TRANSPORT MATRICES OF FINITE BEAMS RESTING ON ELASTIC FOUNDATIONS A SUMMARY BY

GIGER(1) M. W. AND SHIRIMA(2) L. M.

ABSTRACT

A method of derivation of the coefficients of the transport matrix for a finite element resting on one and two-parameter elastic foundation soils is outlined. The coefficients are summarized for both the Benoulli-Euler and the Timoshenko beam model in tabular form, for general reference. In addition, bounds on the parameters describing beam/soil flexure, beam and soil shear, which govern the validity of the two-parameter elastic solutions are provided in graphical form.

INTRODUCTION

Through the availability of powerful micro-computers nowadays available to a large number of design engineers, the application of finite element technique for use in foundation engineering design is becoming increasingly more feasible. A large number of authors [1,2,3,4,10], among many, have formulated solutions to this type of problem. Scott[11] and Selvadurai[12] in their books provide detailed reviews on various models published during the past. Without exceptions, all feasible, numerical solutions on elastic beams resting on elastic foundation soils are based on some system of linear springs representing the foundation soil in question. The majority of solutions to problems on elastic foundations are based on the classical Bernoulli - Euler beam theory wherein the effect of beam shear is neglected. This type of beam has been modelled as being supported by a soil exhibiting zero shear resistance or one that exhibits non-zero shear resistance. The most practical and, thus, the most popular two beam foundation types are generally referred to as "one-parameter elastic foundation" and "two-parameter elastic foundation" both based on linear elastic spring analogy. Since, under certain conditions, neglecting beam shear affects were recognized to lead to significant errors, researchers have developed solutions to the problem of concern herein based on the Timoshenko beam resting on one or two-parameter elastic foundations.

(1) Associate Professor, Department of Civil Engineering, Head Soils Mechanics and Foundation Engineering

(2) Assistant Lecturer, Department of Civil Engineering, currently on PT, Technishe Hochschule, Dortmund, FRG.

This paper deals with the transport matrix which in structural engineering has been subject to treatment already back in the sixties [7,8,9]. Filipkowski[5], in context with the development of a new version of displacement method, known as exact finite element method, showed that the transport matrix bears a link to the force–displacement relationship of a finite element. With this fact in mind the transport matrix for a finite element of the Timoshenko type, resting on a two–parameter elastic foundation, was formulated. It is the objective of this paper to outline the development of the transport matrix pertaining to a finite Timoshenko beam element resting on a one or two–parameter elastic spring foundation, to list the matrix coefficients for the Bernoulli – Euler and Timoshenko beam models and to indicate bounds of parameters governing the validity of the two–parameter elastic foundation solution. The development of the force – displacement relationship for the Timoshenko beam element resting on a two parameter elastic spring foundation is subject to a detailed presentation in another paper which is currently under preparation.

GOVERNING DIFFERENTIAL EQUATIONS OF TIMOSHENKO BEAM

The Timoshenko beam theory [6] includes the effect of transverse shear. It is assumed that (a) lateral deflections are small when compared with the thickness of the beam, (b) planes normal to the neutral axis remain plane but do not, in general, remain normal to the neutral axis and (c) stresses transverse to the beam axis are negligible. Consider Figure 1 below:

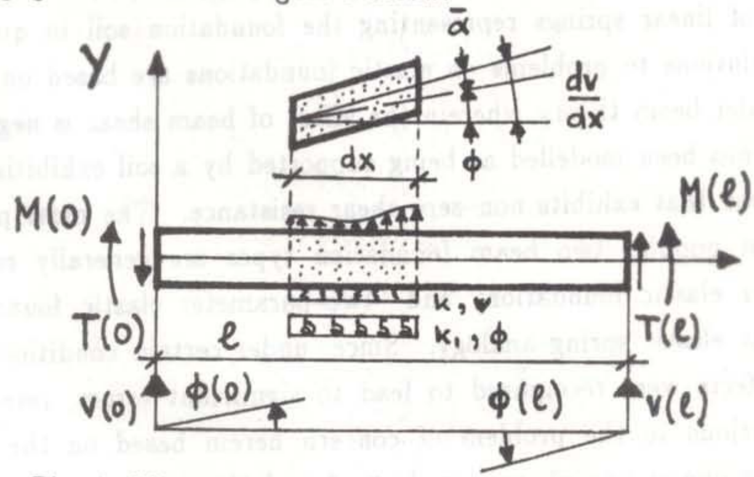


Fig. 1 Kinematics of Deformation of the Beam Element.

from where it can be seen that the angular bending distortion, ϕ, or rotation of the normal to the neutral axis of an element dx is of the form:

$$\phi = \frac{dv}{dx} - \overline{\alpha} \tag{1}$$

in which dv/dx stands for the slope of the elastic curve of the beam element at point x and $\overline{\alpha}$ represents the additional rotation of the neutral beam axis due to transverse shear deformation.

If k denotes the modulus of subgrade reaction and k_1 is a measure of rotational stiffness of the subgrade then, by making reference to Figure 1, the total potential energy of the element – force system, can be written:

$$I(v,\phi) = \tfrac{1}{2}\int_0^l EI(\phi')^2dx + \tfrac{1}{2}\int_0^l \aleph GA(v'-\phi)^2dx + \tfrac{1}{2}\int_0^l kv^2dx$$

$$+ \tfrac{1}{2}\int_0^l k_1\phi^2dx - \int_0^l p(x)vdx - Tv\Big|_0^l - M\phi\Big|_0^l, \tag{2}$$

where EI = flexural rigidity of the beam
GA = shear rigidity of the beam
I = moment of inertia of the cross-section
A = area of cross-section
l = element length
x = coordinate, independent variable along neutral beam axis
$\phi(x)$ = rotation of the normal to the neutral axis
$v(x)$ = vertical displacement of neutral axis, v(x)
$'$ = first derivative
$\aleph$ = warping constant
$p(x)$ = external, continous load normal to the neutral axis
$T(x)$ = section shear forces in y – direction
$M(x)$ = section bending moments.

The application of the principle of virtual work, from which it follows that equilibrium of a deformable system is subject to the condtion:

$$\delta I(v,\phi) = 0, \tag{3}$$

where δ denotes the first variation, leads to the following set of two simultaneous differential equations in v and ϕ for beam elements with constant prismatic cross-sections,

$$-\aleph GA\frac{d^2v}{dx^2} + kv + \aleph GA\frac{d\phi}{dx} = p(x)$$

$$\aleph GA\frac{dv}{dx} + EI\frac{d^2\phi}{dx^2} - (\aleph GA + k_1)\phi = 0 \tag{4}$$

In the absence of geometric boundary conditions, the associated natural boundary conditions are:

$$M(0) = EI\phi'(0) \quad \text{and} \quad T(0) = \aleph GA[v'(0) - \phi(0)] \tag{5}$$

By assuming a function f(x) such that

$$v = -\frac{1}{\aleph GA}[EI\frac{d^2}{dx^2} - (\aleph GA + k_1)]f(x); \qquad \phi = \frac{d\,f(x)}{dx} \tag{6}$$

and by substitution of the expressions in Eq. (6) into Eq. (4), one obtains the governing differential equation of a Timoshenko beam resting on a two-parameter elastic foundation:

$$EI\frac{d^4f(x)}{dx^4} - (k_1 + \frac{kEI}{\aleph GA})\frac{d^2f(x)}{dx^2} + k(1 + \frac{k_1}{\aleph GA})f(x) = p(x). \tag{7}$$

with $\eta = 1/\aleph Ga$, Eq. (6) can be rewritten to read:

$$v(x) = -EI\eta\frac{d^2f(x)}{dx^2} + (1 + k_1\eta)\,f(x); \qquad \phi(x) = \frac{df(x)}{dx} \tag{8}$$

According to Eq. (5) and (6) the bending moment, M, and the shear force, T, are given by Eq. (9) below:

$$M(x) = EI\frac{d^2f(x)}{dx^2} \quad \text{and} \quad T(x) = -EI\frac{d^3f(x)}{dx^3} + k_1\frac{df(x)}{dx} \tag{9}$$

SOLUTION TO HOMOGENEOUS DIFFERENTIAL EQUATION

A solution to the homogeneous differential equation (7) [p(x) = 0] does exist in the form of

$$f^{(n)}(x) = A\phi_1^{(n)}(x) + B\phi_2^{(n)}(x) + C\phi_3^{(n)}(x) + D\phi_4^{(n)}(x) \tag{10}$$

where $^{(n)}$ stands for the nth derivative ($n \geq 0$); A, B, C and D are constants of integration and the functions $\phi_m(x)$ take the form,

$$\begin{aligned} \phi_1(x) &= \sin(a\lambda x)\cdot\sinh(b\lambda x) \\ \phi_2(x) &= \sin(a\lambda x)\cdot\cosh(b\lambda x) \\ \phi_3(x) &= \cos(a\lambda x)\cdot\sinh(b\lambda x) \\ \phi_4(x) &= \cos(a\lambda x)\cdot\cosh(b\lambda x). \end{aligned} \tag{11}$$

The coefficients a, b and λ are defined:

$$a = \sin\frac{\varphi}{2}; \quad b = \cos\frac{\varphi}{2}; \quad \lambda = \left[\frac{k}{EI}(1+k_1\eta)\right]^{1/4} \tag{12}$$

where φ is determined from the relationship,

$$\varphi = \arctan\frac{\left[4kEI(1+k_1\eta)-(k_1+kEI\eta)^2\right]^{1/2}}{(k_1+kEI\eta)}. \tag{13}$$

By defining

$$k_2 = \lambda^2 EI = \left[kEI(1+k_1\eta)\right]^{1/2} \tag{14}$$

and making use of TABLE I below, one can readily

TABLE I. Values of Function ϕ_m and Its Derivatives For $x = 0$

m	$\phi_m(0)$	$\phi_m'(0)$	$\phi_m''(0)$	$\phi_m'''(0)$
1	0	0	$2\lambda^2 ab$	0
2	0	λa	0	$-\lambda^3 a(1-4b^2)$
3	0	λa	0	$\lambda^3 a(1-ab^2)$
4	1	0	$-\lambda^2(a^2-b^2)$	0

express the values at $x = 0$ of the functions given in Eq. (8) and (9) in matrix form:

$$\begin{Bmatrix} v(0) \\ \phi(0) \\ M(0) \\ T(0) \end{Bmatrix} = \begin{bmatrix} -2abk_2\eta & 0 & 0 & 1+k_1\eta+(a^2-b^2)k_2\eta \\ 0 & a\lambda & b\lambda & 0 \\ 2abk_2 & 0 & 0 & -(a^2-b^2)k_2 \\ 0 & (a\lambda[k_1+(1-4b^2)k_2]) & (b\lambda[k_1-(1-4a^2)k_2]) & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \tag{15}$$

By inspection of Eq. (15), it is immediately apparent that the following systems of linear equations can be written and solved in two sets of two simultaneous linear equations to obtain the constants of integration (A,B,C,D) in terms of the boundary parameters shown in Fig. 2(a) and 2(b).

$$\begin{bmatrix} -2abk_2\eta & 1+k_1\eta+(a^2-b^2)k_2\eta \\ 2abk_2 & -(a^2-b^2)k_2 \end{bmatrix} \begin{Bmatrix} A \\ D \end{Bmatrix} = \begin{Bmatrix} v(0) \\ M(0) \end{Bmatrix} = \begin{Bmatrix} v_i \\ -M_i \end{Bmatrix} \tag{16(a)}$$

and

$$\begin{bmatrix} a\lambda & b\lambda \\ a\lambda[k_1+(1-4b^2)k_2] & b\lambda[k_1-(1-4a^2)k_2] \end{bmatrix} \begin{Bmatrix} B \\ C \end{Bmatrix} = \begin{Bmatrix} \phi(0) \\ T(0) \end{Bmatrix} = \begin{Bmatrix} \phi_i \\ -T_i \end{Bmatrix} \tag{16(b)}$$

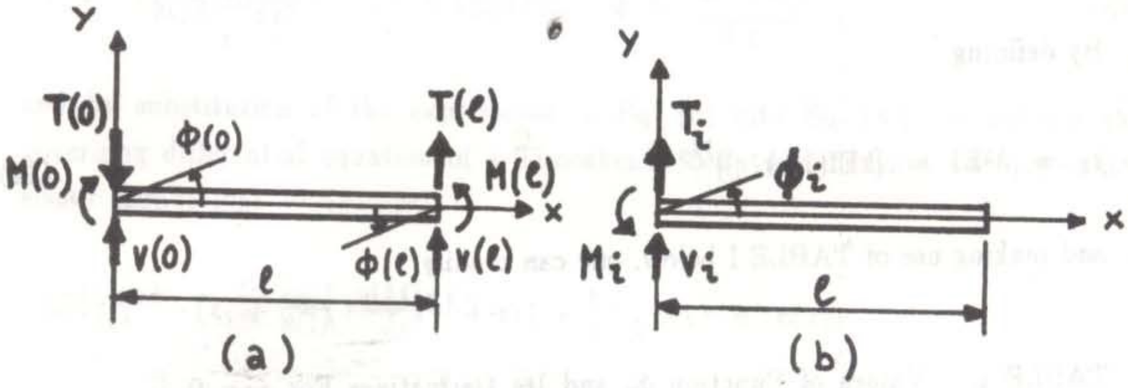


Fig. 2. Boundary Parameters for Beam Element.

Employing Cramer's rule and by making reference to the intial parameters of Figure 2 (b), the constants of integration turn out to be of the form:

$$A = \frac{a^2-b^2}{2ab(1+k_1\eta)} v_i - \frac{1+[k_1+(a^2-b^2)k_2]\eta}{2ab(1+k_1\eta)k_2} M_i\,,$$

$$B = \frac{k_1-(1-4a^2)k_2}{2a\lambda k_2} \phi_i + \frac{1}{2a\lambda k_2} T_i\,, \tag{17}$$

$$C = -\frac{k_1+(1-4b^2)k_2}{2b\lambda k_2} \phi_i - \frac{1}{2b\lambda k_2} T_i\,,$$

$$D = \frac{1}{1+k_1\eta} v_i - \frac{\eta}{1+k_1\eta} M_i\,.$$

Making use of the constants in Eq. (17), the tabulated functions of TABLE II below in combination with Eq. (11) and Eq. (10), permits one to write the solution vector $\{v(x), \phi(x), T(x), M(x)\}^T$ in matrix form with reference to Eq. (8) and (9):

$$\begin{Bmatrix} v(x) \\ \phi(x) \\ T(x) \\ M(x) \end{Bmatrix} = \begin{bmatrix} B_{VV} & B_{V\phi} & B_{VT} & B_{VM} \\ B_{QV} & B_{\phi\phi} & B_{\phi T} & B_{\phi M} \\ B_{TV} & B_{T\phi} & B_{TT} & B_{TM} \\ B_{MV} & B_{M\phi} & B_{MT} & B_{MM} \end{bmatrix} \begin{Bmatrix} v_i \\ \phi_i \\ T_i \\ M_i \end{Bmatrix} \tag{18}$$

TABLE II. Function $\phi_m(x)$ and Its Derivatives For Arbitrary Values of x

m	ϕ_m	ϕ_m'	ϕ_m''	ϕ_m'''
1	ϕ_1	$\lambda(a\phi_3+b\phi_2)$	$-\lambda^2(a^2-b^2)\phi_1+2ab\lambda^2\phi_4$	$\lambda^3 b(1-4a^2)\phi_2-\lambda^3 a(1-4b^2)\phi_3$
2	ϕ_2	$\lambda(a\phi_4+b\phi_1)$	$-\lambda^2(a^2-b^2)\phi_2+2ab\lambda^2\phi_3$	$\lambda^3 b(1-4a^2)\phi_1-\lambda^3 a(1-4b^2)\phi_4$
3	ϕ_3	$\lambda(-a\phi_1+b\phi_4)$	$-\lambda^2(a^2-b^2)\phi_3-2ab\lambda^2\phi_2$	$\lambda^3 b(1-4a^2)\phi_4+\lambda^3 a(1-4b^2)\phi_1$
4	ϕ_4	$\lambda(-a\phi_2+b\phi_3)$	$-\lambda^2(a^2-b^2)\phi_4-2ab\lambda^2\phi_1$	$\lambda^3 b(1-4a^2)\phi_3+\lambda^3 a(1-4b^2)\phi_2$

To provide an insight into how the elements of the transport matrix are calculated, the first of Eq. (8) is considered. This results in the first row of the elements in the matrix, [B(x)], shown in Eq. (18):

$$v(x) = -EI\eta\Big[A\phi_1''(x) + B\phi_2''(x) + C\phi_3''(x) + D\phi_4''(x)\Big]$$

$$+ (1 + k_1\eta)\Big[A\phi_1(x) + B\phi_2(x) + C\phi_3(x) + D\phi_4(x)\Big] \qquad (19)$$

with reference to TABLE II and Eq. (17), one may write

$$v(x) = \left[\frac{a^2-b^2}{2ab(1+k_1\eta)}v_i - \frac{1+[k_1+(a^2-b^2)k_2]\eta}{2ab(1+k_1\eta)k_2}M_i\right]\Big[\lambda^2EI\eta(a^2-b^2)\phi_1 - 2\lambda^2EI\eta ab\phi_4$$

$$-(1+k_1\eta)\phi_1\Big]$$

$$+\left[\frac{k_1-(1-4a^2)k_2}{2a\lambda k_2}\phi_i + \frac{1}{2a\lambda k_2}T_i\right]\Big[\lambda^2EI\eta(a^2-b^2)\phi_2 - 2\lambda^2EI\eta ab\phi_3 + (1+k_1\eta)\phi_2\Big]$$

$$-\left[\frac{k_1-(1-4b^2)k_2}{2b\lambda k_2}\phi_i + \frac{1}{2b\lambda k_2}T_i\right]\Big[\lambda^2EI\eta(a^2-b^2)\phi_3 + 2\lambda^2EI\eta ab\phi_2 + (1+k_1\eta)\phi_3\Big]$$

$$+\left[\frac{1}{1+k_1\eta}v_i - \frac{\eta}{1+k_1\eta}M_i\right]\Big[\lambda^2EI\eta(a^2-b^2)\phi_4 + 2\lambda^2EI\eta ab\phi_1 + (1+k_1\eta)\phi_4\Big] \qquad (20)$$

If all terms in Eq. (20) containing v_i are summed up and recognition of the fact that $a = \sin(\varphi/2)$, $b = \cos(\varphi/2)$ and $a^2+b^2 = 1$ is taken, B_{vv} is found to be of the form:

$$B_{vv} = \frac{(1+k_1\eta)(a^2-b^2) + k_2\eta}{2ab(1+k_1\eta)}\cdot\phi_1(x) + \phi_4(x) \qquad (21)$$

Similarly, all terms containing ϕ_i are collected together to obtain $B_{v\phi}$. Other elements of the transport matrix are calculated in the same way by considering each row with the corresponding equations.

TRANSPORT MATRIX COEFFICIENTS

Transport matrix coefficients are listed in the APPENDIX for both Timoshenko and Bernaulli – Euler beams resting on one and two–parameter elastic foundations.

TABLE IV contains all elements of the transport matrix for a Timoshenko beam resting on a two–parameter elastic foundation. For the Bernoulli–Euler beam resting on a two–parameter elastic foundation, one assumes $\eta = 0$ and obtains all elements of the transport matrix as shown in TABLE V. The one–parameter transport matrix coefficients subject to condition of $k_1 = 0$ are listed in TABLE VI for the Timoshenko beam element whereas those for the Bernoulli–Euler beam are tabulated in TABLE VII (note: both η and k_1 are zero). It should be born in mind that all $\phi_m = \phi_m(x)$. Also, note that the coefficients A and B used in tables bear no relationship with the constants of integration in Eq. (17). The coefficients a and b are defined by Eq. (12) and (13).

PARAMETER RANGE OF APPLICABILITY

For the solution of the homogeneous differential equation for the Timoshenko beam element resting on a two–parameter elastic foundation to be valid, the following condition must hold,

$$k_1 + kEI\eta \leq \sqrt{4kEI(1+k_1\eta)} \tag{22}$$

If one defines the dimensionless quantities Λ, Γ_1 and Γ_2 which represent the influence of the first foundation parameter k, second foundation parameter k_1 and the effect of transverse beam shear, η, respectively, where

$$\Lambda = \sqrt[4]{\frac{k\,l^4}{EI}}\,, \quad \Gamma_1 = \frac{k_1\,l^2}{EI} \quad \text{and} \quad \Gamma_2 = \eta\frac{EI}{l^2} \tag{23}$$

Eq. (22) can be rewritten to read:

$$4(1 + \Gamma_1\Gamma_2)\Lambda^4 \geq (\Gamma_1 + \Gamma_2\Lambda^4)^2\,. \tag{24}$$

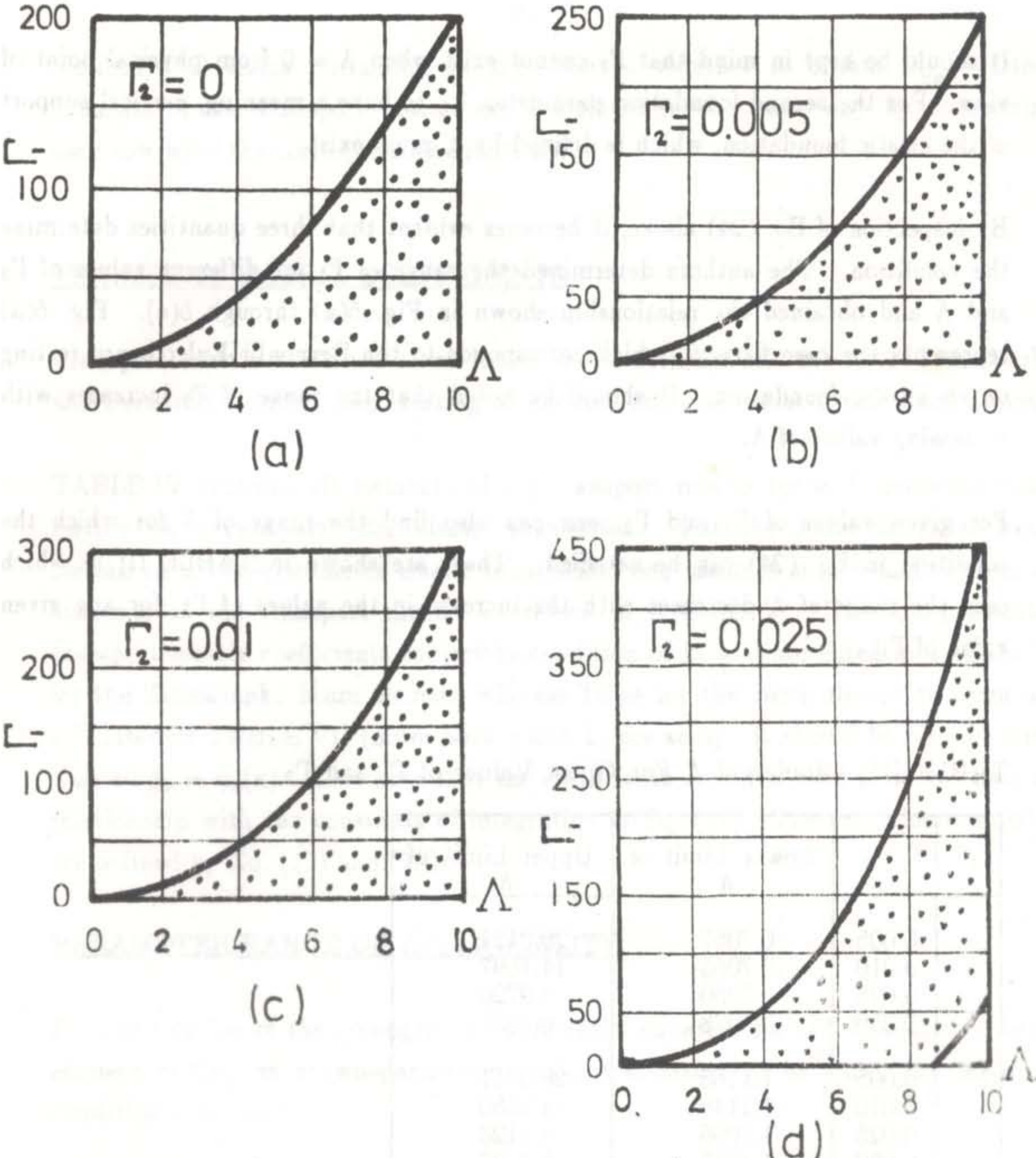
It should be kept in mind that Γ_1 cannot exist when $\Lambda = 0$ from physical point of view. For the second foundation parameter, k_1, to have a meaning, vertical support of the elastic foundation, which is defined by k, must exist.

By inspection of Eq. (24) above, it becomes evident that three quantities determine the condition. The authors determined the range of Γ_1 for different values of Γ_2 and Λ and obtained the relationship shown in Fig. 5(a) through 5(e). Fig. 5(a) represents the case $\Gamma_2 = 0$, which corresponds to the Bernoulli–Euler beam resting on an elastic foundation. It should be noted that the range of Γ_1 increases with increasing values of Λ.

For given values of Γ_1 and Γ_2, one can also find the range of Λ for which the condition in Eq. (24) can be satisfied. These are shown in TABLE III in which case the range of Λ decreases with the increase in the values of Γ_2, for any given value of Γ_1.

TABLE III. Range of Λ For Given Values of Γ_1 and Γ_2

Γ_1	Γ_2	Lower Limit of Λ	Upper Limit of Λ
1.0	0.005	0.7067	20.0124
	0.010	0.7063	14.1597
	0.025	0.7050	8.9720
	0.050	0.7028	6.3634
2.5	0.005	1.1163	20.0311
	0.010	1.1146	14.1850
	0.025	1.1096	9.0128
	0.050	1.1015	6.4197
5.0	0.005	1.5763	20.0620
	0.010	1.5715	14.2791
	0.025	1.5577	9.0788
	0.050	1.5365	6.5085
7.5	0.005	1.9276	20.0926
	0.010	1.9180	14.2770
	0.025	1.8945	9.1427
	0.050	1.8580	6.5918



$$\Lambda = \left[\frac{kl^4}{EI}\right]^{1/4}$$

$$\Gamma_1 = \frac{k_1 l^2}{EI}$$

$$\Gamma_2 = \frac{EI}{l^2}\eta$$

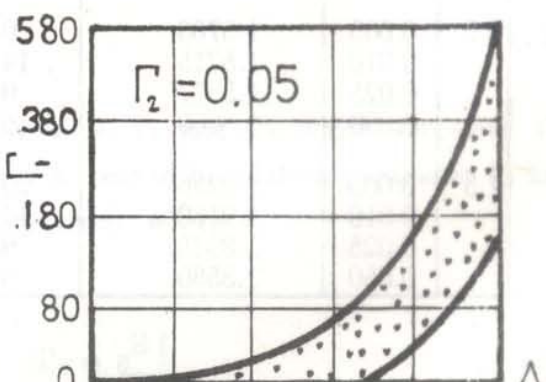


Fig. 5 Upper and Lower Limits of Γ_1 versus Λ

CONCLUSIONS AND RECOMMENDATIONS

Based on the study presented in this paper, the following conclusions and recommendations are set forth.

1. The use of finite element technique in foundation engineering design is becoming increasingly more feasible due to the widespread availability of powerful micro-computers. Thus, the transport matrix gains on important role since it is a key element in the formulation of a finite element solution.

2. The transport matrix provides complete information on the behaviour of the beam at any point x in the range $x_i \leq x \leq x_k$ to the right of the element boundary, x_i, provided the information at $x_i (x = 0)$ represented by the column vector $\{v(0), \phi(0), T(0), M(0)\}^T$ is known.

3. Since the determination of the transport matrix coefficients is quite work-involved and the nature of the transport matrices for the problem of concern in this paper have general validity, transport matrix coefficients were tabulated in TABLES IV through VIII for both Timoshenko and Bernoulli-Euler type of beam models. These coefficients may be used directly in the development of a stiffness matrix for a given beam/foundation soil system if in compliance with the assumptions made in the choice of a particular model.

4. The results shown in Figure 5 and Table III serve as a guidance for the setting up of proper constants in a given problem of a beam resting on an elastic spring foundation. These constitute the computer input for a particular numerical soulution sought by the engineer.

REFERENCES

1. Bing Y., Ting and Eldon. F. Mockry, "Beam on Elastic Foundation Finite Element," Journal of the Structural Division, ASCE, Vol. 110, No.10, Paper No.19229, October 1984.

2. Bing Y. Ting, "Finite Beams on Elastic Foundation with Restraints," Journal of the Structural Division, ASCE, Vol.108, No. ST3, March 1982.

3. Constansio Miranda and Kesharan Nair, "Finite Beams on Elastic Foundation," Journal of the Structural Division, ASCE, Vol.92. No. ST2, December 1964.

4. Feng Zhaohua and Cook R. D. "Beam Elements on Two Parameter Elastic Foundations," Journal of Structural Mechanics, ASCE, Vol. 109, No.6, Paper No.18431, December 1985.

5. Filipkowski, J., "Algorithm for the Analysis of Skeletal Structures," Unpublished Staff Seminar Paper, Civil Engineering Department, University of Dar es Salaam.

6. Hinton, E. and Owen, D. R. J., An Introduction to Finite Element Computations, Pineridge Press Limited, 1979.

7. Livesley, R. K., Matrix Methods of Structural Analysis, Pergamon Press, Oxford, 1964, pp. 177 – 198.

8. Pestel, E. C. and Leckie, F. A., Matrix Methods of Elastostatics, McGraw–Hill, New York, 1963, pp. 130 – 191.

9. Tuwa, J. J., Theory and Problems of Structural Analysis, Schaum's Outline Series, McGraw–Hill, New York, 1969, pp. 138 – 161.

10. Toyoaki Nogami and Oneill, M. W. "Beam on Generalised Two parameter Foundation," Journal of Engineering Mechanics, ASCE Vol.111, No.5, Paper No.19701, May 1985.

11. Scott, R. F. Foundation Analysis, Prentice–Hall, Inc., 1981.

12. Selvadurai A. P. S., Elastic Analysis of Soil–Foundation Interaction, Elsevier Scientific Publishing Company, Amsterdam, Holland 1979.

LIST OF SYMBOLS

A	=	Area of Beam Cross–Section
G	=	Shear Modulus of Beam Material
$I(\phi,v)$	=	Total Potential Energy
$M(x)$	=	Section Bending Moment
M_i	=	$-M(0)$ Applied End Moment

M_k	=	$-M(l)$ Applied End Moment
$p(x)$	=	External, Continuous Load
$v(x)$	=	Total Vertical Displacement of Neutral Axis
$'$	=	First Derivative
$\phi(x)$	=	Rotation of Neutral Axis Due to Bending
$\phi_m(x)$	=	Arbitrary Functions for m = 1, 2, 3, 4
E	=	Young's Modulus of Beam Material
I	=	Area Moment of Inertia of Cross-Section
$T(x)$	=	Section Shear Force in y-direction
T_i	=	$-T(0)$ Applied End Force
T_k	=	$-M(l)$ Applied End Force
k, k_1	=	First and Second Soil Parameters
l	=	Beam Element Length
x	=	Coordinate, Independent Variable Along Neutral Beam Axis
$\aleph$	=	Warping Constant
δ	=	First Variation
$\Gamma_1, \Gamma_2, \Lambda$	=	Dimensionless Parameters

APPENDIX
TRANSPORT MATRIX COEFFICIENTS

TABLE IV: Timoshenko Beam / Two-Parameter Elastic Foundation

$\lambda = \sqrt[4]{\dfrac{(1+k_1\eta)k}{EI}}, \quad k_2 = \lambda^2 EI, \quad A = 1+\eta(k_1+k_2), \; B = 1+\eta(k_1-k_2)$	
B_{vv}	$\dfrac{a^2A-b^2B}{ab(A+B)}\cdot\phi_1 + \phi_4$
$B_{\phi v}$	$\dfrac{-\lambda}{ab(A+B)}\cdot(b\phi_2 + a\phi_3)$
B_{Tv}	$\dfrac{\lambda}{ab(A+B)}\cdot[-b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3]$
B_{Mv}	$\dfrac{-k_2}{ab(A+B)}\cdot\phi_1$
$B_{v\phi}$	$\dfrac{1}{2ab\lambda}\cdot(b\phi_2 + a\phi_3)$
$B_{\phi\phi}$	$\dfrac{a^2(k_1+k_2)+b^2(k_1-k_2)}{2abk_2}\cdot\phi_1 + \phi_4$
$B_{T\phi}$	$\dfrac{a^2(k_1+k_2)^2+b^2(k_1-k_2)^2}{2abk_2}\cdot\phi_1$
$B_{M\phi}$	$\dfrac{1}{2ab\lambda}\cdot[b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3]$
B_{vT}	$\dfrac{1}{2ab\lambda k_2}\cdot(bB\phi_2 - aA\phi_3)$
$B_{\phi T}$	$\dfrac{1}{2abk_2}\cdot\phi_1$
B_{TT}	$\dfrac{a^2(k_1+k_2)+b^2(k_1-k_2)}{2abk_2}\cdot\phi_1 - \phi_4$
B_{MT}	$\dfrac{1}{2ab\lambda}\cdot(b\phi_2 + a\phi_3)$
B_{vM}	$\dfrac{-1}{2abk_2}\cdot\phi_1$
$B_{\phi M}$	$\dfrac{-\lambda}{ab(A+B)k_2}\cdot(bB\phi_2 + aA\phi_3)$
B_{TM}	$\dfrac{-\lambda}{ab(A+B)}\cdot(b\phi_2 - a\phi_3)$
B_{MM}	$\dfrac{a^2A-b^2B}{ab(A+B)}\cdot\phi_1 - \phi_4$

TABLE V: Bernoulli–Euler Beam / Two–Parameter Elastic Foundation

$\lambda = \sqrt[4]{\frac{k}{EI}}$, $k_2 = \lambda^2 EI$, $A = B = 1$ $\eta = 0$	
B_{vv}	$\frac{a^2-b^2}{2ab}\cdot\phi_1 + \phi_4$
$B_{\phi v}$	$\frac{-\lambda}{2ab}\cdot(b\phi_2 - a\phi_3)$
B_{Tv}	$\frac{\lambda}{2ab}\cdot[-b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3]$
B_{Mv}	$\frac{-k_2}{2ab}\cdot\phi_1$
$B_{v\phi}$	$\frac{1}{2ab\lambda}\cdot(b\phi_2 + a\phi_3)$
$B_{\phi\phi}$	$\frac{a^2(k_1+k_2)+b^2(k_1-k_2)}{2abk_2}\cdot\phi_1 + \phi_4$
$B_{T\phi}$	$\frac{a^2(k_1+k_2)^2+b^2(k_1-k_2)^2}{2abk_2}\cdot\phi_1$
$B_{M\phi}$	$\frac{1}{2ab\lambda}\cdot[b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3]$
B_{vT}	$\frac{1}{2ab\lambda k_2}\cdot(b\phi_2 - a\phi_3)$
$B_{\phi T}$	$\frac{1}{2abk_2}\cdot\phi_1$
B_{TT}	$\frac{a^2(k_1+k_2)+b^2(k_1-k_2)}{2abk_2}\cdot\phi_1 - \phi_4$
B_{MT}	$\frac{1}{2ab\lambda}\cdot(b\phi_2 + a\phi_3)$
B_{vM}	$\frac{-1}{2abk_2}\cdot\phi_1$
$B_{\phi M}$	$\frac{-\lambda}{2abk_2}\cdot(b\phi_2 + a\phi_3)$
B_{TM}	$\frac{-\lambda}{2ab}\cdot(b\phi_2 - a\phi_3)$
B_{MM}	$\frac{a^2-b^2}{2ab}\cdot\phi_1 - \phi_4$

TABLE VI: Timoshenko Beam / One–Parameter Elastic Foundation

$\lambda = \sqrt[4]{\frac{k}{EI}}$, $k_2 = \lambda^2 EI$, $A = 1+k_2\eta$, $B = 1-k_2\eta$	
B_{vv}	$\frac{a^2-b^2+k_2\eta}{2ab}\cdot\phi_1 + \phi_4$
$B_{\phi v}$	$\frac{\lambda}{2ab}\cdot(b\phi_2 - a\phi_3)$
B_{Tv}	$\frac{\lambda}{2ab}\cdot(b\phi_2 + a\phi_3)$
B_{Mv}	$\frac{-k_2}{2ab}\cdot\phi_1$
$B_{v\phi}$	$\frac{1}{2ab\lambda}\cdot(b\phi_2 + a\phi_3)$
$B_{\phi\phi}$	$\frac{a^2-b^2}{2ab}\cdot\phi_1 + \phi_4$
$B_{T\phi}$	$\frac{k_2}{2ab}\cdot\phi_1$
$B_{M\phi}$	$\frac{k_2}{2ab\lambda}\cdot(-b\phi_2 + a\phi_3)$
B_{vT}	$\frac{1}{2ab\lambda k_2}\cdot[b(1-k_2\eta)\phi_2 - a(1+k_2\eta)\phi_3)$
$B_{\phi T}$	$\frac{1}{2abk_2}\cdot\phi_1$
B_{TT}	$\frac{a^2-b^2}{2abk_2}\cdot\phi_1 - \phi_4$
B_{MT}	$\frac{1}{2ab\lambda}\cdot(b\phi_2 + a\phi_3)$
B_{vM}	$\frac{-1}{2abk_2}\cdot\phi_1$
$B_{\phi M}$	$\frac{-\lambda}{2abk_2}\cdot[b(1-k_2\eta)\phi_2 + a(1+k_2\eta)\phi_3]$
B_{TM}	$\frac{-\lambda}{2ab}\cdot(b\phi_2 - a\phi_3)$
B_{MM}	$\frac{a^2-b^2+k_2\eta}{2ab}\cdot\phi_1 - \phi_4$

TABLE VII: Bernoulli–Euler Beam / One–Parameter Elastic Foundation

$\lambda = \sqrt[4]{\frac{k}{EI}}$, $k_2 = \lambda^2 EI$, $A = B = 1$ $a = b = \frac{1}{\sqrt{2}}$ $k_1 = 0$	
B_{vv}	ϕ_4
$B_{\phi v}$	$\frac{-\lambda}{\sqrt{2}}\cdot(\phi_2 - \phi_3)$
B_{Tv}	$\frac{-\lambda k_2}{\sqrt{2}}\cdot(\phi_2 + \phi_3)$
B_{Mv}	$-k_2\cdot\phi_1$
$B_{v\phi}$	$\frac{1}{\lambda\sqrt{2}}\cdot(\phi_2 + \phi_3)$
$B_{\phi\phi}$	ϕ_4
$B_{T\phi}$	$k_2\cdot\phi_1$
$B_{M\phi}$	$\frac{-k_2}{\lambda\sqrt{2}}\cdot(\phi_2 - \phi_3)$
B_{vT}	$\frac{1}{\lambda k_2\sqrt{2}}\cdot(\phi_2 - \phi_3)$
$B_{\phi T}$	$\frac{1}{k_2}\cdot\phi_1$
B_{TT}	$-\phi_4$
B_{MT}	$\frac{1}{\lambda\sqrt{2}}\cdot(\phi_2 + \phi_3)$
B_{vM}	$\frac{-1}{k_2}\cdot\phi_1$
$B_{\phi M}$	$\frac{-\lambda}{\sqrt{2}k_2}\cdot(\phi_2 + \phi_3)$
B_{TM}	$\frac{-\lambda}{\sqrt{2}}\cdot(\phi_2 - \phi_3)$
B_{MM}	$-\phi_4$