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GAIN MARGIN AND PHASE MARGIN INTERRELATIONSHIPS

by

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ABSTRACT

The gain margin and the phase margin of a control system are important parameters in the determination of the relative stability of control system. In this paper we derive general interrelationships between these two parameters.

1.0 DEFINITIONS

The phase margin and the gain margin of a control system are defined below with the aid of Fig.1 which is a polar plot of the open-loop transfer function $G(\omega)$ of a system which has positive values of gainmargin and phase margin.

The gain margin and phase margin are defined as:

Gain margin (in dB) $= -20\log_{10}\alpha$

Phase margin (in degrees) $= \phi$

Observe that here is a frequency, ω_{π}, called the phase cross-over frequency at which lm $|G(j\omega_{\pi})| = 0$.

Then $-\alpha = G(j\omega_{\pi})$.

Another frequency of interest is ω_{1} called the gain cross-over frequency at which $|G(j\omega_{1})| = 1$

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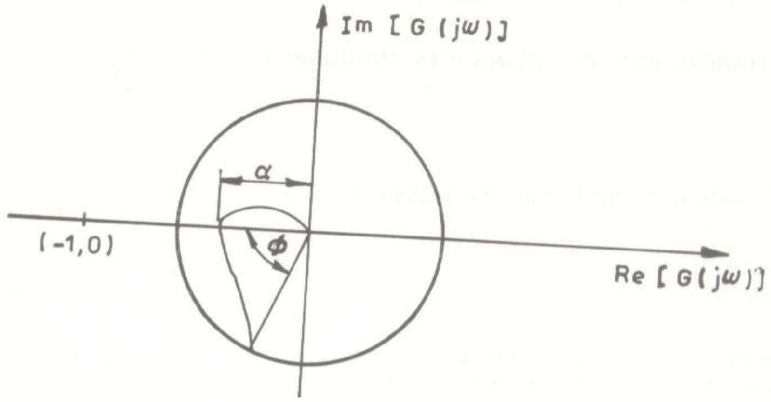

Fig.1: Gain margin and phase margin of a system

2. INTERRELATIONS

In this section we shall first seek to establish an interrelationship between the gain and phase margins of control systems which is as general as possible and then later we shall do the same for second-order systems.

Since a general result on the phase margin and gain margin interrelationship is not system-specific we have only been able to establish the following:

2.1 Proposition 1:

Let a unity-feedback control system with open-loop transfer function $G_1(s)$ have a gain margin $\zeta = 20\log_{10}\alpha dB$. If another unity feedback system with open-loop transfer function $G_2(s)$ has Re $\{G_2(j\omega_1)\}$, then the phase margin ϕ of the second system is related to α

Proof: By definition of the gain margin, we obtain the following result for the first system

$$-\alpha = G_1(j\omega_g)$$

Using the definition of ω_p for the second system we get

Re $\{G_2(j\omega_p)\} = -\cos\phi$

Finally, since Re $\{G_1(j\omega_p)\} = Re\{G_2(j\omega_p)\}$ we get $\alpha = \cos\phi$.

We next establish the results for second-order systems. These results are important because most practical control systems can be approximately modelled by second-order systems. For second-order systems we can say considerably more about their phase margin and gain margin interrelationship.

We assume we have a unity-feedback system with open-loop transfer function G(s) given by

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \qquad (1)$$

where ζ is the system damping coefficient.

2.2 Proposition 2:

The gain margin, η dB, of the system whose open-loop transfer function is given by equation (1) above is related to its phase margin, ϕ degrees, by

$$\eta = 20\log_{10}(\sin\phi\tan\phi)$$

Proof: Since $G(j\omega) = \dfrac{\omega_n^2(-\omega^2 - j2\zeta\omega\omega_n)}{\omega^4 + 4\zeta^2\omega^2\omega_n^2}$

we get $\omega_p = \omega_n\left\{\sqrt{4\zeta^4+1} - 2\zeta^2\right\}^{\frac{1}{2}}$

and $\omega_g = 0$

Hence $\eta = 20\log_{10}(4\zeta^2)$ (2)

and

$$\phi = \tan^{-1} \frac{2\zeta}{\left\{ (4\zeta^4 + 1)^{\frac{1}{2}} - 2\zeta^2 \right\}^{\frac{1}{2}}} \tag{3}$$

so, by eliminating ζ from equation (2) and (3) we get

$$\eta = 20 \log_{10}(\sin\phi \tan\phi)$$

2.3 Remark 1:

Fig.2 is a plot of η versus ζ and ϕ versus ζ. We see that so long as $\zeta < 0.5$ the phase margin versus ζ curve can be approximated very accurately by straight lines yielding.

$$\phi = 106\zeta \tag{4}$$

and

$$\eta = 65\zeta \tag{5}$$

These approximations are resonably accurate for $\zeta < 0.5$ and are a useful index linking the frequency response of second order systems to their transient performance.

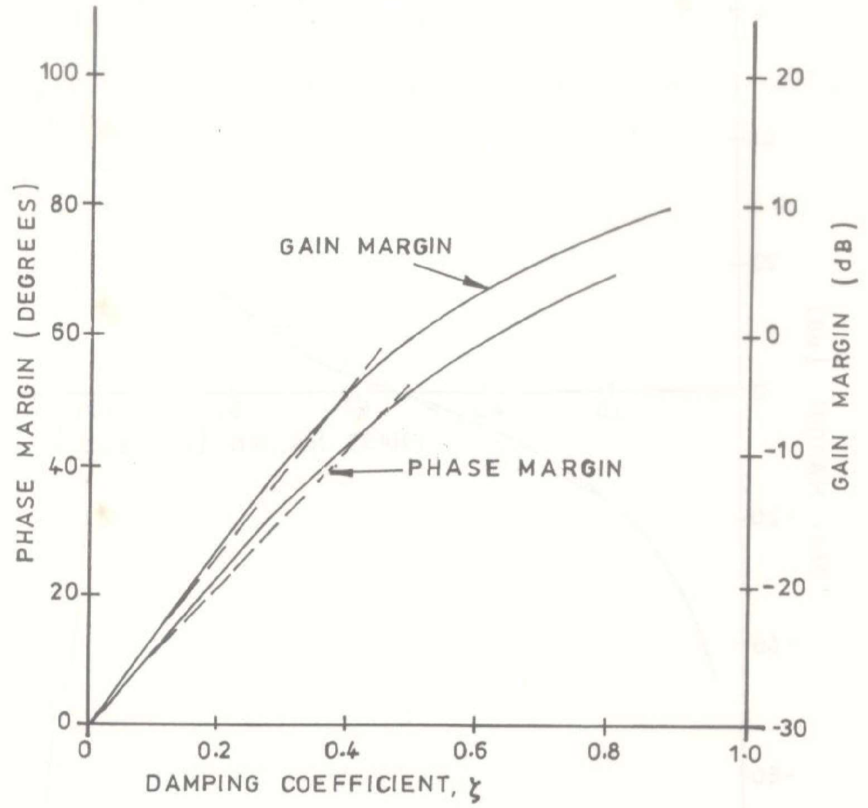

Fig. 2: Gain and phase margin versus ζ

2.4 Remark 2:

A plot of the gain margin in dB versus the phase margin in degrees for second-order systems is shown in Fig.3. For $20° < \phi < 80°$ we can approximate the gain margin versus phase margin relations by a straight line:

$$\eta = 0.57\phi - 29.7 \qquad (6)$$

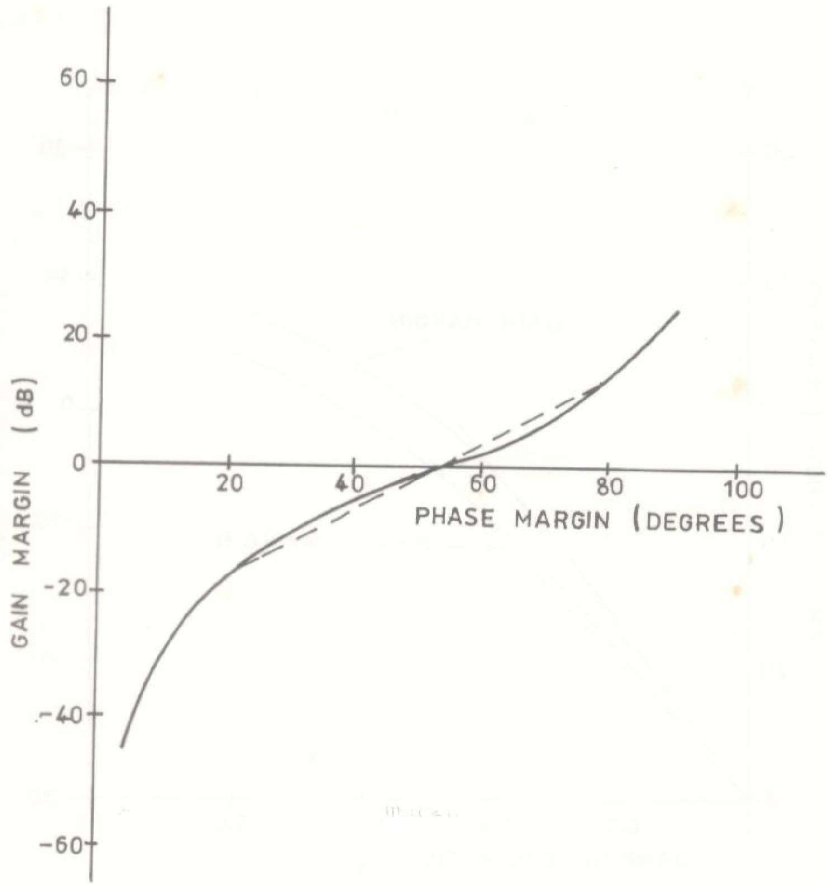

Fig. 3: Gain margin versus phase margin

3.0 CONCLUSIONS

This paper has derived interrelationships between the gain margin and the phase margin of control systems. For second-order systems we show that over gain margin and phase margin ranges of practical interest the interrelationships are approximately linear.

4. REFERENCES

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2. J.J. D'AZZO and C. HUPIS, Feedback Control System Analysis, and Synthesis, McGraw-Hill, 1966.