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Fixed Point Theorems and Sensitivity Analysis in Solving Nonlinear Matrix Equations: A Study Involving Partially Ordered Sets

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Abstract

Nonlinear matrix equations of the form $X - A^*X^{\alpha}A = P$, where X represents an unknown matrix, and A and P are given square matrices, are encountered in various fields. Understanding the characteristics and behaviour of these equations is essential for developing efficient computational methods and obtaining reliable solutions. This paper explores the use of partially ordered sets within fixed point schemes to solve the targeted equations. Furthermore, it derives perturbation estimates of the solutions and evaluates them computationally. Finally, the numerical simulations are provided to validate our theoretical claims and affirm the effectiveness of the proposed fixed-point scheme.

Keywords: Fixed point, partially ordered sets, Nonlinear matrix equation, Positive definite solution, Perturbation

Introduction

In this paper, we investigate the nonlinear matrix equation:

$$X - A^*X^{\alpha}A = P, \quad (1)$$

where P and A are square positive definite matrices, and $\alpha \in (0,1)$. Over the past decades, extensive research has been conducted on matrix equations similar to (1) (see Engwerda et al. 1993, El-Sayed and Ramadan 2001, Ran and Reurings 2002, Ramadhan and El-Shazly 2006, Gao and Yang 2008, Duan and Liao 2009 and the references therein). These equations are critically important across multiple scientific and engineering fields, including control theory, optimization, and signal processing (Chacha and Kim 2019; Chacha 2021a, 2021b). Understanding the behaviour and properties of nonlinear matrix equations is crucial for developing effective computational methods and obtaining reliable solutions. Notably, the sensitivity of these equations to perturbations in their input parameters has received considerable attention (Chacha 2022a, Chacha 2022b).

In this paper, we investigate the intriguing realm of partially ordered sets (POS) within the context of fixed-point schemes for nonlinear matrix equations. Partially ordered sets provide a mathematical framework for characterizing relationships and dependencies among elements in a set. By incorporating POS concepts into fixed point schemes, we aim to enhance our understanding of the underlying dynamics and intricacies of nonlinear matrix equations (Ran and Reurings 2004).

Our primary objective is to explore the sensitivity of a specific family of nonlinear matrix equations using the framework of partially ordered sets. Sensitivity analysis plays a crucial role in assessing the stability and robustness of mathematical models, enabling us to evaluate the impact of parameter variations on the system's overall behaviour. By incorporating partially ordered sets into the sensitivity analysis, we aim to unveil the interplay between order structures and the sensitivity of nonlinear matrix equations.

To achieve this goal, we adopt a comprehensive approach that combines theoretical analysis, numerical simulations, and illustrative examples. Through rigorous investigation, we seek to uncover new insights into the behaviour and properties of nonlinear matrix equations, shedding light on the role of partially ordered sets in their fixed-point schemes.

Furthermore, previous studies have examined perturbation estimates of matrix equations similar to equation (1). For example, Hasanov and Ivanov 2004 conducted research on the solution and perturbation estimates for the matrix equation $X \pm A^{*}X^{-n}A = Q$, as studied by Hasanov 2010. Additionally, El-Shazly 2016 investigated the perturbation estimates of the maximal solution of the matrix equation $X + A^{T}\sqrt{X^{-1}}A = P$.

To the best of our knowledge, equation (1) for the case $\alpha \epsilon (0,1)$ and positive definite matrix A has not been done using the concept of fixed points in partially ordered sets.

The following notations will be used throughout this paper: M_n stands for a set of positive definite matrices; $N > 0$ means N is a positive definite matrix; $N \geq 0$ means N is a positive semi-definite matrix. If $N, M \in M_n$ such that $N \leq M$, then $[N, M]$ represents a set of all matrices $R \in M_n$ such that $N \leq R \leq M$; A^{T} is the transpose of matrix A; A^{*} stands for transpose if A is real and complex conjugate transpose of A in the complex case; $\rho(A)$ stands for spectral radius o matrix A; $\|\blacksquare\|$ is spectral norm, where $\|A\|^2 = \lambda_{\max}(A^{*}A)$ and $\lambda_{\max}(A^{*}A)$ stands for maximum eigenvalue of $A^{*}A$.

The remainder of this paper is organized as follows: The next section, titled Preliminaries, provides a brief overview of the foundational concepts and theoretical background related to nonlinear matrix equations, partially ordered sets, and fixed-point schemes. Following the preliminaries, we present our methodology for incorporating partially ordered sets into the sensitivity analysis of the considered family of equations. This is followed by a section presenting the results of our numerical simulations and discussing the insights gained from the analysis. The final section concludes the paper by summarizing our findings.

Preliminaries

In this section, we provide some lemmas that we will be applied in the next sections.

Lemma 1: (Bhatia 1997) If $0 < \theta \leq 1$, and P and Q are positive definite matrices of the same order with $P, Q \geq bI > 0$, then

$$\left\|P^{\theta} - Q^{\theta}\right\| \leq \theta b^{\theta-1}\|P - Q\|,$$

and

$$\left\|P^{-\theta} - Q^{-\theta}\right\| \leq \theta b^{-(\theta+1)}\|P - Q\|.$$

Lemma 2: (Horn and Johnson 2012) If $A \geq B > 0$ $(A > B > 0)$, then $A^{\alpha} \geq B^{\alpha} > 0$ (or $A^{\alpha} > B^{\alpha}$) for all $\alpha \in (0,1]$, and $B^{\alpha} \geq A^{\alpha} > 0$ (or $B^{\alpha} > A^{\alpha}$) for all $\alpha \in [-1, 0)$.

Lemma 3: Let S be a partially ordered set such that every pair $x, y \in S$ has a lower bound and an upper bound. Furthermore, let d be a metric on S such that (S, d) is a complete metric space. If F is a continuous monotone (that is either order preserving or order reserving) map from S into S such that there exists

(1) $c \in (0,1)$: $d\big(F(x), F(y)\big) \leq cd(x, y)$ $x \geq y$, and

(2) $x_0 \in S : x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$, then F has a unique fixed point x^{*} in S. Moreover, for every $x \epsilon S$,

$$\lim_{n\to\infty} F^{n} = x^{*}.$$

Proof: The proof is the same as in Ran and Reurings 2004. So, it is omitted here.

Methodology

In this section, we will employ Lemma 3, along with the concept of partially ordered sets, to demonstrate the existence of a fixed point that serves as a solution to equation (1). By employing these techniques, we are able to establish a rigorous framework for finding a solution to equation (1), thereby enhancing our understanding of its properties and implications. From equation (1), we have

$F(X) = X - A^*X^\alpha A - P.$ (2)

Then, from (2), we define a map $\mathcal{F}(X) = P + A^*X^\alpha$A.

We make the assumption that both $\mathcal{F}$ and X^α are well-defined in M_n and exhibit order-preserving properties. Since P is a positive definite matrix, it follows that $\mathcal{F}(X) \geq P$ for all $X(k) \in M_n$, k=0, 1, 2, $\cdots$. Notably, the matrix sequence $\{\mathcal{F}^k(P)\}_{k=0}^{\infty}$ is observed to be increasing, as $\mathcal{F}(P) \geq P$.

Thus, we ascertain that condition 2 of Lemma 3 is satisfied. However, to guarantee the existence of a unique fixed point, we must establish an additional condition on $\mathcal{F}$, specifically pertaining to condition 1 of Lemma 3. This condition will play a crucial role in our analysis, enabling us to determine the presence of a solitary fixed point that fulfils the desired criteria.

Theorem 1: Suppose there exists an initial guess $X(0)$ such that $\mathcal{F}(X(0)) \leq X(0)$.Then, $\mathcal{F}$ maps the set $S = [P, X(0)]$ into itself, the limit $X^- = lim_{k\to\infty} \mathcal{F}^k(P)$ exists and it is the minimal solution of equation (1). Moreover, the sequence $\{\mathcal{F}^k(X(0))\}_{k=0}^{\infty}$ decreases to a solution X^+ which is the maximal solution in S.

Proof: From the assumption that $\mathcal{F}(X(0)) \leq X(0)$, we have inequality $P \leq \mathcal{F}(X(0) \leq X(0)$. A recursive application of the map $\mathcal{F}$, the matrix sequence $\{\mathcal{F}^k(P)\}_{k=0}^{\infty}$ increases and it is bounded above by $\mathcal{F}^m(X(0))$ for any m. Moreover, the matrix sequence $\{\mathcal{F}^k(X(0))\}_{k=0}^{\infty}$ is decreasing and bounded below by P. Therefore, both sequences converge.

Now, let $X^{\text{sol.}}$ be the solution of equation (1). Then, $P \leq X^{\text{sol.}} = \mathcal{F}(X^{\text{sol.}})$, and a recursive application of $\mathcal{F}$ yield $X^- \leq X_{\text{sol}}$. If $X \in S$, that is if $X_{\text{sol}}. \leq X(0)$, we see also that a recursive application of the map $\mathcal{F}$, yield $X_{\text{sol}}. \leq X^+$.

The Fixed Point Algorithm

The following steps are employed in this algorithm:

(1) Choose $X_0 \geq P$;

(2) $X(k+1) = \mathcal{F}(X(k)) \quad \forall k = 0, 1, 2, \cdots$;

(3) Check if $\|F(X_k)\|_{\text{F}} \leq n.$eps, then stop, otherwise go step 2;

(4) Display the solution X.

Theorem 2: Unique Fixed Point Existence in Partially Ordered Sets

Let $P \in S \subset M_n$, where S represents a subset of M_n. Suppose there exists a positive number $\omega = ab^{\alpha-1}\|A\|^2$ satisfying $0 < \omega < 1$, such that $X \leq Y$ for any $X, Y \in S$.

Under these conditions, we assert that equation (1) possesses a solitary fixed point within the set S, and that this fixed point is unique in M_n.

This theorem establishes a significant result regarding the existence and uniqueness of a fixed point in a given partially ordered set. The condition involving ω provides a crucial constraint that guarantees the presence of a single solution to equation (1). Moreover, the ordered relationship between elements in the set S further strengthens the theorem's assertion by ensuring the uniqueness of this fixed point across the entire matrix space M_n. Here follows the proof.

Proof: Let $X, Y \in S, \mathcal{F}(X) = P + A^*X^\alpha A$ and $\mathcal{F}(Y) = P + A^*Y^\alpha A$. Then,

$$\begin{aligned}\|\mathcal{F}(Y) - \mathcal{F}(X)\| &= \|A^*Y^\alpha A - A^*Y^\alpha A\| \\ &= \|A^*Y^\alpha A - A^*Y^\alpha A\| \\ &\leq \|A^2\|\|Y^\alpha - X^\alpha\| \\ &\leq ab^{\alpha-1}\|A^2\|\|Y - X\|.\end{aligned}$$

Therefore, condition 1 of Lemma 3 is satisfied and $\mathcal{F}$ has a fixed point in S.

Given that $\alpha b^{\alpha-1}\|A\|^2 = \omega \in (0,1)$ belongs to the interval (0, 1), we can conclude that condition 1 of Lemma 3 is indeed satisfied. As a result, we can assert that the function $\mathcal{F}$ has a fixed point within the subset $S = [P, X(0)] \subset M_n$.

This statement highlights the significance of ω falling within the range (0,1) as it establishes the fulfilment of condition 1 of Lemma 3, which is crucial for guaranteeing the existence of a fixed point. By defining the subset S as $[P, X(0)]$, we effectively narrow down the search space for the fixed point within the matrix space M_n. Consequently, we can confidently conclude that $\mathcal{F}$ possesses a unique fixed point within this specified subset.

Now, let's consider a sequence of matrices:

$$X(0) = P,$$
$$X(k+1) = P + A^*X(k)^\alpha A, \quad \text{for all } k = 0, 1, 2, \cdots.$$

By iterating this sequence, we obtain a series of matrices where each term is obtained by applying a specific formula involving the previous term. This formulation allows us to explore the evolution of the matrix sequence and study its properties.

$$\begin{aligned}\|X(k+1) - X_{\text{sol.}}\| &= \|P + A^*X(k)^\alpha A - (P + A^*X_{\text{sol.}}{}^\alpha A)\|,\\ &= \|A^*X(k)^\alpha A - A^*X_{\text{sol.}}{}^\alpha A\|,\\ &\le \alpha b^{\alpha-1}\|A^2\|\|X(k) - X_{\text{sol.}}\|.\end{aligned}$$

Thus, as k approaches infinity, the matrix sequence $X(k+1)$ converges linearly to a unique fixed point, denoted as $X_{\text{sol.}}$. This fixed point represents the solution of equation (1), signifying the equilibrium state of the iterative process. The convergence of the sequence implies that as the number of iterations increase, the values of $X(k)$ gradually approach and stabilize at $X_{\text{sol.}}$, providing valuable insights into the behavior and properties of the system described by equation (1).

Perturbation Estimates of the Solution for the Nonlinear Matrix Equation $X - A^*X^\alpha A = P$

In this section, our focus shifts towards examining perturbation estimates associated with the matrix equation (1). By analysing the effects of perturbations on the equation, we gain valuable insights into the stability and robustness of the maximal solution. Through this investigation, we aim to quantify the extent to which small variations or disturbances in the system parameters impact the solution. These perturbation estimates shed light on the sensitivity and reliability of the solution in practical scenarios, enabling a deeper understanding of the equation's behaviour and its implications.

Lemma 4: Suppose A is a non-singular matrix. Let $\lambda_{\max}(A^*A)$ denote the largest eigenvalue of A^*A , $X_{\text{sol.}}$ be the maximal positive definite solution of the matrix equation (1), and P be a given matrix satisfying $\lambda_{\max}(A^*A)X^\alpha < P$. Under these conditions, the following conclusions hold:

(i) $P \le X_{\text{sol.}} < 2P$, that is the maximal solution $X_{\text{sol.}}$ lies within the interval $[P, 2P)$. In other words, $X_{sol.}$ is bounded above by $2P$ and is greater than or equal to P.

(ii) $\rho(A) < \sqrt{\frac{\|P\|}{\|X^\alpha\|}}$, meaning that the spectral radius of A, denoted as $\rho(A)$, satisfies the inequality $\rho(A) < \sqrt{\frac{\|P\|}{\|X^\alpha\|}}$. Hence, the largest absolute value of any eigenvalue of A is strictly less than the square root of the ratio between the norms of P and X^α.

By establishing these relationships, Lemma 4 provides important insights into the properties and bounds of the maximal positive definite solution $X_{\text{sol.}}$ in relation to the given matrix equation (1) and its parameters.

Proof: From equation(1), we have

$$\begin{aligned}X(k) &= P + A^*X^\alpha(k)A,\\ X_{\text{sol.}} &= P + A^*X_{\text{sol.}}^\alpha A,\\ X_{\text{sol.}} &\le P + \lambda_{\max}(A^*A)X_{\text{sol.}}^\alpha < 2P.\end{aligned}$$

It is easy to see that $P \le X_{\text{sol.}}$. Also, from the assumption that $\lambda_{\max}(A^*A)X^\alpha < P$, and the fact that $\rho(A) = \lambda_{\max}(A)$, for Hermitian positive definite matrix A ,we get $\rho(A) = \|A\| < \sqrt{\frac{\|P\|}{\|X^\alpha\|}}$.

This concludes the proof.

Theorem 2: Suppose that $\|A\| < \sqrt{\frac{\|P\|}{\|X^{\alpha}\|}}$, $\frac{\|P\|}{\|X_{\text{sol.}}\|} \leq 1$, and $X_{\text{sol.}}$ is the maximal solution of equation (1) . Then, $\frac{\|\Delta X_{\text{sol.}}\|}{\|X_{\text{sol.}}\|} \leq \frac{1}{\beta}\left(\frac{\|\Delta P\|}{\|P\|} + \frac{2\|\Delta A\|}{\|A\|}\right)$.

Proof: Let $\Delta A = \tilde{A} - A$, $\Delta X_{\text{sol.}} = \widetilde{X_{\text{sol.}}} - X_{\text{sol.}}$, and $\Delta P = \tilde{P} - P$.

Then,

$$\begin{aligned}
\triangle P &= \tilde{P} - P \\
&= \widetilde{X_{\text{sol.}}} - \widetilde{A^*}\widetilde{X_{\text{sol.}}}^{\alpha}\tilde{A} - (X_{\text{sol.}} - A^* X_{\text{sol.}}^{\alpha} A) \\
&= \Delta X_{sol.} - (A + \Delta A)^* \widetilde{X_{\text{sol.}}}^{\alpha}(A + \Delta A) + A^* X_{\text{sol.}}^{\alpha} A \\
&= \Delta X_{\text{sol.}} - A^*\widetilde{X_{\text{sol.}}}^{\alpha} A - A^*\widetilde{X_{\text{sol.}}}^{\alpha}\Delta A - \Delta A^*\widetilde{X_{\text{sol.}}}^{\alpha} A - \Delta A^*\widetilde{X_{\text{sol.}}}^{\alpha}\Delta A + A^* X_{\text{sol.}}^{\alpha} A \\
&= \Delta X_{\text{sol.}} - A^*(\widetilde{X_{\text{sol.}}}^{\alpha} - X_{\text{sol.}}^{\alpha})A - A^*\widetilde{X_{\text{sol.}}}^{\alpha}\Delta A - \triangle A^*\widetilde{X_{\text{sol.}}}^{\alpha} A. \qquad (3)
\end{aligned}$$

Because both $\Delta A^* \to 0$ and $\Delta A \to 0$ in equation (3), the term $\Delta A^*\widetilde{X_{\text{sol.}}}^{\alpha}\Delta A$ is neglected.

Now, for convenience, let $N = A^*\left(\widetilde{X_{\text{sol.}}}^{\alpha} - X_{\text{sol.}}^{\alpha}\right)A$ and $H = A^*\widetilde{X_{\text{sol.}}}^{\alpha}\Delta A - \triangle A^*\widetilde{X_{\text{sol.}}}^{\alpha} A$, we have,

$$\|\triangle P + N + H\| = \|\Delta X_{\text{sol.}}\|. \qquad (4)$$

Now, from equation (4), we have

$$\begin{aligned}
\|\Delta P\| &\geq \|\Delta X_{\text{sol.}}\| - \|A\|^2\alpha b^{\alpha-1}\|\Delta X_{\text{sol.}}\| - 2\|A\|\|\Delta A\|\left\|\widetilde{X_{\text{sol.}}}^{\alpha}\right\|, \\
&= \|\Delta X_{\text{sol.}}\|(1 - \|A\|^2\alpha b^{\alpha-1}) - 2\|A\|\|\Delta A\|\left\|\widetilde{X_{\text{sol.}}}^{\alpha}\right\|.
\end{aligned}$$

$$\|\Delta X_{sol.}\| \leq \frac{1}{(1 - \|A\|^2\alpha b^{\alpha-1})}(\|\Delta P\| + 2\|A\|\|\Delta A\|\left\|\widetilde{X_{\text{sol.}}}^{\alpha}\right\|.$$

$$\frac{\|\Delta X_{\text{sol.}}\|}{\|X_{\text{sol.}}\|} \leq \frac{1}{1 - \|A\|^2\alpha b^{\alpha-1}}\left(\frac{\|\Delta P\|}{\|P\|}\frac{\|P\|}{\|X_{\text{sol.}}\|} + \frac{2\|\Delta A\|\left\|\widetilde{X_{\text{sol.}}}^{\alpha}\right\|}{\|A\|}\frac{\|A\|^2}{\|X_{\text{sol.}}\|}\right).$$

Recall that $\rho(A) = \|A\| < \sqrt{\frac{\|P\|}{\|X^{\alpha}\|}}$ and $\frac{\|P\|}{\|X_{\text{sol.}}\|} \leq 1$. So, we have

$$\|A\|^2 < \frac{\|P\|}{\left\|\widetilde{X_{\text{sol.}}}^{\alpha}\right\|}.$$

Then, $$\frac{\|\Delta X_{\text{sol.}}\|}{\|X_{\text{sol.}}\|} \leq \frac{1}{\beta}\left(\frac{\|\Delta P\|}{\|P\|} + \frac{2\|\Delta A\|}{\|A\|}\right),$$

where $\beta = 1 - \|A\|^2\alpha b^{\alpha-1} > 0$.

This completes the proof.

Numerical Experiments

In this section, we present two illustrative examples to assess the properties and performance of our proposed methodology. The experiments conducted aim to shed light on the convergence behaviour of the matrix sequence and evaluate the perturbation estimates derived from our theoretical analysis. The numerical experiments were carried out using MATLAB R2015a, employing a stopping condition defined as tol $= n \times$ eps, where n represents the size of the matrix A, and eps = 2.2204×10^{-16} denotes the machine epsilon.

In Example 1, we explore the convergence of the matrix sequence for two different initial solutions. By comparing these cases, we gain insights into the influence of initial conditions on the convergence behaviour. The analysis provides valuable observations regarding the stability and robustness of the iterative process. In Example 2, we focus on assessing the perturbation estimates derived from our theoretical derivations. By introducing controlled perturbations in the system, we evaluate the accuracy and reliability of the obtained estimates. This evaluation enables a deeper understanding of the sensitivity and practical applicability of the theoretical results.

Through these numerical experiments, we aim to validate the theoretical findings, assess the algorithm's performance, and provide practical insights into the behaviour and properties of the matrix equation.

Example 1

Case I: To illustrate the convergence of our fixed-point algorithm, we consider a symmetric matrix A and specific parameter values. Here are the details of the case:

Matrix $A=\begin{bmatrix} 0.1500 & 0.1750 & 0.1550 \\ 0.1750 & 0.2500 & 0.1750 \\ 0.1550 & 0.1750 & 0.4000 \end{bmatrix}$,

matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, exponent $\alpha = {}^1/_2$ and initial solution matrix $X\ (0)\ = P$.

In this case, our fixed-point algorithm requires 20 iterations for the matrix sequence to converge to a unique fixed point $X_{\text{sol.}}$, denoted as $X(20)$. The algorithm aims to minimize the error between successive iterations, ultimately reaching a stable solution.

By considering this specific case, we can observe the behaviour of the algorithm and evaluate the convergence properties. The resulting fixed point $X_{\text{sol.}}$ represents the final solution of the iterative process, indicating the equilibrium state of the system under consideration with the error $\|X(20) - A^*X(20)^{1/2}A - P\| = 7.97x10^{-17}$.

Table 1: Convergence of Fixed Point Algorithm for $X(0) = P$

X_{ij}	X_{11}	X_{12}	X_{13}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	X_{33}
$X_{(1)}$	1.0772	0.0971	0.1159	0.0971	1.1238	0.1409	0.1159	0.1409	1.2147
$X_{(2)}$	1.0904	0.1136	0.1377	0.1136	1.1442	0.1679	0.1377	0.1679	1.2511
$X_{(3)}$	1.0925	0.1163	0.1412	0.1163	1.1476	0.1723	0.1412	0.1723	1.2570
$X_{(4)}$	1.0929	0.1167	0.1418	0.1167	1.1481	0.1731	0.1418	0.1731	1.2580
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$X_{(20)}$	1.0929	0.1168	0.1419	0.1168	1.1482	0.1732	0.1419	0.1732	1.2581

Case II: Building upon Example 1, we investigate a scenario where the initial solution exceeds the value of P. The following parameters are considered: Initial solution $X(0) = 2P, A$ and α are as defined in **Example 1**. In this case, our Fixed Point Algorithm requires 19 iterations to converge to a unique solution $X_{\text{sol.}}$, denoted as $X(19)$ with the residual
$\|X(19) - A^*X(19)^{1/2}\, A - P\| = 5.49 \times 10^{-16}$.
The nature of convergence is shown in Table 2.

Table 2: Convergence of Fixed Point Algorithm for $X(0) = 2P$

X_{ij}	X_{11}	X_{12}	X_{13}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	X_{33}
$X_{(1)}$	1.1091	0.1374	0.1639	0.1374	1.1750	0.1992	0.1639	0.1992	1.3036
$X_{(2)}$	1.0953	0.1197	0.1458	0.1197	1.1519	0.1780	0.1458	0.1780	1.2646
$X_{(3)}$	1.0933	0.1172	0.1425	0.1172	1.1488	0.1740	0.1425	0.1740	1.2592
$X_{(4)}$	1.0930	0.1168	0.1420	0.1168	1.1483	0.1733	0.1420	0.1733	1.2583
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$X_{(19)}$	1.0929	0.1168	0.1419	0.1168	1.1482	0.1732	0.1419	0.1732	1.2581

Note that X_{ij} denotes the elements of matrix $X(k)$, and the indices i, j = 1:3 and k stand for an iteration number.

Remarks: As we can see in Table 1, the matrix sequence

$$P = X(0) \leq X(1) \leq X(2) \leq \cdots \leq X_{\text{sol.}} = X(20),$$

is increasing and converges to the maximal solution $X_{\text{sol.}}$. On the other hand, the matrix sequence in Table 2,

$$2P = X(0) > X(1) \geq X(2) \geq \cdots \geq X_{\text{sol.}} = X(19),$$

decreases to the maximal solution of equation (1).

This means that $P \leq X_{\text{sol.}} = X(19) < X(0) = 2P$, as evident from the observations, all of our theoretical assumptions align with our expectations.

Example 2

Consider equations $X - A^*X^{\alpha}A = P$ and $\tilde{X} - \tilde{A}^*\tilde{X}^{1/2}\tilde{A} = \tilde{P}$. Suppose that $P =$eye$(3) = X(0), \alpha = 1/2,$ A is the same as in **Example 1,** where

$$\beta = 1 - \|A\|^2\alpha b^{\alpha-1} = 0.7245, \qquad b = 0.5, \qquad \Delta P = \tilde{P} - P = 0.44\,\gamma I.$$

using the proposed Fixed Point Algorithm, we obtain $X_{\text{sol.}} = \begin{bmatrix} 1.0929 & 0.1168 & 0.1419 \\ 0.1168 & 1.1482 & 0.1732 \\ 0.1419 & 0.1732 & 1.2581 \end{bmatrix}$,

$\Delta X_{\text{sol.}} = 10^{-07} \times \begin{bmatrix} 0.1792 & 0.1969 & 0.1996 \\ 0.1969 & 0.2366 & 0.2196 \\ 0.1996 & 0.2196 & 0.3298 \end{bmatrix}$ and

$\|A\| = 0.6245 < \sqrt{\frac{\|P\|}{\|X^{\alpha}\|}} = 0.9076$. All assumptions are satisfied. Now, let

$C_1 = \frac{\|\Delta X_{\text{sol.}}\|}{\|X_{\text{sol.}}\|}$ and $C_2 = \frac{1}{\beta}\left(\frac{\|\Delta P\|}{\|P\|} + \frac{2\|\Delta A\|}{\|A\|}\right)$.

Table 3: Computed values of $\boldsymbol{C_1}, \boldsymbol{C_2}$, $\|\Delta \boldsymbol{P}\|$ and $\|\Delta \boldsymbol{A}\|$ for different $\boldsymbol{\gamma}$ values

γ	C_1	C_2	$\|\Delta P\|$	$\|\Delta A\|$
2.0000×10^{-8}	4.5307×10^{-8}	1.0059×10^{-7}	8.8000×10^{-9}	2.0000×10^{-8}
4.0000×10^{-8}	9.0614×10^{-8}	2.0118×10^{-7}	1.7600×10^{-8}	4.0000×10^{-8}
6.0000×10^{-8}	1.3592×10^{-7}	3.0177×10^{-7}	2.6400×10^{-8}	6.0000×10^{-8}
8.0000×10^{-8}	1.8123×10^{-7}	4.0237×10^{-7}	3.5200×10^{-8}	8.0000×10^{-8}
1.0000×10^{-7}	2.2653×10^{-7}	5.0296×10^{-7}	4.4000×10^{-8}	1.0000×10^{-7}

The data in Table 3 consistently reveals that C_1 is always less than or equal to C_2 for all values of γ. Consequently, C_2 serves as the upper bound for C_1.

Conclusion

This paper applied the concept of fixed points in partially ordered sets to solve equation (1). Perturbation estimates of the solution were also derived and experimentally verified. The results strongly validate our theoretical assertions and confirm the effectiveness of the proposed iterative method. The findings not only offer a robust solution to the equation but also demonstrate the practical applicability and efficiency of the approach. Consequently, this research significantly enhances our understanding of partially ordered sets in the context of solving nonlinear matrix equations and establishes a reliable framework for addressing similar challenges in various scientific and engineering fields.

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