the remainder

$$\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} La(X_s)du\, dz\, ds \tag{2.9}$$

for $t \in [t_0, T]$. For a general times differentiable function $f: \mathcal{R} \to \mathcal{R}$, this method gives the classical Taylor formula in integral form as:

$$f(X_t) = f(X_{t_0}) + \sum_{i=1}^{r} \frac{(t-t_0)^i}{i!} L^i f(X_{t_0}) + \int_{t_0}^{t} \ldots\ldots \int_{t_0}^{s_2} L^{r+1} f(X_{s_2}) ds_1 \ldots ds_{r+1} \tag{2.10}$$

for $t \in [t_0, T]$ and r = 1,2,3, ...

The Taylor's formula presented above has been a vital tool for both practical and theoretical investigation in numerical analysis as it allows for the approximation of sufficiently smooth functions in the neighbourhood of a given point to and desired level of accuracy. We can therefore use this method too to expand increment of smooth function of Ito process which will allow us to derive relevant formulae for the numerical solutions of stochastic differential equations, [7].

Consider the integral form of the Ito process, which is given as :

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s \tag{2.11}$$

for $t \in [t_0, T]$ where the second integral in (11) is a stochastic integral and the coefficients a and b are sufficiently smooth real valued functions satisfying a linear growth bound. For any twice continuously differentiable function $f: \mathcal{R} \to \mathcal{R}$ the Ito formula is given as :

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)\frac{\partial}{\partial x} f(X_s) + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2} f(X_s)\right) ds + \int_{t_0}^{t} b(X_s)\frac{\partial}{\partial x} f(X_s)dW_s$$

$$= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)\, ds + \int_{t_0}^{t} L^1 f(X_s)dW_s$$

for $t \in [t_0, T]$ where

$$L^0 = a\frac{\partial}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2} \tag{2.12}$$

and $L^1 = b\frac{\partial}{\partial x}$ (2.13)

Clearly, for $f(x) \equiv x$ we have $L^0 f = a$ and $L^1 f = b$ which mean that equation (12) reduces to the original Ito equation for X_t, that is

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s \qquad (2.14)$$

Just like the deterministic case presented earlier, if we apply the formula (12) to the function $f = a \; and \; f = b$ in (3.2.14), we obtain :

$$X_t = X_{t_0} + \int_{t_0}^{t}\left(a(X_{t_0}) + \int_{t_0}^{t} L^0 a(X_s)dz + \int_{t_0}^{t} L^1 a(X_s)dW_s\right)ds + \int_{t_0}^{t}\left(b(X_{t_0}) + \int_{t_0}^{t} L^0 b(X_s)dz + \int_{t_0}^{t} L^1 b(X_s)dW_s\right)dW_s$$

$$= X_{t_0} + a(X_{t_0})\int_{t_0}^{t} ds + b(X_{t_0})\int_{t_0}^{t} dW_s + R$$

with remainder

$$R = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_s)dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_s)dW_z ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_s)dz\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_s)dW_z dW_s$$

This is the simplest nontrivial Ito-Taylor's expansion. We can continue it, for example, by applying the formula (2.5.12) in which we shall have :

$$X_t = a\,(X_{t_0})\int_{t_0}^{t} ds + b(X_{t_0})\int_{t_0}^{t} dW_s + L^1\, b(X_{t_0})\int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s + R \qquad (2.15)$$

with remainder

$$R = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_s)dz\; ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_s)dW_z ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_s)dz\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0 L^1 b(X_u)du\, dW_z dW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1 L^1 b(X_u)dW_u\, dW_z dW_s$$

The Stratonovich counterpart which is called the Stratonovich-Taylor's expansion can similarly be presented as follows. The 1-dimensional Stratonovich stochastic differential equation in its integral form is of the form :

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(X_s)ds + \int_{t_0}^{t} b(X_x) \circ dW_s \qquad (2.16)$$

for $t \in [t_0, T]$ where the second integral is a Stratonovich stochastic integral and the coefficients $\underline{a}$ and $d$are sufficiently smooth real valued function satisfying the linear growth bound and when $\underline{a} = a - \frac{1}{2}bb'$ the Ito equation in (3.5.1) and the Stratonovich equation in (18) have the same solutions. The solution of the Stratonovich SDE for a function $f: \mathcal{R} \rightarrow \mathcal{R}$ where f is any twice continuously differentiable function transforms according to the deterministic chain rule to

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \underline{a}(X_s)\frac{\partial}{\partial x}f(X_s)ds + \int_{t_0}^{t} b(X_s)\frac{\partial}{\partial x}f(X_s) \circ dW_s \qquad (2.17)$$

$$= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)ds + \int_{t_0}^{t} L^1 f(X_s) \circ dW_s$$

for $t \in [t_0, T]$ with the operators

$$\underline{L}^0 = \underline{a}\frac{\partial}{\partial x} \tag{2.18}$$

and

$$\underline{L}^1 = b\frac{\partial}{\partial x} \tag{2.19}$$

For $f(x) \equiv x$ we have $\underline{L}^0 f = \underline{a}$ and $\underline{L}^1 f = b$ in which case (19) reduces to integrand function $f = \underline{a}$ and $f = b$ in (20) and this gives

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(X_s)\,ds + \int_{t_0}^{t} b\,(X_s) \circ dW_s \tag{2.20}$$

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(\underline{a}(X_{t_0}) + \int_{t_0}^{t} L^0\,\underline{a}f(X_s)ds + \int_{t_0}^{t} L^1\,\underline{a}f(X_s) \circ dW_s\right) d + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{t} L^0\,b(X_s)dz + \int_{t_0}^{t} L^1\;b\,(X_s) \circ dW_s\right) \circ dW_s$$

$$= X_{t_0} + \underline{a}(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} \circ\, dW_s + R \tag{2.21}$$

with remainder

$$R = \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0\underline{a}(X_s)dz\;ds + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1\,\underline{a}(X_s) \circ dW_z ds + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 b(X_s)dz \circ\; dW_s + \int_{t_0}^{t} \int_{t_0}^{t} \underline{L}^1 b(X_s) \circ dW_z \circ dW_s$$

and this is the simplest nontrivial Stratonovich-Taylor's expansion of $f(X_t)$. We can continue expanding, for example, if we apply (2.18) to the integrand $\underline{L}^1 f = b$ in (23), we shall have

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j\;(s, X_s)dW_s^j \tag{2.22}$$

for $t \in [t_0, T]$ and the equivalent Stratonovich equation

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(s, X_s)ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j\;(s, X_s) \circ dW_s^j \tag{2.23}$$

for $t \in [t_0, T]$ where

$$\underline{a}^i = a^i - \frac{1}{2}\sum_{j=1}^{m} \sum_{k=1}^{d} b^{k,j} \frac{\partial b^{i,j}}{\partial x^k} \tag{2.24}$$

for $i = 1,2,3,\ldots d$.In both SDEs, $W_t = t \in [0, T]$ is a standard m-dimensional Wiener process adapted to an increasing family of σ-algebra $\{A_t, t \in [0, T]\}$.

3.0 Strong Convergent Schemes for Simulation of Stochastic Differential Equations

It is well known that only a few stochastic differential equations have explicitly known solutions. This makes simulation techniques to be a very important method for solving SDEs. This is can be done through the use of the discrete time approximation methods. The discrete time approximation methods provide suitable algorithm that can be used to recursively compute the solutions of SDE at its discretization points with the values at intermediary points computed through interpolation, [7].

Through this, an appropriate sample path of the driving SDE can be obtained for the Wiener process, which is usually simulated with an appropriate pseudo-random number generator. Consequently, we shall now consider numerical schemes for simulation of stochastic differential equations.

3.1 The Euler Maruyama Scheme

One of the simplest discrete approximations of an Ito process is the *Euler* approximations also called the *Euler-Maruyama* approximations. We consider the Ito process $X = \{X_t, t_0 \le t \le T\}$ satisfying the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \qquad (.3.1)$$

on $t_0 \le t \le T$ with the initial value

For a given discretization $t_0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$ of the time interval $[t_0, T]$, an Euler-Maruyama approximation is a continuous time stochastic process $Y = \{Y(t), t_0 \le t \le T\}$ satisfying the iterative scheme

$$X_{t_0} = X_0 \qquad (3.2)$$

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)(\tau_{n+1} - \tau_n) + b(\tau_n, Y_n)(W_{\tau_n} - W_{\tau_n}) \qquad (3.3)$$

for $n = 0, 1, 2, \dots N-1$ with the initial value $Y_{t_0} = Y_0$. If we write

$\Delta_n = \tau_{n+1} - \tau_n$ for the nth increment and $\delta = max_n \Delta_n$ for the maximum step size, then $\Delta = (T - t_0)/N$ for some integer N. If the diffusion coefficient $b \equiv 0$, then the stochastic iterative scheme reduces to the ordinary deterministic Euler's scheme so that the sequence of $Y_n, n = 0, 1, \dots$ is computed in the usual manner. The

$$Y_{n+1} = Y_n + a\Delta_n + b\Delta W_n \qquad (3.4)$$

for $n = 0, 1, \dots N-1$. The recursive structure of the scheme makes it suitable to be implemented on a digital computer. Each of the approximation method determines the value of the approximating process at the discretization times only. When required, intermediate values can be determined using the method of interpolation. In general, the paths of an Ito process inherit the irregularity of the sample paths of the driving Wiener process such as the non-differentiability.

approach is similar for the stochastic case, that is for $b \neq 0$ only that in this case we need to generate the random increment $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$ for $n = 0, 1, \dots$ by using a suitable pseudo random number generator. The Euler-Maruyama scheme can be written in a simpler form as

3.2 The Order 1.5 Strong Taylor's Scheme

The multiple stochastic integrals provide additional information about the sample path of the driving Wiener process with the discretization subinterval. The necessity of their inclusion in higher order schemes is a fundamental difference between the numerical analysis of deterministic differential equations and that of stochastic differential equations. By including more terms from the Ito Taylor's scheme for an autonomous 1-dimensional case $d = m = 1$, we obtain the order 1.5 strong Taylor's scheme, [3]:

$$Y_{n+1} = Y_n + a\Delta_n + b\Delta W_n + \frac{1}{2} bb'\{(\Delta W_n)^2 - \Delta_n\}$$

$$+ a' b \Delta Z_n + \frac{1}{2}\left(aa' + \frac{1}{2} b^2 a''\right) \Delta_n^2$$

$$+ \left(ab' + \frac{1}{2} b^2 b''\right)\{\Delta W_n \Delta_n - \Delta Z_n\}$$

$$+ b(bb'' + (b')^2)\{(\Delta W_n)^2 - \Delta_n\}\Delta W_n \quad (3.4)$$

An additional random variable $\Delta Z_n = \int_{t_n}^{t_{n-1}} \int_{t_n}^{s_2} dW_{s_1} ds_2$ is required. ΔZ_n is a random variable that is normally distributed with $E(\Delta Z_n) = 0$ and variance $E((\Delta Z_n)^2) = \frac{1}{3} {\Delta_n}^3$ and covariance $E(\Delta Z_n \Delta W_n) = \frac{1}{2} {\Delta_n}^2$ and can be generated with ΔW_n from two independently N(0,1) distributed random variables. It should be noted that the last term of (1) above contains the triple Ito integral

$$I_{(1,1,1)} = \frac{1}{2}\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right\}\Delta W_n \quad (3.5)$$

In general, the multidimensional case with $d, m = 1,2,3 \ldots k$ of the order 1.5 strong Taylor's scheme has the form :

$$Y_{n+1}^k = Y_n^k + a^k \Delta_n + \frac{1}{2} L^0 a^k {\Delta_n}^2$$

$$+ \Sigma_{j=1}^m \left(b^{k,j} \Delta W_n^{\,j} + L^0 b^{k,j} I_{(0,j)}\right)$$

$$+ \Sigma_{j_1,j_2=1}^m L^{j_1} b^{k,j_2} I_{(j_1,j_2)}$$

$$+ \Sigma_{j_1,j_2,j_3=1}^m L^{j_1} L^{j_2} b^{k,j_3} I_{(j_1,j_2,j_3)} \quad \text{for}$$

$$k = 1,2,3, \ldots d. \quad (3.6)$$

3.3 The Order 2.0 Strong Taylor's Scheme

Higher order schemes could be derive by including more terms from the stochastic Taylor's expansion. Platen (2003) observed that the scheme becomes more complicated in the general case but reasonably simple for special cases. For the multidimensional case with d = 1, 2, 3, … with scalar noise, i.e., m = 1, the k^{th} component of the order 2.0 strong Taylor's scheme is given by:

$$Y_{n+1}^k = Y_n^k + \underline{a}^k \Delta_n + b^k \Delta W_n + \frac{1}{2!} \underline{L}^1 b^k (\Delta W_n)^2 + \underline{L}^1 \underline{a}^k \Delta Z_n$$

$$+ \frac{1}{2} \underline{L}^0 \underline{a}^k {\Delta_2}^2 + \underline{L}^0 b^k \{\Delta W_n \Delta_n - \Delta Z_n\}$$

$$+ \frac{1}{3!} \underline{L}^1 \underline{L}^1 b^k (\Delta W_n)^2 + \frac{1}{4!} \underline{L}^1 \underline{L}^1 \underline{L}^1 b^k (\Delta W_n)^4$$

$$\underline{+\, L^0} \underline{L}^1 b^k j_{(0,1,1)} + \underline{L}^1 \underline{L}^0 b^k j_{(1,0,1)}$$

$$+ \underline{L}^1 \underline{L}^1 \underline{a}^k j_{(1,1,0)} \quad (3.7)$$

Here the Gaussian random variable ΔW_n and ΔZ_n has the usual meaning.

3.4 Explicit Strong Convergence Schemes

One of the major disadvantage of the strong Taylor's approximations is that it requires that the derivatives of the various orders of the drift and diffusion coefficients must be determined and evaluated at each steps in addition to the coefficient themselves. The explicit strong schemes avoid the use of derivatives just like the Runge-Kutta method of the ordinary deterministic differential equations does, [8]. Below are the explicit strong convergence schemes for the simulation of stochastic differential equations.

3.5 Explicit Order 1.0 Strong Scheme

The explicit order 1.0 scheme for a 1-dimension case with d=m=1 proposed by Platen is given by

$$Y_{n+1} = Y_n + \underline{a}\Delta_n + b\Delta W_n + \frac{1}{2\sqrt{\Delta_n}}\{b(t_n, \bar{Y}_n) - b\}(\Delta W_n)^2$$

with the supporting value

$$\bar{Y}_n = Y_n + \underline{a}\Delta_n + b\sqrt{\Delta_n} \quad (3.9)$$

where $\underline{a}$ is the Stratonovich corrected drift. The ratio

$$\frac{1}{2\sqrt{\Delta_n}}\{b(t_n, Y_n + \underline{a}\Delta_n + b\sqrt{\Delta_n}) - b(t_n, Y_n)\}$$

is the forward difference for $b\frac{\partial b}{\partial x}$ at (t_n, Y_n). Hence, (3,8) is a derivative free counterpart of the Milstein scheme. The general multidimensional case for $d = m = 1, 2, 3, \ldots,$ for the explicit order 1.0 strong scheme has the kth component as:

$$Y_{n+1}^k = Y_n^k + \underline{a}^n \Delta_n + \sum_{j=1}^m b^{j,k} \Delta W_n^j$$

$$+ \frac{1}{2\sqrt{\Delta_n}} \sum_{j_1, j_2=1}^m \{b^{k,j_2}(t_n, \tilde{\Upsilon}_n^{j_1}) - b^{k,j_2}\} J_{j_1, j_2} \quad (3.10)$$

with the vector supporting values

$$\tilde{\Upsilon}_n^j = Y_n^k + \underline{a}^n \Delta_n + \frac{1}{2} \sum_{j=1}^m b^j \Delta W_n^j$$

There are various explicit order 1.0 schemes involving the Ito drift coefficient a. For instance for general noise, Platen proposed the scheme

$$Y_{n+1}^k = Y_n^k + a^n \Delta_n + \sum_{j=1}^m b^{j,k} \Delta W_n^j$$

$$+ \frac{1}{2\sqrt{\Delta_n}} \sum_{j_1, j_2=1}^m \{b^{j_2}(t_n, \tilde{\Upsilon}_n^{j_1}) - b^{j_2}\} J_{j_1, j_2}$$

with $\tilde{\Upsilon}_n^j = Y_n + \underline{a}^n \Delta_n + b^j \sqrt{\Delta_n}$

The above schemes all converge with order $\gamma = 1.0$ under conditions similar to that of the Milstein scheme.

3.6 Explicit Order 1.5 Strong Scheme

The explicit order 1.5 strong scheme results when the derivatives in the order 1.5 Taylor's scheme are replaced by the corresponding finite difference. For notational simplicity we shall simply state the scheme for the autonomous case.

In the autonomous 1-dimensional case $d = m = 1$ such an explicit order 1.5 scheme proposed by Platen (1980) has the form:

$$Y_{n+1} = Y_n + b\Delta W_n + \frac{1}{2\sqrt{\Delta}}\{a(\tilde{\Upsilon}_+) - a(\tilde{\Upsilon}_-)\}\Delta Z$$

$$+ \{a(\tilde{\Upsilon}_+) + 2a + a(\tilde{\Upsilon}_-)\}\Delta$$

$$+ \frac{1}{4\sqrt{\Delta}}\{b(\tilde{\Upsilon}_+) - b(\tilde{\Upsilon}_-)\}\{(\Delta W)^2 - \Delta\}$$

$$+ \frac{1}{2\sqrt{\Delta}}\{b(\tilde{\Upsilon}_+) - 2b + b(\tilde{\Upsilon}_-)\}\{\Delta W \Delta - \Delta Z\}$$

$$+ \frac{1}{4\sqrt{\Delta}}\{b(\tilde{\Phi}_+) - b(\tilde{\Phi}_-) - b(\tilde{\Upsilon}_+) + b(\tilde{\Upsilon}_-)\}$$

$$(\tfrac{1}{3}\{(\Delta W)^2 - \Delta\}\Delta W)$$

(3.11)

with $\tilde{\Upsilon}_\pm = \Upsilon_n + a\Delta \pm b\sqrt{\Delta}$ and $\tilde{\Phi}_\pm = \tilde{\Upsilon}_+ + b(\tilde{\Upsilon}_+)\sqrt{\Delta}$

Here ΔZ is the multiple Ito integral. I could be noted that a must be evaluated at three points and b at five supporting values for each time step.

In the general multi-dimensional autonomous case with $d = 1, 2, \ldots$ the *kth* component of the explicit order 1.5 strong Taylor's scheme due to Platern satisfies

$$Y_{n+1} = Y_n + a^k \Delta_n + \sum_{j=1}^m b^{k,j} \Delta W_n^j$$

$$+ \frac{1}{2\sqrt{\Delta_n}} \sum_{j_2=0}^m \sum_{j_1=1}^m \{b^{k,j_2}(\tilde{\Upsilon}_+^{j_1}) - b^{k,j_2}(\tilde{\Upsilon}_-^{j_1})\} I_{(j_1, j_2)}$$

$$+ \frac{1}{2\sqrt{\Delta_n}} \sum_{j_2=0}^m \sum_{j_1=1}^m \{b^{k,j_2}(\tilde{\Upsilon}_+^{j_1}) - 2b^{k,j_2} + b^{k,j_2}(\tilde{\Upsilon}_-^{j_1})\} I_{(0, j_2)}$$

$$+ \frac{1}{2\sqrt{\Delta_n}} \sum_{j_1, j_2, j_3=1}^m [b^{k,j_3}(\Phi_+^{j_1, j_2}) - b^{k,j_3}(\Phi_-^{j_1, j_2}) - b^{k,j_3}(\tilde{\Upsilon}_+^{j_1}) + b^{k,j_3}(\tilde{\Upsilon}_-^{j_1})] I_{(j_1, j_2, j_3)}$$

3.12)

with $\tilde{Y}_{\pm}^{j} = Y_n + \frac{1}{m} a\Delta_n \pm b^j \sqrt{\Delta_n}$ and $\Phi_{+}^{j_1,j_2} = \tilde{Y}_{\pm}^{j} \pm b^{j_2}(\tilde{Y}_{+}^{j_1})\sqrt{\Delta_n}$

where $b^{k,0}$ is written for a^k in the summation terms. The multiple Ito integral here can be approximated by multiple Stratonovich.

3.7 Explicit Order 2.0 Strong Scheme

The explicit order 1.0 and 1.5 are usually derived by replacing the derivatives in the Taylor's scheme by the equivalent finite difference. This procedure is good enough for low order explicit schemes. However, for higher order schemes, the procedure becomes too complicated. This can be avoided by taking advantage of some special structures of the equations under consideration to derive relatively simple higher order explicit schemes which do not involve the derivatives of the drift and the diffusion coefficients [8]. Here we shall consider a special case in which the noise term is additive. That is $b(x,t) = b(t)$ for all x and t.

For the autonomous 1-dimensional case $d = m = 1$ an explicit order 2.0 strong scheme for additive noise due to Chang (1987) has the form

$$Y_{n+1} = Y_n + \frac{1}{2}\{\underline{a}(\tilde{Y}_+) + \underline{a}(\tilde{Y}_-)\}\Delta + b\Delta W \quad (3.13)$$

with

$$\tilde{Y}_{\pm} = Y_n + \frac{1}{2}\underline{a}\Delta + \frac{1}{\Delta}b\left\{\Delta Z \pm \sqrt{J_{(1,1,0)}\Delta - (\Delta Z)^2}\right\}$$

where the random variable ΔW, ΔZ and $J_{(1,1,0)}$ can be approximated in the usual manner.

In the multi-dimensional case d = 1, 2, … with m = 1 the explicit order2.0strong scheme for scalar additive noise has kth component

$$Y_{n+1}^k = Y_n^k + \frac{1}{2}\left\{a^k\left(\frac{1}{2}\Delta_n, \tilde{Y}_+\right) + \underline{a}^k\left(\frac{1}{2}\Delta_n, \tilde{Y}_-\right)\right\}\Delta_n$$

$$+ b^k\Delta W_n + \frac{1}{\Delta_n}\{b^k(t_{n+1}) - b^k\}\{\Delta W_n\Delta_n - \Delta Z_n\} \quad (3.14)$$

with

$$\tilde{Y}_{\pm} = Y_n + \frac{1}{2}\underline{a}\Delta_n + \frac{1}{\Delta_n} b\{a\Delta Z_n \pm \sqrt{2J_{(1,1,0)}\Delta_n - (Z_n)^2}\}$$

for k = 1, … d.

3.8 Stochastic Multistep Schemes

From the experience of the deterministic ordinary differential equations, multistep schemes are more known to be computationally efficient that the one step scheme of the same order. Also in most cases, such multi step schemes are often more stable for larger time step. Some stochastic multistep methods are hare presented.

3.9 A Two-Step order 1.0 Strong Scheme

Though it is not easy to write out and investigate higher order multi-step schemes in the most general case, we can take advantage of the structure of certain types of stochastic differential equations to obtain relatively simple multi-step schemes. For example, consider the 2-dimensional Ito system

$$dX_t^1 = X_t^2 dt$$

$$dX_t^2 = \{-a(t)X_t^2 + b(t, X_t^1)\}dt + \sum_{j=1}^{m} c^j(t, X_t^1)dW_t^j \quad (3.15)$$

The Milstein scheme for (1) is of the form

$$Y_{n+1}^1 = Y_n^2 + Y_n^2\Delta$$

$$Y_{n+1}^2 = a(t_n)Y_n^2\Delta + b(t_n, Y_n^1\Delta) + \sum_{j=1}^{m} c^j(t_n, Y_n^1)\Delta W_n^j \quad (3.16)$$

and has a strong order 1.0. It can be noted that a simplifying feature of this scheme is the absence of the Ito double integral $I_{(j_1,j_2)}$. Moreover the first equation can be solved for

$$Y_n^2 = \frac{1}{\Delta}(Y_{n+1}^1 - Y_n^1)$$

and substituted into the second equation to obtain a two step scheme for the first component Y^1provided we use an equidistant discretization. This resulting two step scheme

$$Y_{n+2}^1 = \{2 - a(t_n)\Delta\}Y_{n+1}^1 - \{1 - a(t_n)\Delta\}Y_n^1 + b(t_n, Y_n^1)\Delta^2 + \sum_{j=1}^m c^j(t_n, Y_n^1)\Delta W_n^j \Delta \quad (3.17)$$

This scheme was due to Lépingale, D and Ribémont B (1991). Thus we have a multistep scheme for the first component of the approximation which is equivalent to the 2-dimensional Milstein's scheme for the system, [7].

3.10 A Two-Step order 1.5 Strong Scheme

For the 1-dimensional case d=m=1 we propose the two step order 1.5 strong scheme

$$Y_{n+1} = Y_{n-1} + 2a\Delta - a'(Y_{n-1})b(Y_{n-1})\Delta W_{n-1}\Delta + V_n + Y_{n-1} \quad (3.18)$$

With $V_n = b\Delta W_n + \left(ab' + \frac{1}{2}b^2b''\right)\{\Delta W_n\Delta - \Delta Z_n\} + a'b\Delta Z_n + \frac{1}{2}bb'\{(\Delta W_n)^2 - \Delta\}$

$$+\frac{1}{2}b(bb')'\left\{\frac{1}{3}(\Delta W_n)^2 - \Delta\right\}\Delta W_n$$

where the random variable ΔW_n, ΔZ_n are take as usual.

In the general multidimensional case d=m=1, 2, … we have in vector form the two-step order1.5 scheme

$$Y_{n+1} = Y_{n-1} + 2a\Delta - \sum_{j=1}^m L^j a(t_{n-1}, Y_{n-1})\Delta W_{n-1}^j + V_n + V_{n-1} \quad (3.19)$$

with

$$V_n = \sum_{j=1}^m [b^j \Delta W_n^j + L^0 b^j\{\Delta W_{n1}^j \Delta - \Delta Z_n^j\} + L^j a \Delta Z_n^j]$$

$$+ \sum_{j_1,j_2=1}^m L^{j_1} b^{j_2} I_{(j_1,j_2),t_n,t_{n+1}} + \sum_{j_1,j_2,j_3=1}^m L^{j_1} L^{j_2} b^{j_3} I_{(j_1,j_2,j_3),t_n,t_{n+1}}$$

where the Ito Integrals here can be approximated in the usual manner.

4.0 Implicit Strong Taylor's Schemes

The implicit strong Taylor's scheme is obtained by adapting the corresponding strong Taylor's schemes.

The implicit Euler Scheme is the simplest implicit strong scheme and it has a strong order $\gamma = 0.5$. In the one dimensional case $d = m = 1$ has the form

4.1 The Implicit Euler's Scheme

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + b\Delta W \quad (4.1)$$

where we $b = (t_n, Y_n)$. A family of implicit Euler schemes

$$Y_{n+1} = Y_n + \{\alpha a(t_{n+1}, Y_{n+1}) + (1-\alpha)a\}\Delta + b\Delta W$$

where the parameter $\alpha \epsilon [0,1]$ characterises the degree of implicitness.

In the general multidimensional case $d, m = 1,2, \ldots$ the family of implicit Euler schemes has the kth component

$$Y_{n+1}^k = Y_n^k + \{\alpha_k a^k(t_{n+1}, Y_{n+1}) + (1-\alpha_k)a^k\}\Delta + \sum_{j=1}^m b^{k,j}\Delta W^j \tag{4.32}$$

with the parameter $\alpha_k \epsilon [0,1]$ for $k = 1, \ldots, d$.

4.2 The Implicit Milstein Scheme

The implicit Milstein scheme is the counter part of the order 1.5 strong Milstein scheme earlier discussed. In the 1-dimensional case $d = m = 1$ it has the Ito version

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + b\Delta W + \frac{1}{2}bb'\{(\Delta W)^2 - \Delta\} \tag{4.3}$$

and the Stratonovich version

$$Y_{n+1} = Y_n + \underline{a}(t_{n+1}, Y_{n+1})\Delta + b\Delta W + \frac{1}{2}bb'(\Delta W)^2 \tag{4.4}$$

where $\underline{a} = a - \frac{1}{2}bb'$.

As with the Euler scheme, we can interpolate between the explicit and the implicit Milstein scheme of the same type. In the general multi-dimensional case d= m= 1,2, … with cumulative noise the Ito version of the family of implicit Milstein scheme with the kth component is

$$Y_{n+1}^k = Y_n^k + \{\alpha_k a^k(t_{n+1}, Y_{n+1}) + (1-\alpha_k)a^k\}\Delta + \sum_{j=1}^m b^{k,j}\Delta W^j$$

$$+ \frac{1}{2}\sum_{j_1,j_2=1}^m L^{j_1} b^{k,j_2}\{\Delta W^{j_1}\Delta W^{j_2} - \delta_{j_1 j_2}\Delta\} \tag{4.5}$$

where $\delta_{j_1 j_2}$ is the Kronecker delta symbol

$$\delta_{j_1 j_2} = \begin{cases} 1: j_1 = j_2 \\ 0: otherwise \end{cases}$$

and a Stratonovich version with kth component

$$Y_{n+1}^k = Y_n^k + \{\alpha_k \underline{a}^k(t_{n+1}, Y_{n+1}) + (1-\alpha_k)\underline{a}^k\}\Delta$$

$$+ \sum_{j=1}^m b^{k,j}\Delta W^j + \frac{1}{2}\sum_{j_1,j_2=1}^m L^{j_1} b^{k,j_2}\,\Delta W^{j_1}\Delta W^{j_2} \tag{4.6}$$

Here the parameter $\alpha \epsilon [0,1]$ indicates the degree of the implicitness of the kth component for k = 1, …, d. When $\alpha_k = 0$ we have the explicit Milstein scheme in the kth component, and the implicit Milstein scheme when it equals 1.0. The case $\alpha_k = 0.5$ gives an order $\gamma = 1.0$ generalization of the deterministic trapezoidal method, [7].

4.3 The Implicit Order 1.5 Strong Taylor's Scheme

The simplest version of the implicit order 1.5 strong Taylor's scheme for the autonomous case d=m=1is of the form

$$Y_{n+1} = Y_n + \frac{1}{2}\{a(Y_{n+1}) + a\}\Delta + \left(ab' + \frac{1}{2}b^2b''\right)\{\Delta W\Delta - \Delta Z\}$$
$$+a'b\left\{\Delta Z - \frac{1}{2}\Delta W\Delta\right\} + \frac{1}{2}bb'\{(\Delta W)^2 - \Delta\}$$
$$+ \frac{1}{2}b(bb')'\left\{\frac{1}{3}(\Delta W)^2 - \Delta\right\}\Delta W \qquad (4.7)$$

where ΔW and ΔZ are Gaussian random variables with zero mean and $E((\Delta W)^2) = \Delta, E((\Delta Z)^2) = \frac{1}{3}\Delta^3$ and $E(\Delta W\Delta Z) = \frac{1}{2}\Delta^2$.

In the general multi-dimensional case we obtain the family of implicit order 1.5 strong Taylor's schemes with the kth components

$$Y_{n+1}^k = Y_n^k + \{\alpha_k a^k(t_{n+1}, Y_{n+1}) + (1-\alpha_k)a^k\}\Delta + \left(\frac{1}{2} - \alpha_k\right)\{\beta_k L^0 a_k(t_{n+1}, Y_{n+1}) + (1 -$$
$$\beta_k)L^0 a_k\}\Delta^2$$
$$+ \sum_{j=1}^{m}(b^{k,j}\Delta W^j + L^o\, b^{k,j} I_{(0,j)} + L^j a^k\{I_{(0,j)} - a_k \Delta W^j\Delta\})$$
$$+ \sum_{j_1,j_2=1}^{m} L^{j_1} b^{k,j_2} I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=1}^{m} L^{j_1} L^{j_2} b^{k,j_3} I_{(j_1,j_2,j_3)} \qquad (4.8)$$

where the parameters $\alpha_k, \beta_k \in [0,1]$ for $k = 1, \dots, d$.

4.4 The Implicit Order 2.0 Strong Taylor's Scheme

In the 1-dimensional case $d = m = 1$, the Stratonovich version has the form

$$Y_{n+1} = Y_n + \frac{1}{2}\{\underline{a}(Y_{n+1}) + \underline{a}\}\Delta + b\Delta W + \underline{a}b'\{\Delta W\Delta - \Delta Z\} + \underline{a}'b\left\{\Delta Z - \frac{1}{2}\Delta W\Delta\right\}$$
$$+\frac{1}{2}bb'(\Delta W)^2 - \frac{1}{3!}b(bb')'(\Delta W)^2 + \frac{1}{4!}(b(bb')')'(\Delta W)^4 + \underline{a}(bb')'J_{(0,1,1)} + b(b')'J_{(1,0,1)}$$
$$+b(\underline{a}'b)'\left\{J_{(1,1,0)} - \frac{1}{4}(\Delta W)^2\Delta\right\} \qquad (4.9)$$

The multiple integrals appearing here can be approximated in the usual manner.

In the general multi-dimensional case d=m=1, 2, … the kth component of the implicit order 2.0 strong Taylor's scheme is given by

$$Y_{n+1}^k = Y_n^k + \{\alpha_k \underline{a}^k(t_{n+1}, Y_{n+1}) + (1-\alpha_k)\underline{a}^k\}\Delta$$
$$+ \left(\frac{1}{2} - \alpha_k\right)\{\beta_k \underline{L}^0 \underline{a}^k(t_{n+1}, Y_{n+1}) + (1-\beta_k)\underline{L}^0\underline{a}^k\}\Delta^2$$
$$+ \sum_{j=1}^{m}[(b^{k,j}\Delta W^j + \underline{L}^o\, b^{k,j} I_{(0,j)} + \underline{L}^j \underline{a}^k\{I_{(j,0)} - a_k\Delta W^j\Delta\})]$$

$$+\sum_{j_1,j_2=1}^{m}[\ \underline{L}^{j_1} b^{k,j_2} I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=1}^{m} \underline{L}^{0} L^{j_1} b^{k,j_3} I_{(0,j_1,j_2)}$$

$$+\underline{L}^{j_1}\underline{L}^{0} b^{k,j_3} I_{(j_1,0,j_2)} + \underline{L}^{j_1},\underline{L}^{j_2}\underline{a}^{k}\{J_{(j_1,j_2,0)} - \alpha_k \Delta J_{(j_1,j_2)}\}]$$

$$+\sum_{j_1,j_2=1}^{m} \underline{L}^{j_1}\ \underline{L}^{j_2} b^{k,j_3} J_{(j_1,j_2,j_3)} + \sum_{j_1,j_2,j_3,j_4=1}^{m} \underline{L}^{j_1}\ \underline{L}^{j_2}\ \underline{L}^{j_3} b^{k,j_4} J_{(j_1,j_2,j_3,j_4)} \qquad (4.10)$$

where the parameters $\alpha_k, \beta_k \in [0,1]$ for $k = 1, \dots, d$.

5.0 Numerical Experiments

Computer has remained a vital tool for solving mathematical problems ever since its introduction. In other to do justices to this work, MATLAB which is designed for high level numerical simulation and visualization. Whereas MAPLE is ideally suited for manipulations with stochastic calculus and stochastic differential equations particularly in the aspect of derivation of numerical schemes for SDEs and finding analytic solutions where it exist. Higham D. J et.al (2001) pointed out that although numerical simulations are possible in MAPLE, it is more convenient in MATLAB. Conversely, although MATLAB permits symbolic computations through its symbolic Math Toolbox, theses facilities are only a subset of those offered in Maple. Consequently, MATLAB was chosen as the software toll because the interest of this work is numerical solutions of the stochastic differential equations rather than the symbolic solution.

We considered the SDE

$$dX_t = \mu_t dt + \sigma_t dW_t ; X(0) = 5;\ t_0 = 0 \le t \le 10 \qquad (5.1)$$

A Matlab scripts file that implements the Euler and Milstein schemes were written, compiled and implemented. Below is the result of the simulation

.

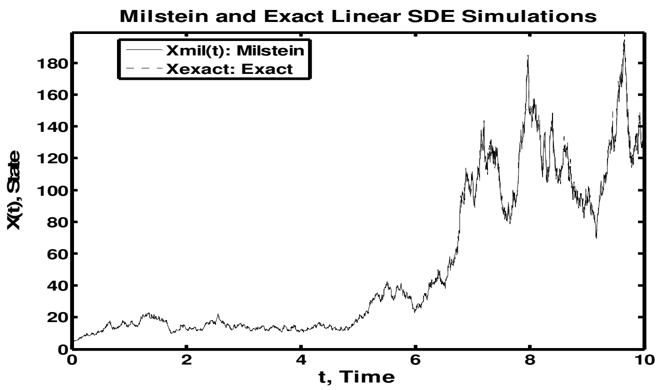

Figure 1: Milstein and Euler Simulation of SDE

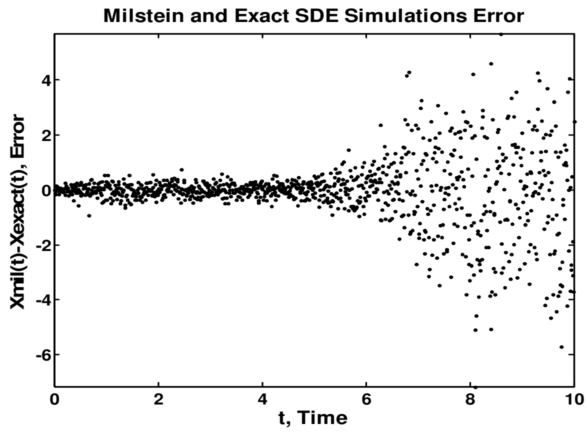

Figure 3: Milstein and Exact Simulation Error

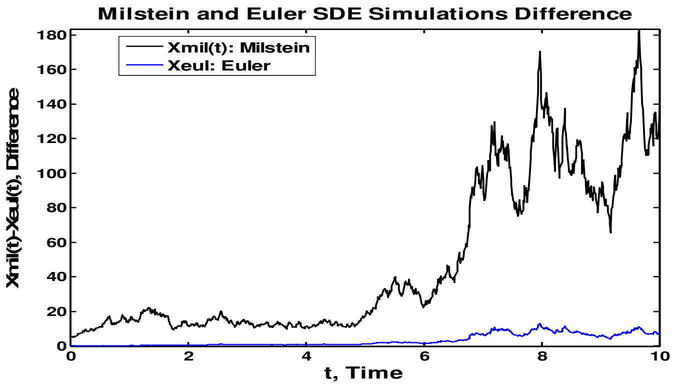

Figure 2: Milstein and Euler Simulation of SDE

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