# NUMERICAL INVESTIGATION OF A NON-NEWTONIAN MATERIAL THROUGH A SUDDEN PIPE EXPANSION

by

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**ABSTRACT**

**A numerical method for solving two dimensional elliptic boundary-value problems of fourth order is used for finding stationary solutions of the equations of motion for a laminar flow through a sudden pipe expansion of an incompressible non- Newtonian fluid with a constitutive equation of the Reiner-Rivlin (R-R) type. Results for the Newtonian fluid, which is a special case, are in excellent agreement with those by previous investigators and they demonstrate the reliability, superiorit and accuracy of the method. Results for the non- Newtonian cases could be of value for fixing ideas about visco-elastic behaviour although the R-R fluid allows only inelasticity: this is particularly useful since the R-R model is mathematically the least complicated allowing the introduction of non-zero normal stress differences.**

## 1. Introduction

Flow in pipes has long been a subject of experimental and theoretical investigations. Results from such work provide essential data for construction of efficient fluid handling systems.

Rheologically complex fluids occur in industrially important contexts such as polymer and chemical process plants, food processing and textile industries, thermal power plants, tubular exchangers, geological piping systems, sewage systems and polymer extrusion pipes. Understanding the behaviour of such fluids in non-trivial situations is a matter of practical importance and concern.

In industrial situations fluids are often exposed to sudden geometry changes and one of the unresolved problems is the theoretical prediction of fluid behaviour in situations involving abrupt geometry changes, especially when fluids involve long-range memory of past history of deformation such as visco-elastic fluids.

Experimental investigations and computational studies |2,3,4| for flow through abrupt geometry changes show the need for application of simple constitutive models for a better understanding of the flow mechanism. The Reiner-Rivlin constitutive equation is mathematically the least complicated allowing the introduction of non-zero normal stress differences (i.e. viscometric functions) and can be of value for fixing ideas about viscoelasticity.

In this analysis, the flow of a special Reiner-Rivlin fluid formulated by Masanja |6| through a circular pipe with sudden expansion, is investigated numerically.

The numerical method is a difference procedure based on a 9-point star which allows fourth order approximation of all differential operators up to the third derivative. This method developed by Wagner |1| **is being extended** for non-Newtonian fluids for solving 2-dimensional elliptic boundary-value problems of fourth order. Hence it utilises the stream-function formulation of the equations of motion, which is a non-linear differential equation (d.e) of 4th order. This d.e. is linearised with the help of the Newton-Chord Method and the resulting linear d.e. is discretized using the difference procedure and solved using a direct-solver.

## 2. Formulation of the Problem

Consider flow of an incompressible non-Newtonian fluid with negligible temperature and memory effects, the equation of motion for the velocity field $\underline{v}(\underline{x},t)$ is

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\underline{\nabla})\underline{v}\right] = \underline{\nabla}.\underline{T}, \qquad \nabla.\,\underline{v} = \underline{0} \tag{1}$$

where $\rho$ is the constant density and $\underline{T}(\underline{x},t)$ is the symmetric stress tensor. The latter can be written as

$$\underline{T} = -p\underline{I} + \underline{\tau} \tag{2}$$

where the pressure $p(\underline{x},t)$ is to be determined from the condition of incompressibility, and the deviatoric stress tensor $\underline{\tau}$ is related by the constitutive equation to the rate of strain tensor

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (3)$$

If this relation is linear,

$$\underline{\tau} = 2\eta\underline{d} \qquad (4)$$

where $\eta$ is the shear viscosity, we deal with an incompressible Newtonian or Navier-Stokes fluid. It has been shown by Reiner |1| and Rivlin |2| from euclidean invariance that the general nonlinear constitutive equation must be of the form

$$\underline{\tau} = g(I_2, I_3) + h(I_1, I_2)\underline{d}^2 \qquad (5)$$

where $I_2$ and $I_3$ are the invariants

$$I_2 = \text{tr}\,\underline{d}^2, \quad I_3 = \text{tr}\,\underline{d}^3 \qquad (6)$$

The invariant $I_1 = \text{tr}\,\underline{d}$ vanishes due to incompressibility. Fluids characterized by the constitutive equation (5) are called purely viscous or Reiner-Rivlin (R-R) fluids. A special case of the R-R fluid obtained by Masanja |6|,

$$\underline{\tau} = -2\mu_1\underline{d} - 2\mu_2\left(\frac{I_2}{3}\underline{I} - I_3\underline{d}^{-1}\right) \qquad (7)$$

is used in this investigation. The material constant $\mu_1$ and $\mu_2$ are called the viscosity coefficients. Consider a stationary state, i.e. $\frac{\partial}{\partial t}() = 0$. The suitable physical coordinates for the flow through a circular pipe are cylindrical polar $(\bar{x}, \theta, \bar{r})$ with the velocity vector field $\underline{v} = (\bar{u}, 0, \bar{v})$ in the axial-, azimuthal- and radial directionals respectively. The deviatoric stress tensor is thus

$$\begin{bmatrix} \tau_{\bar{x}\bar{x}} & 0 & \tau_{\bar{r}\bar{x}} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{\bar{x}\bar{r}} & 0 & \tau_{\bar{r}\bar{r}} \end{bmatrix} = -2\mu_1 \begin{bmatrix} d_{\bar{x}\bar{x}} & 0 & d_{\bar{x}\bar{r}} \\ 0 & d_{\theta\theta} & 0 \\ d_{\bar{x}\bar{r}} & 0 & d_{\bar{r}\bar{r}} \end{bmatrix}$$

$$-2\mu_2 \begin{bmatrix} d_{\theta\theta}d_{\bar{x}\bar{x}} - \frac{I_2}{3} & 0 & -d_{\theta\theta}d_{\bar{x}\bar{r}} \\ 0 & d_{\bar{x}\bar{x}}d_{\bar{r}\bar{r}} - d_{\bar{r}\bar{x}}^2 - \frac{I_2}{3} & 0 \\ -d_{\theta\theta}d_{\bar{x}\bar{r}} & 0 & d_{\theta\theta}d_{\bar{r}\bar{r}} - \frac{I_2}{3} \end{bmatrix}$$

where the components of $\underline{d}$ are

$$d_{\bar{x}\bar{x}} = \frac{\partial\bar{u}}{\partial\bar{x}}, \quad d_{\bar{x}\bar{r}} = \frac{1}{2}\left(\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{\partial\bar{v}}{\partial\bar{x}}\right), \quad d_{\bar{r}\bar{r}} = \frac{\partial\bar{v}}{\partial\bar{r}} \quad \text{and}$$

$$d_{\theta\theta} = \frac{\bar{v}}{\bar{r}}$$

and the scalar invariant $I_2$ is

$$I_2 = \frac{1}{2}\left(\frac{\partial\bar{v}}{\partial\bar{r}}\right)^2 + \left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)^2 + \left(\frac{\bar{v}}{\bar{r}}\right)^2 - \left(\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{\partial\bar{v}}{\partial\bar{x}}\right)^2 \qquad (10)$$

As common practice, the following dimensionless quantities are introduced:

$$u = \frac{\bar{u}}{u_0}, \quad v = \frac{\bar{v}}{u_0}, \quad r = \frac{\bar{r}}{a}, \quad x = \frac{\bar{x}}{a} \quad \text{and} \quad p = \frac{p - p_0}{\frac{1}{2}\rho u_0^2} \tag{11}$$

where a is the constant pipe diameter, $u_o$ and $p_o$ respectively are the velocity and pressure at the entrance cross-section.

In view of the mass balance $\nabla.\underline{v} = 0$, there exists a stream-function such that

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial r}, \quad v = \frac{1}{r}\frac{\partial\psi}{\partial x} \tag{12}$$

By taking the curl of $\rho(\underline{v}.\nabla)\underline{v} = \nabla.\underline{T}$ and making use of (12) a non-linear 4th order elliptic partial differential equation in $\psi$ is obtained:

$$A(\psi)\left(1 + \frac{W}{r^2}\frac{\partial\psi}{\partial x}\right) - R_e B(\psi) = 0 \tag{13}$$

with

$$A(\psi) = \frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial r^4} + 2\frac{\partial^4\psi}{\partial r^2\partial x^2} - \frac{2}{r}\frac{\partial^3\psi}{\partial x^2\partial r} - \frac{2}{r}\frac{\partial^3\psi}{\partial r^3} +$$

$$+ \frac{3}{r^2}\frac{\partial^2\psi}{\partial r^2} - \frac{3}{r^3}\frac{\partial\psi}{\partial r} \tag{14}$$

and

$$B(\psi) = \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial r}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2}\right) - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2}\right)$$

$$- \frac{1}{r}\frac{\partial\psi}{\partial x}\left(2\frac{\partial^2\psi}{\partial x^2} + 3\frac{\partial^2\psi}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial x\partial r} + \frac{3}{r^3}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial r} \qquad (15)$$

where $W$, the non-Newtonian parameter and $R_e$ the Reynolds number are given by

$$W = \frac{\mu_1}{\mu_2}\frac{\mu_0}{a}, \quad R_e = \frac{\rho a u_0}{\mu_1} \qquad (16)$$

The differential equation (13) is to be solved subject to imposed boundary conditions.

Consider a circular pipe with constant cross-section of diameter a. The pipe is sharply expanded as shown in Fig.1 below.

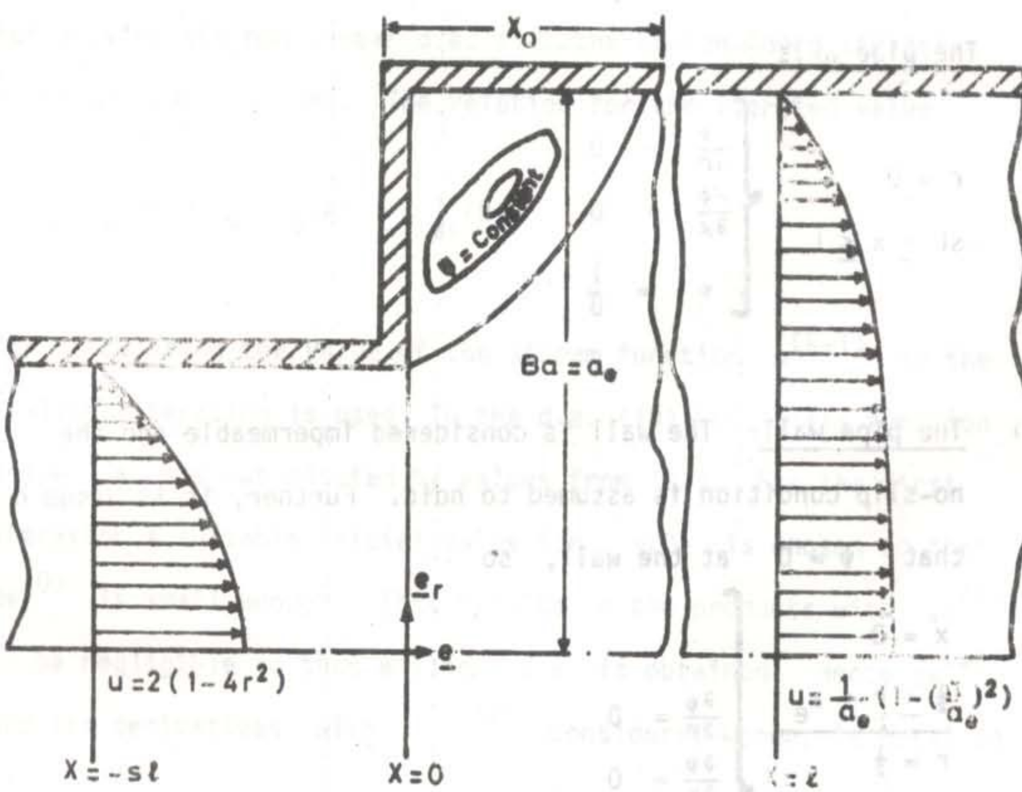


Fig. 1: A schematic representation of flow after an abrupt expansion

Taking the origin to be the intersection of the entrance cross-section and the axis of symmetry, the boundaries and their conditions are

(i) the inlet: Assume the flow regime to be fully developed, then

$$0 \le r \le \tfrac{1}{2},\quad x = -sL \qquad \begin{cases} \dfrac{\partial \psi}{\partial r} = 2r\,(1 - 4r^2) \\[2mm] \dfrac{\partial \psi}{\partial x} = 0 \\[2mm] \psi = r^2(1 - 2r^2) + \dfrac{1}{8} \end{cases} \tag{17}$$

where $s > 0$ is a constant, $L$ is the pipe length.

(ii) The pipe axis

$$r = 0,\quad -sL \le x \le L \qquad \begin{cases} \dfrac{\partial \psi}{\partial r} = 0 \\[2mm] \dfrac{\partial \psi}{\partial x} = 0 \\[2mm] \psi = \dfrac{1}{8} \end{cases} \tag{18}$$

(iii) The pipe wall: The wall is considered impermeable and the no-slip condition is assumed to hold. Further, it is assumed that $\psi = 0$ at the wall, so

$$\left.\begin{array}{l} x = 0 \\ \tfrac{1}{2} \le r \le a_e \\ \hline r = \tfrac{1}{2} \\ -sL \le x \le 0 \\ \hline r = a_e \\ 0 \le x \le L \end{array}\right\} \qquad \begin{cases} \dfrac{\partial \psi}{\partial r} = 0 \\[2mm] \dfrac{\partial \psi}{\partial x} = 0 \\[2mm] \psi = 0 \end{cases} \tag{19}$$

(iv) <u>The exit</u>: A sufficiently long distance behind the expansion the flow is assumed to have resumed the form of a fully developed profile, i.e. $u = k\,(1 - (\frac{r}{a_e})^2)$; $\psi(a_e) = 0$ leads to $k = -\frac{1}{2a_e^2}$ so that

$$0 \le r \le a_e \qquad x = L \qquad \begin{cases} \frac{\partial\psi}{\partial r} = \frac{1}{2}\,\frac{r}{a_e^2}\,(1 - (\frac{r}{a_e})^2) \\ \frac{\partial\psi}{\partial x} = 0 \\ \psi = \frac{1}{4}\,(\frac{r}{a_e})^2\,(1 - (\frac{r}{a_e})^2) + \frac{1}{8} \end{cases} \qquad (20)$$

## 3. The Numerical Solution

For solving the non-linear d.e. (13) the Newton-Chord iteration procedure |8| is used. The relation for the iterated value

$$\psi^{(k+1)} = \psi^{(k)} + \delta\psi^{(k)} \qquad (21)$$

for calculating the value of the stream function $\psi^{(k+1)}$ in the (k+1)th iteration is used. In the d.e. (13) the stream function value $\psi$ is substituted by values from (21). For the first iteration a suitable initial value for $\psi(0)$ is chosen so that $\delta\psi^{(0)}$ is small enough. This results in the products with $\delta\psi^{(k)}$ to be negligible so that a linear d.e. is obtained. Hence $\delta\psi^{(k)}$ and its derivatives, with $\psi^{(k)}$ considered known, is obtained from

$$F(A_{m,n}\ ;\ G\ ;\ x\ ;\ r) \;=\; A_{40}\,\delta\,\frac{\partial^4\psi}{\partial x^4}^{(k)} + A_{22}\,\delta\,\frac{\partial^4\psi}{\partial x^2\partial r^2}^{(k)} +$$

$$+\; A_{04}\,\delta\,\frac{\partial^4\psi}{\partial r^4}^{(k)} + A_{30}\,\delta\,\frac{\partial^3\psi}{\partial x^3}^{(k)} + A_{21}\,\delta\,\frac{\partial^3\psi}{\partial x^2\partial r}^{(k)} +$$

$$+\; A_{12}\,\delta\,\frac{\partial^3\psi}{\partial x\partial r^2}^{(k)} + A_{03}\,\delta\,\frac{\partial^3\psi}{\partial r^3}^{(k)} + A_{20}\,\delta\,\frac{\partial^2\psi}{\partial x^2}^{(k)} +$$

$$+\; A_{11}\,\delta\,\frac{\partial^2\psi}{\partial x\partial r}^{(k)} + A_{02}\,\delta\,\frac{\partial^2\psi}{\partial r^2}^{(k)} + A_{10}\,\delta\,\frac{\partial\psi}{\partial x}^{(k)} + A_{01}\,\delta\,\frac{\partial\psi}{\partial r}^{(k)} +$$

$$+\; G \;=\; 0 \qquad (22)$$

where

$$A_{40} = A_{04} = \frac{1}{2}A_{22} = \frac{W}{r^2}\,\frac{\partial\psi}{\partial x}^{(k)} + 1 \qquad (23)$$

$$A_{30} = A_{12} = \frac{R_e}{r}\,\frac{\partial\psi}{\partial r}^{(k)} \qquad (24)$$

$$A_{21} = A_{03} = \frac{1}{r}\,\frac{\partial\psi}{\partial x}^{(k)}\left(\frac{2W}{r^2} + R_e\right) \qquad (25)$$

$$A_{20} = -\frac{R_e}{r^2}\,\frac{\partial\psi}{\partial x}^{(k)} \qquad (26)$$

$$A_{11} = -\frac{R_e}{r^2}\,\frac{\partial\psi}{\partial r}^{(k)} \qquad (27)$$

$$A_{02} = \frac{3}{r^2}\,\frac{\partial\psi}{\partial x}^{(k)}\left(\frac{W}{r^2} + R_e\right) + \frac{3}{r^4} \qquad (28)$$

$$A_{10} = \frac{W}{r^2}\left(\frac{\partial^4\psi}{\partial x^4}^{(k)} + 2\,\frac{\partial^4\psi}{\partial x^2\partial r^2}^{(k)} + \frac{\partial^3\psi}{\partial r^3}^{(k)}\right) -$$

$$-\,2\left(\frac{W}{r^3} + \frac{R_e}{r}\right)\left(\frac{\partial^3\psi}{\partial x^2\partial r}^{(k)} + \frac{\partial^3\psi}{\partial r^3}^{(k)}\right) + \frac{2}{r^2}\,R_e\,\frac{\partial^2\psi}{\partial x^2}^{(k)}$$

$$+\,\frac{3}{r^2}\left(\frac{W}{r^2} + R_e\right)\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\,\frac{\partial\psi}{\partial r}^{(k)}\right) \qquad (29)$$

$$A_{01} = -\frac{3}{r^3}\frac{\partial\psi}{\partial x}^{(k)}\left(\frac{W}{r^2} + R_e + \frac{R_e}{r}\right)\left(\frac{\partial^3\psi}{\partial x^3}^{(k)} + \frac{\partial^3\psi}{\partial x\partial r^2}^{(k)} + \frac{1}{r}\,\frac{\partial^2\psi}{\partial r\partial x}^{(k)}\right)$$

and

$$G(x,r) = A_{40}\left(\frac{\partial^4\psi}{\partial x^4}^{(k)} + 2\,\frac{\partial^4\psi}{\partial x^2\partial r^2}^{(k)} + \frac{\partial^4\psi}{\partial r^4}^{(k)}\right)$$

$$+\,A_{30}\,\frac{\partial^3\psi}{\partial x^3}^{(k)} + A_{21}\left(\frac{\partial^3\psi}{\partial x^2\partial r}^{(k)} + \frac{\partial^3\psi}{\partial r^3}^{(k)}\right)$$

$$+\,A_{12}\,\frac{\partial^3\psi}{\partial x\partial r^2}^{(k)} - A_{20}\,\frac{\partial^2\psi}{\partial x^2}^{(k)} + A_{11}\,\frac{\partial^2\psi}{\partial x\partial r}^{(k)}$$

$$+\,A_{02}\left(\frac{\partial^2\psi}{\partial r^2}^{(k)} - \frac{1}{r}\,\frac{\partial\psi}{\partial r}^{(k)}\right). \qquad (31)$$

The iteration is stopped when $\delta\psi^{(k)}$ is small enough or if the prescribed number of iterations is attained.

In order to increase the accuracy we transform the $xr$-plane into a $\xi\eta$- plane in such a way that the mesh net is dense near regions of high velocity changes and it spreads out elsewhere.

For a pipe expansion big velocity changes occur behind the expansion step. Following |9| we assume the transformations

$$X(\xi) = \alpha_x (\xi-\xi_0) + \theta \frac{(\xi-\xi_0)^3}{\gamma - \xi} \tag{32}$$

and

$$r(\eta) = a\eta^3 + b\eta^2 + c\eta \tag{33}$$

in the x- and r- directions respectively with $\alpha_x$, $\xi_0$, $\theta$, $\gamma$, a, b, c constants. For suitability to the pipe expansion certain conditions must be satisfied. For the x- direction transformation, conditions for determining $\theta$ are:

$$X(\xi_0) = 0; \quad X(0) = -sL; \quad \frac{dx}{d\xi}(\xi_0) = \alpha_x \tag{34}$$

from this and (32) follows

$$\theta = \frac{\gamma}{\xi_0^3} (SL - \alpha_x\xi_0) \tag{35}$$

which when substituted in (32) gives

$$X(\xi) = \alpha_x (\xi-\xi_0) + \frac{\gamma}{\xi_0^3} (SL - \alpha_x\xi_0) \frac{(\xi-\xi_0)^3}{\gamma - \xi} \tag{36}$$

whose derivative with respect to (w.r.t) $\xi$ is

$$\frac{dx}{d\xi}(\xi) = \alpha_x + \frac{\gamma}{\xi_0^3} (SL - \alpha_x\xi_0) \frac{3(\xi-\xi_0)(\gamma-\xi) - (\xi-\xi_0)^3(-1)}{(\gamma - \xi)^2} \tag{37}$$

So that the condition $\frac{dx}{d\xi}(\xi_0) = \alpha_x$ is satisfied since for $\xi = \xi_0$ the last term in (37) is zero. $\xi \in [0,1]$ so the value of $\gamma \in (1,1.1]$. The nearer to 1 the value of $\gamma$ is, the more stretched out the mesh net is, in the x,r-plane far down the pipe expansion. Small values of $\alpha_x$ of the equation (36) effect the compression of the net in the x-direction in the region of the expansion step. The mesh net in the x,r-plane is given in Figure 2.

In order the pipe wall in the expansion to be always in one mesh point, the transformation (33) must fulfill the conditions

$$r(1) = \frac{\varepsilon}{2} \; ; \; r(\eta_0) = \frac{1}{2}, \; \frac{dr}{d\eta}(\eta_0) = \alpha_r \tag{38}$$

from which the constants a, b, c are determined as

$$b = \frac{1}{2} k_1 \left(\frac{1-\varepsilon\eta_0}{k_2}\right) - \frac{1}{k_1}\left(\alpha_r - \frac{\varepsilon}{2}\right). \; k_1 \; \frac{(\eta_0^3 - \eta_0)}{k_2} \tag{39}$$

$$a = \frac{1}{k_1}\left(\alpha_r - \frac{\varepsilon}{2} - (2\eta_0 - 1)b\right) \tag{40}$$

$$c = \frac{\varepsilon}{2} - (a + b) \tag{41}$$

where

$$k_1 = 3\eta_0^2 - 1 \; , \; k_2 = (\eta_0^2 - \eta_0)k_1 - (2\eta_0 - 1)(\eta_0^3 - \eta_0) \tag{42}$$

The transformations (32) and (33) are used to transform equation (22) i.e.

$$F(A_{mn}\ (x,r);\ \ G(x,r);\ x;\ r) = 0 \quad \text{into}$$

$$\bar{F}(a_{mn}(\xi,\eta);\ \ g(\xi,\eta);\ \xi,\eta) = 0 \qquad (43)$$

with $a_{mn}(\xi,\eta)$ and $g(\xi,\eta)$ respective transformations of $A_{mn}(x,r)$ and $G(x,r)$. $a_{mn}$ and $g$ are obtained using formulae (A1) to (A8) and (B1) to (B8) given in the appendix.

## 4. Results and Conclusion

Numerical results are illustrated graphically in diagrams Fig.3 to 7 and tabulated in plots Tab.1 to III. The parameters W and $R_e$ are the non-Newtonian parameter and the Reynolds number respectively. $R_e$, the measure of the importance of inertia, is based on the pipe diameter and average velocity. The non-Newtonian character of the fluid is displayed by W. The larger W is the more non-Newtonian the fluid is. The parameter $\beta$ is the ratio of pipe diameter after and before the expansion.

Most results were obtained for $\beta = 2$ using a grid of maximum size of 31 x 31 in the radial and axial directionals.

Tab.1 shows the effect of the Reynolds number on the re-attachment length $x_0$ for 3 reprsentative non-Newtonian fluid parameters. Also shown in Tab.II is the excellent agreement of this work with the results obtained by Halmos et al. |10| and Macagno et al. |2| for a Newtonian fluid with the same geometry. For vanishing W, the re-attachment length increases with increasing $R_e$.

Fig. 3 illustrates the increasing size of the secondary cell which is associated with increasing values of $R_e$. As a result, the development length down stream of the expansion is also found to be longer the larger the value of $R_e$ is. The longer secondary cell allows the main flow to decelerate gradually and behaves not only as a dissipator of energy but also as a mechanism which shapes the flow through the expansion and dampens the effect of the discontinuity in geometry. Intuitively it can be seen that the greater the inertia, the more difficult it is for the fluid to follow the contour of the expansion and therefore, the larger will be the size of the cell.

The effect of the non-Newtonian parameter W on the flow field is illustrated in Fig. 4 for $R_e = 5$ and $W = 0.0$, $0.1$ and $1.0$. The secondary cell can be seen to be longer for the larger W value. The intensity of the secondary cell decreases as W increases. This decrease is caused by an increase in the non-Newtonian viscosity of the secondary motion as W departs away from Newtonian behaviour. The main flow therefore drives the circulation in the cell at a lower rate.

The axi-symmetric flow produced by the sudden expansion can be analysed in terms of axial and radial velocity components. Both u and v are functions of $R_e$ and W. Fig.5 illustrates the development of the axial velocity profile for $W = 0.0$ and $1.0$ for $R_e = 10$. The dependence of u on the two flow parameters W and $R_e$ is best described by the centre line values of this variable as illustrated by Fig.6. The length necessary for fully developed flow to be attained increases with increasing $R_e$ but decreases with increasing W.

Fig. 7 display the dependence of the centre line axial velocity on the pi[pe] expansion ratio $\beta$. It is observed that after expansion, the entrance length depends strongly on $\beta$. Tab.III shows the dependence of re-attachmen[t] lengths on W and $\beta$.

| W / $R_e$ | | 0.0 | 0.1 | 0.4 |
|---|---|---|---|---|
| 10 | $x_0$ = | 0.64 | 0.66 | 0.67 |
| 20 | $x_0$ = | 0.89 | 1.15 | 1.19 |

Tab.1: Dependence of re-attachment length $x_0$ on W and $R_e$ for $\beta = 2$.

| $R_e$ | $R_e$-attachment Halmos et.al. \|10\| Numerical (from graphs) | lengths obtained by Macagno et.al. \|2\| Experimental (from graphs) | This study |
|---|---|---|---|
| 5 | 0.2 | 0.3 | 0.25 |
| 20 | 0.9 | 0.9 | 0.87 |
| 40 | 1.8 | 1.6 | 1.70 |
| 70 | 3.1 | 3.0 | 2.94 |
| 100 | 4.4 | 4.3 | 4.17 |
| 150 | 6.5 | 6.5 | 6.30 |
| 200 | - | 8.6 | 8.47 |

Tab.II: Effect of $R_e$ on the re-attachment length for $W = 0.0$ and $\beta = 2.0$.

| W / β | 0.0 | 0.1 | 0.4 |
|---|---|---|---|
| 2.0 | 0.64 | 0.66 | 0.67 |
| 3.0 | 1.25 | 1.29 | 1.33 |
| 4.0 | 1.89 | 1.99 | 2.11 |

Tab. III. Re-attachment lengths for $R_e = 10$ and various W and β parameters.

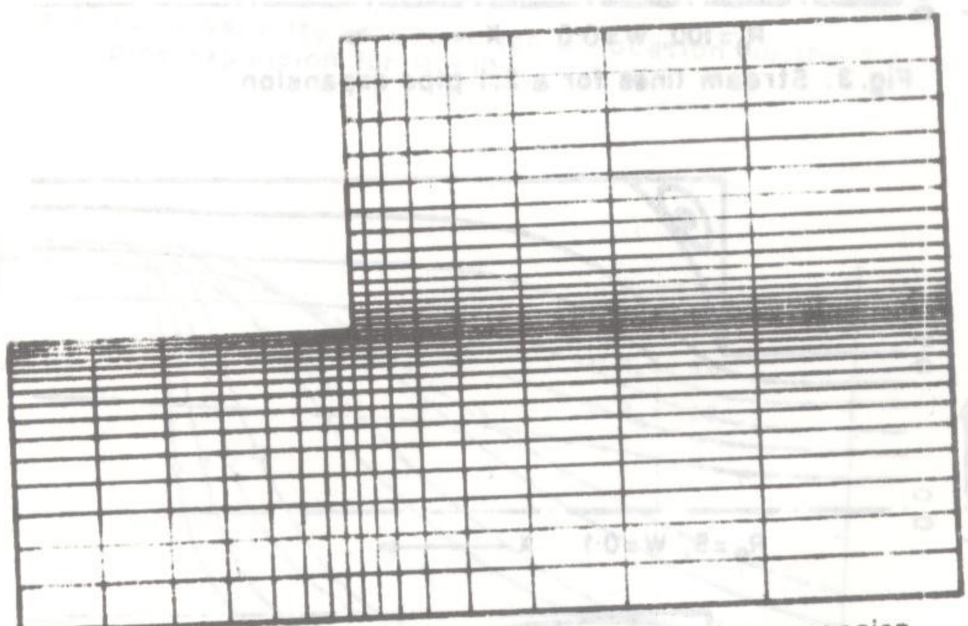

Fig. 2: A transformed coordinate mesh for the pipe expansion

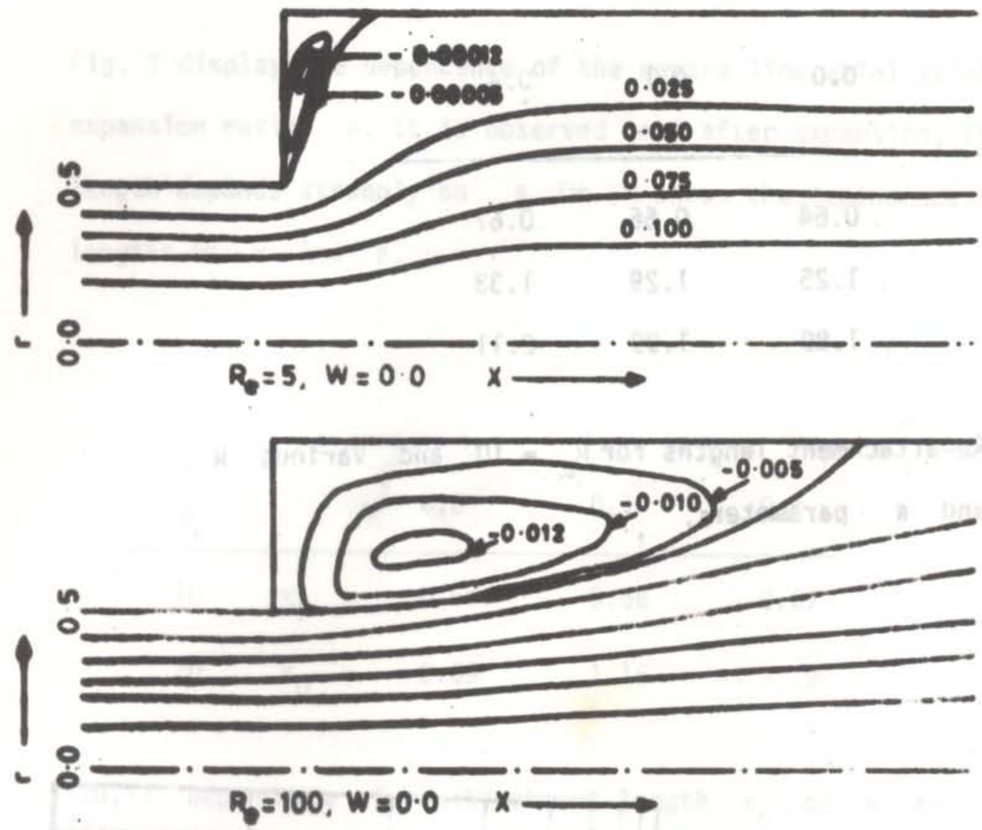


**Fig. 3: Stream lines for a 2:1 pipe expansion**

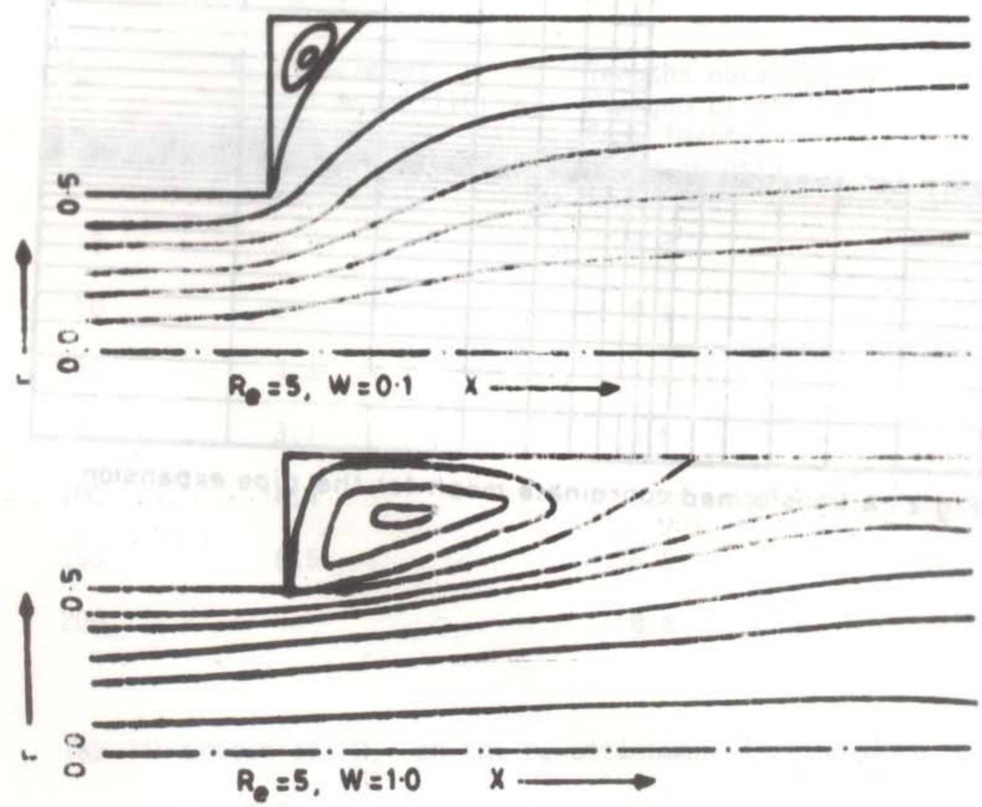


**Fig. 4: Stream lines for a 2:1 pipe expansion**

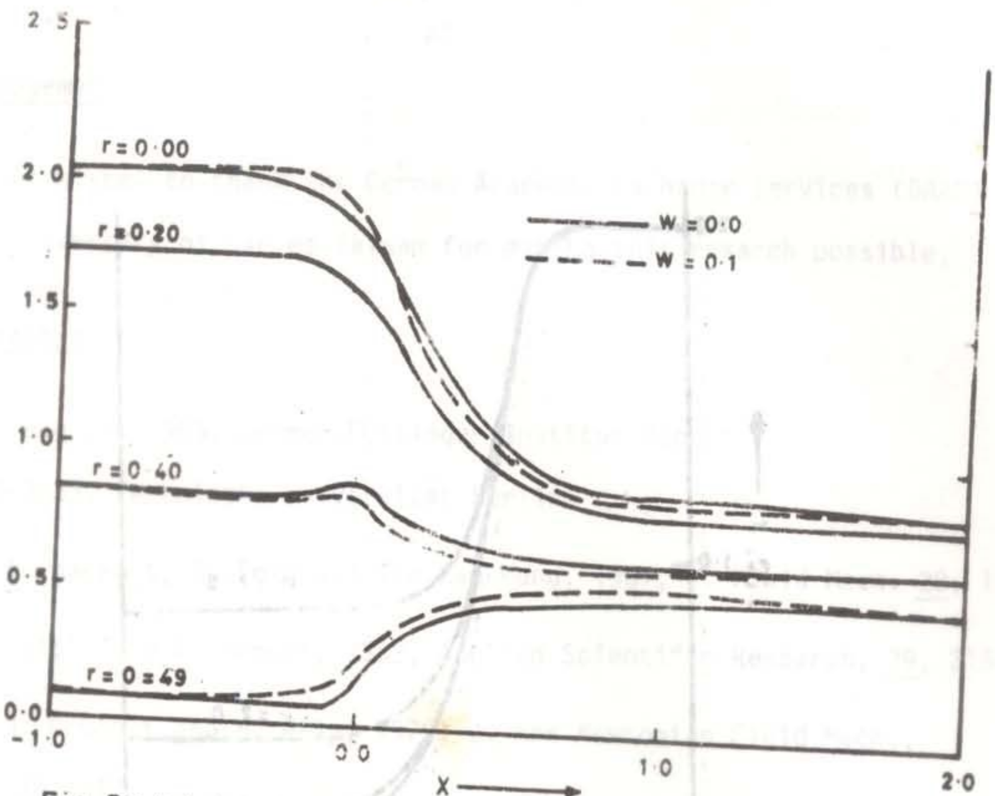


Fig. 5: Axial velocity as a function of location for the 2:1 pipe expansion for $R_e = 10$.

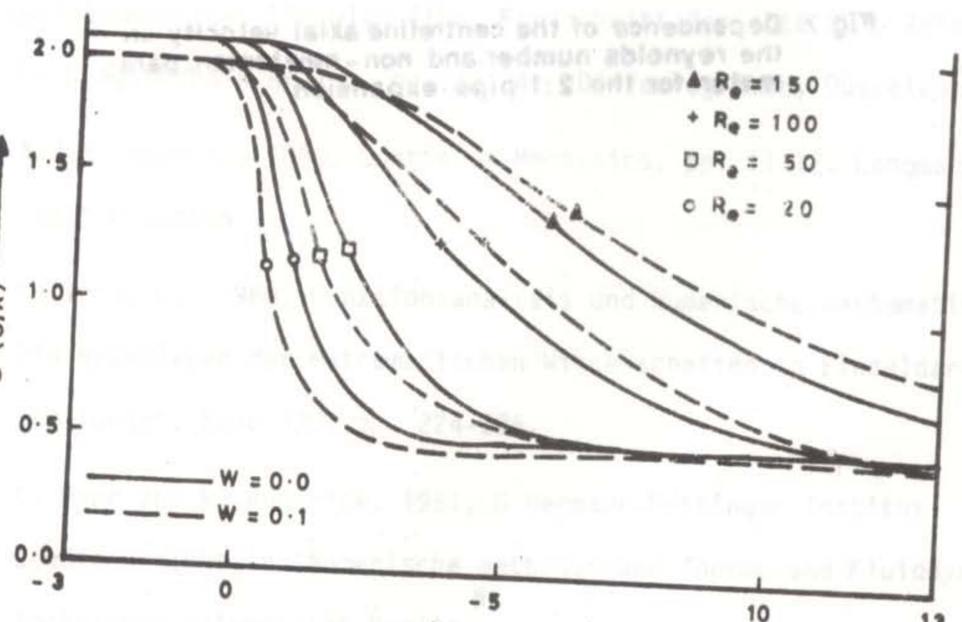


Fig. 6: Variation of centreline axial velocity with the pipe expansion ratios.

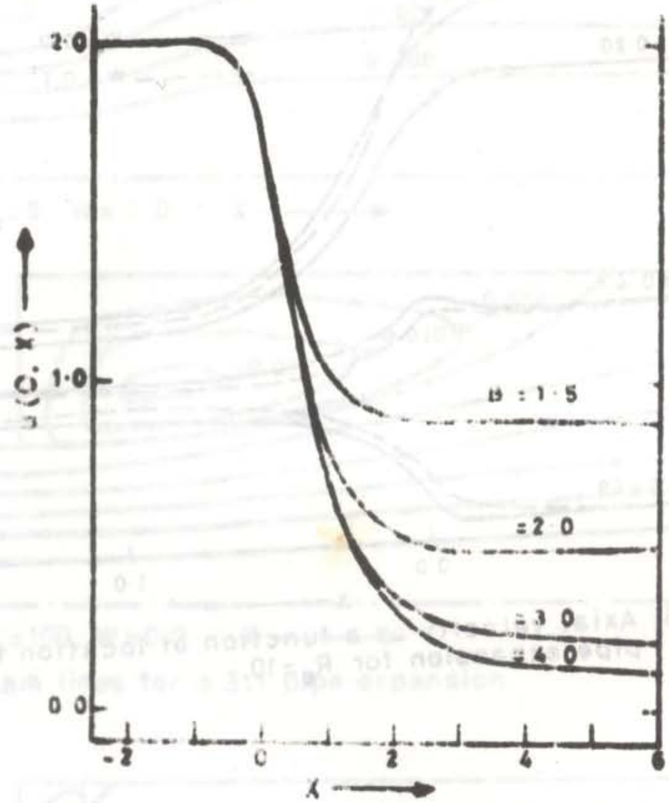


**Fig. 7 Dependence of the centreline axial velocity on the reynolds number and non-newtonian parameter for the 2:1 pipe expansion.**

## Acknowledgement

The author wishes to thank the German Academic Exchange Services (DAAD) and the University of Dar es Salaam for making this resarch possible.

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## APPENDIX

**A. Transformation equations for $\psi = \psi(x(\xi), r(\eta))$**

**We use subscripts for partial differentiation, e.g.**

$$\frac{\partial}{\partial x}(\ ) = (\ )_x$$

$$\psi_x = \frac{1}{x_\xi}\ \psi_\xi \tag{A1}$$

$$\psi_{xx} = \left(\frac{1}{x_\xi}\right)^2\ \psi_{\xi\xi} - \frac{x_{\xi\xi}}{(x_\xi)^2}\ \psi_\xi \tag{A2}$$

$$\psi_{xxx} = \frac{1}{(x_\xi)^3}\psi_{\xi\xi\xi} - 3\frac{x_{\xi\xi\xi}}{(x_\xi)^4}\ \psi_{\xi\xi} + \left[3\frac{(x_{\xi\xi})^2}{(x_\xi)^5} - \frac{x_{\xi\xi\xi}}{(x_\xi)^4}\right]\psi_\xi . \tag{A3}$$

$$\psi_{xxxx} = \frac{1}{(x_\xi)^4}\psi_{\xi\xi\xi\xi} - 6\frac{x_{\xi\xi}}{(x_\xi)^5}\psi_{\xi\xi\xi} + \left[15\frac{(x_{\xi\xi})^2}{(x_\xi)^6} - 4\frac{x_{\xi\xi\xi}}{(x_\xi)^5}\right]\psi_{\xi\xi}$$

$$+ \left[10\frac{x_{\xi\xi}.x_{\xi\xi\xi}}{(x_\xi)^6} - 15\frac{(x_{\xi\xi\xi})^3}{(x_\xi)^7} - \frac{x_{\xi\xi\xi\xi}}{x_\xi^{\ 5}}\right]\psi_\xi . \tag{A4}$$

**For partial derivatives w.r.t r similar expressions to (A1) - (A4) are obtained by interchanging x with r and $\xi$ with $\eta$. Mixed derivatives are**

$$\psi_{xr} = \frac{1}{r_\eta\ x_\xi}\psi_{\xi\eta} \tag{A5}$$

$$\psi_{xxr} = \frac{1}{r_\eta(x_\xi)^2}\ \psi_{\xi\xi\eta} - \frac{x_{\xi\xi}}{r_\eta(x_\xi)^3}\ \psi_{\xi\eta} \tag{A6}$$

$$\psi_{xrr} = \frac{1}{(r_\eta)^2 x_\xi}\ \psi_{\xi\eta\eta} - \frac{r_{\eta\eta}}{(r_\eta)^3 x_\xi}\ \psi_{\xi\eta} \tag{A7}$$

$$\psi_{xxrr} = \frac{1}{(r_\eta\, x_\xi)^2}\,\psi_{\xi\xi\eta\eta} - \frac{r_{\eta\eta}}{(r_\eta)^3 (x_\xi)^2}\,\psi_{\xi\xi\eta} - \frac{x_{\xi\xi}}{(r_\eta)^2 (x_\xi)^3}\,\psi_{\xi\eta\eta} +$$

$$+ \frac{x_{\xi\xi} r_{\eta\eta}}{(r_\eta\, x_\xi)^3}\,\psi_{\xi\eta} \qquad \text{(A8)}$$

B. <u>**Derivatives of transformation functions**</u>

$$x_\xi = \alpha_x + \frac{\theta 3(\xi-\xi_0)^2\,(\gamma-\xi) + (\xi-\xi_0)^3}{(\gamma-\xi)^2} \qquad \text{(B1)}$$

$$x_{\xi\xi} = \frac{\theta 6(\xi-\xi_0)(\gamma-\xi)^2 + 6(\xi-\xi_0)^2\,(\gamma-\xi) + 2\,(\xi-\xi_0)^3}{(\gamma-\xi)^3}$$

$$x_{\xi\xi\xi} = \frac{\theta 6(\gamma-\xi)^3 + 18\,(\xi-\xi_0)(\gamma-\xi)^2 + 18\,(\xi-\xi_0)^2\,(\gamma-\xi) + 6\,(\xi-\xi_0)^3}{(\gamma-\xi)^4} \qquad \text{(B3)}$$

$$x_{\xi\xi\xi\xi} = \frac{\theta 24(\gamma-\xi)^3 + 72(\xi-\xi_0)(\gamma-\xi)^2 + 72(\xi-\xi_0)^3(\gamma-\xi) + 24(\xi-\xi_0)^4}{(\gamma-\xi)^5} \qquad \text{(B4)}$$

$$r_\eta = 3a\eta^2 + 2b\eta + c \qquad \text{(B5)}$$

$$r_{\eta\eta} = 6a\eta + 2b \qquad \text{(B6)}$$

$$r_{\eta\eta\eta} = 6a \qquad \text{(B7)}$$

$$r_{\eta\eta\eta\eta} = 0 \qquad \text{(B8)}$$