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A NOTE ON THE POCHHAMMER FREQUENCY EQUATION

by

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ABSTRACT

The Pochhammer frequency equation for both a convex and a concave cylinder is analysed in the limiting case when the propagation constant tends to zero. It is found that below a certain value of the radius the displacements involve a combination of ordinary and modified Bessel functions instead of just one type of Bessel functions. This value of the radius is calculated and displayed for various materials.

Keywords : Pochhammer, propagation constant.

INTRODUCTION

The propagation of time-harmonic waves in an elastic circular cylinder of infinite extent is governed by the Pochhammer equation. The traditional derivation of this equation is through the Helmholtz decomposition of the Navier equation of classical elasticity, (Kolsky, 1963; Achenbach, 1984). The displacements are first worked out, and imposing the condition of no stress at the surface then yields the dispersion relation or frequency equation known as the Pochhammer frequency equation. Owing to its complexity only limiting cases are usually considered. Thus, for the case when the wavelength becomes small, Bancroft (1941) has shown that the phase velocity approaches that of Rayleigh surface waves.

In this note we obtain the Pochhammer equation, without using the Helmholtz decomposition, for both the interior and exterior problems (i.e., for both a convex cylinder and a concave cylinder or cylindrical cavity) and consider the limiting case when the radius and wave number are both small. It is found that there exists a value of the radius below which the displacements involve a combination of the ordinary and modified Bessel functions, instead of just the latter. The critical radius at which this “flip-over” occurs is calculated for different materials and displayed graphically.

THE GOVERNING EQUATIONS

The starting point for the treatment of elastic modes of propagation in a circular cylinder in the absence of body forces is the integration of the Navier equation

$$(\lambda + 2\mu)\nabla\ \nabla\mathbf{.U} - \mu\nabla \wedge \nabla \wedge \mathbf{U} = \rho\ddot{\mathbf{U}}, \qquad \textbf{(1)}$$

where **U** is the displacement vector, λ and μ are the Lamé constants and ρ is the mass density. We work in cylindrical polar coordinates (R, θ, Z), and denote the components of **U** by *U, V, W,* and the radius of the cylinder by *A*. We shall consider only longitudinal waves characterized by the presence of only the displacement components *U and W,* both with symmetry about the *z*-axis so that there is no θ dependence. It is also convenient to non-dimensionalize the variables in order that the problem has the smallest number of parameters overtly involved. The dimensionless variables are introduced as follows :

$$T = t/\omega, \quad (A, R, U, W, Z) = \sqrt{\frac{\mu}{\rho\omega^2}}\,(a, r, u, w, z),$$

where ω is the angular frequency of the wave, which is considered to be imposed in this problem. We also introduce a material parameter α defined by

$$\alpha = \mu/(\lambda + 2\mu)\ ,$$

which is related to Poisson's ratio ν by

$$\alpha = \frac{1-2\nu}{2-2\nu}\ .$$

We note that $0 \le \alpha \le 0.5\,\cdot$

In terms of these dimensionless variables the governing equations are

$$, \qquad \textbf{(2a)}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \alpha\frac{\partial^2 w}{\partial z^2} + (1-\alpha)\left(\frac{\partial^2 u}{\partial r\partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) = \alpha\frac{\partial^2 w}{\partial t^2}\ . \qquad \textbf{(2b)}$$

The conditions for a traction-free surface become

$$\frac{\partial u}{\partial r} + (1-2\alpha)\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right) = 0\ , \qquad \textbf{(3a)}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0\ \cdot \qquad \textbf{(3b)}$$

THE FREQUENCY EQUATION FOR THE INTERIOR PROBLEM

We are interested in the propagation of an infinite train of sinusoidal waves along the cylinder such that the displacement at each point is a simple harmonic function of z as well as of t. Accordingly, we seek solutions of the form

$$u = U \exp[i(kz - t)], \qquad \textbf{(4a)}$$

, **(4b)**

where the amplitudes U and W are functions of r only. Substitution of eqns. (4) into eqns. (2) results in the following coupled differential equations :

, **(5a)**

$$\alpha r^2 \frac{d^2W}{dr^2} + \alpha r \frac{dW}{dr} - p^2 r^2 W = -ik(1-\alpha)[r^2 \frac{dU}{dr} + rU], \qquad \textbf{(5b)}$$

where $p^2 = k^2 - \alpha$, and $q^2 = k^2 - 1$.

Eqns. (5) are of the modified Bessel type and since for the interior problem the field variables should be finite at the centre of the cylinder, their solutions involve only I_0 and I_1, the modified Bessel functions of the first kind. It is easily shown that the displacements are given by

$$u = \exp[i(kz - t)][C_0 I_1(pr) + C_1 I_1(qr)], \qquad \textbf{(6a)}$$

$$w = i \exp[i(kz - t)]\left[\frac{k}{p} C_0 I_0(pr) + \frac{q}{k} C_1 I_0(qr)\right], \qquad \textbf{(6b)}$$

where C_0, C_1 are arbitrary constants.

The boundary conditions (3) then yield a system of homogeneous equations which for consistency results in the Pochhammer frequency equation :

$$2p \frac{I_1(pa)}{I_0(pa)} \frac{I_1(qa)}{I_0(qa)} + a(2k^2 - 1)^2 \frac{I_1(qa)}{I_0(qa)} - 4ak^2 pq \frac{I_1(pa)}{I_0(pa)} = 0. \qquad \textbf{(7)}$$

We now analyse eqn. (7) in the limiting case when the propagation constant *ka* is small. To this end, we pose a perturbation expansion for *k* of the form

$$k = k_0 + a^2 k_1 + \mathrm{O}(a^4).$$

Using expansions of Bessel functions for small arguments and the above *k* in eqn. (7)

we obtain

$$k \sim \sqrt{\frac{1-\alpha}{3-4\alpha}}\left[1+\frac{(1-2\alpha)^2}{16(1-\alpha)(3-4\alpha)}\,a^2+\ldots\right], \qquad ka \to 0\,. \tag{8}$$

Eqn. (8) shows that k_0 and k_1 are always less than one so that there is a value of a below which q becomes imaginary; in other words there is a critical radius a_c below which the modified Bessel functions get changed into the standard ones.
To determine a_c we examine the frequency equation as k reduces towards unity; accordingly, we set the following expansion for k :

$$k = 1+\varepsilon + \mathrm{K}\,, \quad \varepsilon \to 0\,.$$

Upon substitution into Eqn. (7) we obtain

$$\alpha = 1 - \tfrac{1}{6} X \frac{I_0(X)}{I_1(X)}\,, \tag{9}$$

where $X = a\sqrt{1-\alpha}$. By assigning suitable values to X, α and hence a_c can be obtained. The variation of a_c with the material parameter α is displayed in the graph below (Fig. 1). Hence for values of a less than a_c , and of k greater than α, the displacements as given in eqns. (6) then involve a combination of modified and ordinary Bessel functions of the first kind.

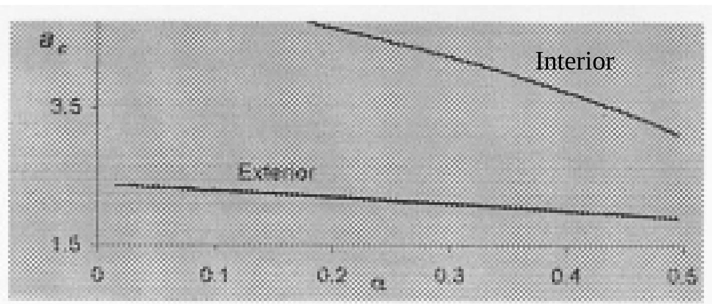

Fig. 1. Variation of the critical radius a_c with the material parameter α.

THE FREQUENCY EQUATION FOR THE EXTERIOR PROBLEM

For the exterior problem the solutions must be bounded at infinity so that the displacements now involve K_0 and K_1, the modified Bessel functions of the second kind. The corresponding frequency equation for longitudinal waves becomes

$$2p\,\frac{K_1(pa)}{K_0(pa)}\,\frac{K_1(qa)}{K_0(qa)} - a(2k^2-1)^2\,\frac{K_1(qa)}{K_0(qa)} + 4ak^2pq\,\frac{K_1(pa)}{K_0(pa)} = 0\,. \qquad \textbf{(10)}$$

By using a similar analysis as above the critical radius a_c can be determined from the following equation

$$\alpha = 1 - \tfrac{1}{2}X\,\frac{K_0(X)}{K_1(X)}\,.$$

The variation of a_c with α is displayed in Fig. 1. We note that for a particular material the critical radius is much smaller for the concave cylinder than for the convex one.

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