FULL LENGTH RESERCH ARTICLE

NEW GAUSSIAN POINTS FOR THE SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

*CHOLLOM, J. P[1]. & MADUGU, S. E[2]

[1]*Department of Mathematics*
University of Jos, Nigeria.
[2]*Department of Primary Studies*
College of Education Gindiri,
Plateau State, Nigeria
*(Corresponding author)
chollomj@yahoo.com

ABSTRACT

The Gauss Radau and the Lobatto points make use of the roots of the Legendre polynomial located within the step $[-1,1]$. In this paper, a new set of Gaussian points has been proposed and used as collocation points for the construction of block numerical methods for the solution of first order IVP through transformation within the step $[x_n, x_{n+2}]$. The new points resulted into stable numerical block methods of order 2m suitable for solving both stiff and non-stiff IVP. Numerical experiments carried out using the new Gaussian points revealed there efficiency on stiff differential equations. The results also reveal that methods using the new Gaussian points are more accurate than those using the standard Gaussian points on non-stiff initial value problems.

Keywords: Gaussian points, Collocation points, Legendre polynomial, Gauss,Lobatto, Block integrators, stiff and non-stiff IVP's

INTRODUCTION

Various studies have been undertaken on methods of solutions of ordinary Differential equations. Recently, interest has shifted to the solution of Ode's using numerical block methods because of their computational advantages. They are self starting since they are used simultaneously for parallel integration (Onumanyi *et al.,* 1994) and useful for dense output. These methods are based on the multi step collocation approach (Lie & Norsett, 1986). Because of the advantages offered by collocation at Gaussian points (Ascher & Baders, 1986), several authors have been involved in the study of schemes with collocation at Gaussian points.

It has been established (Burrage & Butcher, 1979) that all collocation schemes at Gauss points satisfy stability for the test equation while all collocation schemes at Lobatto points do not. This is because collocation at Gaussian points leads to Runge-Kutta schemes which are algebraically stable whereas collocation at Lobatto points lead to schemes which are not algebraically stable (Burage & Butcher 1979; Hairer & Wanner, 1981). Furthermore, Ascher & Butcher (1986) observed that symmetric algebraic stable collocation scheme has to be at Gaussian points. Yakubu (2005) further considered and investigated Gaussian points used as collocation points within the interval $[x_n, x_{n+1}]$ which provided block multi-finite difference methods for simultaneous solutions of ODE's.

In this paper, a new set of Gaussian points is presented within the step $[x_n, x_{n+2}]$.The new points used as collocation points yield a single continuous formula which is evaluated at some distinct points along with its first derivative to yield a multi-finite difference formula for simultaneous application to the Ordinary differential Equations with either an initial, boundary or mixed condition.

Derivation of the new Gaussian points

The solution of the first order system of ODE y'= f(x, y) where y satisfies a set of conditions which are either initial or boundary has been sought using various methods. The multi-step collocation formular using a polynomial of the form

$$y(x) = \sum_{i=0}^{m-1} \phi_i(x) y_{n+i} + h \sum_{i=0}^{n-1} \psi_i(x) f(x_i, y(x)), x_n \le x \le x_{n+k} \quad \ldots (1)$$

has been studied intensively by lie and Norset (1986) and Onumanyi,etal (1994). The idea of the block methods using the MC approach has yielded very efficient numerical methods for the solution of ODE's. This was further investigated (Chollom & Onumanyi 2004; Chollom & Donald 2009). This paper uses the Gaussian points as collocation points as follows:
Consider the Lengendre polynomial of degree n

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, n = 1,2,.. \quad \ldots (2)$$

The roots of this polynomial are sought between the interval $[x_n, x_{n+2}]$ instead of the interval $[-1,1]$.thus the polynomial (2) is transformed from the interval $x \in [-1,1]$ to $x \in [x_n, x_{n+2}]$ linearly. Using the linear transformation

$$\bar{x} = \alpha x + \beta \quad \dots (3)$$

Where α *and* β are real constants.

$$x_n = -\alpha + \beta, x_{n+2} = \alpha + \beta \quad \dots (4)$$

Solving equations (3) and (4) gives

$$\bar{x} = hx + \frac{1}{2}\left(x_n + x_{n+2}\right) \quad \dots (5)$$

For two collocation points and n=2, the Legendre polynomial (2) becomes

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \quad \dots (6)$$

Solving (6) and substituting for x in (5) yields the two Gaussian points for n=2 as,

$$U = \frac{3+\sqrt{3}}{3}, V = \frac{3-\sqrt{3}}{3} \quad \dots (7)$$

Similarly, for n=3 the polynomial (2) becomes

$$P_3(x) = \frac{1}{48}\frac{d^3}{dx^3}\left(x^2 - 1\right) \quad \dots (8)$$

Solving (8) for the values of x and substituting into (5) produces the three Gaussian points;

$$U = 1, V = \frac{5-\sqrt{15}}{5}, W = \frac{5-\sqrt{15}}{5} \quad \dots (9)$$

Derivation of the new continuous schemes

In this section using the multi-collocation approach ,the continuous schemes are derived using the equation

$$y(x) = \sum_{i=0}^{m} a_i x^p, \; p = 0, \dots, n \quad \dots (10)$$

CASE I. One collocation point

Expanding (10) for one collocation point gives the equation

$$y(x) = a_0 + a_1 x + a_2 x^2 \quad \dots (11)$$

The Gaussian point u=1 used as collocation point in (11) results in (12)

$$\begin{aligned} y_n &= a_0 + a_1 x_n + a_2 x_n^2 \\ y_{n+1} &= a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 \\ f_{n+1} &= a_1 + 2a_2 x_{n+1} \end{aligned} \quad \dots (12)$$

The equation (12) expressed in matrix form gives the matrix coefficients D in (13).

$$D = \begin{pmatrix} 1 & x_n & x_n^2 \\ 1 & x_{n+1} & x_{n+1}^2 \\ 0 & 1 & 2x_{n+1} \end{pmatrix} \quad \dots (13)$$

Inverting the matrix (13) using the maple software and simplifying produces the following continuous coefficients.

$$\left.\begin{aligned} \alpha_0(x) &= \left\{\frac{h^2 + (x-x_n)^2 - 2h(x-x_n)}{h^2}\right\} \\ \alpha_1(x) &= \left\{\frac{2h(x-x_n) - (x-x_n)^2}{h^2}\right\} \\ \beta_1(x) &= \left\{\frac{(x-x_n)^2 - h(x-x_n)}{h^2}\right\} \end{aligned}\right\} \quad \dots (14)$$

Substituting (14) into

$$\bar{y}(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+p} + h\beta_0(x) f_{n+p} \quad \dots (15)$$

gives the new continuous interpolant (16).

$$\bar{y}(x) = \left[\frac{h^2 + (x-x_n)^2 - 2h(x-x_n)}{h}\right] y_n + \left[\frac{2h(x-x_n) - (x-x_n)^2}{h^2}\right] y_{n+1} + \left[\frac{(x-x_n)^2 - h(x-x_n)}{h}\right] f_{n+1} \quad \dots (16)$$

Evaluating (16) at $x = x_{n+2}, x = x_{n+\frac{3}{4}}, x = x_{n+\frac{1}{4}}$ produces the following methods used as block integrators.

$$\begin{aligned} y_{n+2} &= y_n + 2hf_{n+1} \\ y_{n+\frac{3}{4}} &= \frac{1}{16} y_n + \frac{15}{16} y_{n+1} - \frac{3}{16} hf_{n+1} \\ y_{n+\frac{1}{4}} &= \frac{7}{16} y_{n+1} + \frac{9}{16} y_n - \frac{3}{16} hf_{n+1} \end{aligned}$$

Differentiating (16) once and evaluating at $x = x_n, x = x_{n+\frac{1}{4}}, x = x_{n+\frac{3}{4}}$ *and* $x = x_{n+2}$ gives the discrete schemes.

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left(f_{n+1} + f_n\right) \\ y_{n+1} &= y_n + \frac{h}{3}\left(3f_{n+1} + f_{n+\frac{1}{4}}\right) \\ y_{n+1} &= y_n + \frac{h}{3}\left(-f_{n+1} + f_{n+\frac{3}{4}}\right) \\ y_{n+2} &= y_n + \frac{h}{2}\left(-f_{n+2} + 3f_{n+1}\right) \end{aligned}$$

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CASE II. Two collocation points

Expanding (10) for **n=2** gives the equation

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad \ldots (17)$$

Expressing the polynomial (17) in the form (18) and writing it in matrix form gives the collocation matrix (19)

$$\begin{aligned} y_n &= a_0 + a_1 x_n + a_2 x_n^{\ 2} + a_3 x_n^{\ 3} + a_4 x_n^{\ 4} \\ y_{n+u} &= a_0 + a_1 x_{n+u} + a_2 x^2_{\ n+u} + a_3 x^3_{\ n+u} + a_4 x^4_{\ n\cdot} \\ y_{n+v} &= a_0 + a_1 x_{n+v} + a_2 x^2_{\ n+v} + a_3 x^3_{\ n+v} + a_4 x^4_{\ n+v} \\ f_{n+u} &= a_1 + 2a_2 x_{n+u} + 3a_3 x^2_{\ n+u} + 4a_4 x^3_{\ n+u} \\ f_{n+v} &= a_1 + 2a_2 x_{n+v} + 3a_3 x^2_{\ n+v} + 4a_4 x^3_{\ n+v} \end{aligned} \quad \ldots (18)$$

$$D = \begin{pmatrix} 1 & x_n & x^2_{\ n} & x^3_{\ n} & x^4_{\ n} \\ 1 & x_{n+u} & x^2_{\ n+u} & x^3_{\ n+u} & x^4_{\ n+u} \\ 1 & x_{n+v} & x^2_{\ n+v} & x^3_{\ n+v} & x^4_{\ n+v} \\ 0 & 1 & 2x_{n+u} & 3x^2_{\ n+u} & 4x^3_{\ n+u} \\ 0 & 1 & 2x_{n+v} & 3x^2_{\ n+v} & 4x^3_{\ n+v} \end{pmatrix} \quad \ldots (19)$$

Inverting the matrix (19) using the maple software and simplifying gives the continuous coefficients of the method in (20).

$$\left.\begin{aligned} \alpha_0(x) &= \left\{ \frac{h^4 - 6h^3(x-x_n) + 12h^2(x-x_n)^2 - 9h(x-x_n)^3 + \frac{9}{4}(x-x_n)^4}{h^4} \right\} \\ \alpha_u(x) &= \left\{ \frac{h^3\frac{(6-3\sqrt{3})}{2}(x-x_n) + h^2\frac{(9\sqrt{3}-24)}{4}(x-x_n)^2 + h\frac{(18-3\sqrt{3})}{4}(x-x_n)^3 - \frac{9}{8}(x-x_n)^4}{h^4} \right\} \\ \alpha_v(x) &= \left\{ \frac{h^3\frac{(6+3\sqrt{3})}{2}(x-x_n) - h^2\frac{(24+9\sqrt{3})}{4}(x-x_n)^2 + h\frac{(18+3\sqrt{3})}{4}(x-x_n)^3 - \frac{9}{8}(x-x_n)^4}{h^4} \right\} \\ \beta_u(x) &= \left\{ \frac{h^3\frac{(\sqrt{3}-2)}{2}(x-x_n) + h^2\frac{(15-7\sqrt{3})}{4}(x-x_n)^2 - h\frac{(15-6\sqrt{3})}{4}(x-x_n)^3 + 3\frac{(3-\sqrt{3})}{8}(x-x_n)^4}{h^3} \right\} \\ \beta_v(x) &= \left\{ \frac{h^3\frac{(-2-\sqrt{3})}{2}(x-x_n) + h^2\frac{(15+7\sqrt{3})}{4}(x-x_n)^2 - h\frac{(15+6\sqrt{3})}{4}(x-x_n)^3 + 3\frac{(3+\sqrt{3})}{8}(x-x_n)^4}{h^3} \right\} \end{aligned}\right\} \quad \ldots (20)$$

Equation (20) substituted into $\bar{y}(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+p} + h\beta_0(x) f_{n+p}$ gives the continuous interpolant (21).

$$\bar{y}(x)=\left[\frac{h^4-6h^3\left(x-x_n\right)+12h^2(x-x_n)^2-9h(x-x_n)^3+\frac{9}{4}\left(x-x_n\right)^4}{h^4}\right]y_n$$

$$+\left[\frac{h^3\left(\frac{6-3\sqrt{3}}{2}\right)\left(x-x_n\right)+h^2\left(\frac{9\sqrt{3}-24}{4}\right)(x-x_n)^2+h\left(\frac{18-3\sqrt{3}}{4}\right)(x-x_n)^3-\frac{9}{8}\left(x-x_n\right)^4}{h^4}\right]y_{n+u}$$

$$+\left[\frac{h^3\left(\frac{6+3\sqrt{3}}{2}\right)\left(x-x_n\right)-h^2\left(\frac{24+9\sqrt{3}}{4}\right)(x-x_n)^2+h\left(\frac{18+3\sqrt{3}}{4}\right)(x-x_n)^3-\frac{9}{8}\left(x-x_n\right)^4}{h^4}\right]y_{n+v} \quad \ldots (21)$$

$$+\left[\frac{h^3\left(\frac{\sqrt{3}-2}{2}\right)\left(x-x_n\right)+h^2\left(\frac{15-7\sqrt{3}}{4}\right)(x-x_n)^2-h\left(\frac{15-6\sqrt{3}}{4}\right)(x-x_n)^3+\frac{3\left(3-\sqrt{3}\right)}{8}\left(x-x_n\right)^4}{h^3}\right]f_{n+u}$$

$$+\left[\frac{h^3\left(\frac{-2-\sqrt{3}}{2}\right)\left(x-x_n\right)+h^2\left(\frac{15+7\sqrt{3}}{4}\right)(x-x_n)^2-h\left(\frac{15+6\sqrt{3}}{4}\right)(x-x_n)^3+\frac{3\left(3+\sqrt{3}\right)}{8}\left(x-x_n\right)^4}{h^3}\right]f_{n+v}$$

Evaluating the continuous interpolant (21) at $x=x_{n+2}, x=x_{n+1}$ yields respectively the discrete formulae.

$$y_{n+2}=y_n+h\left(f_{n+u}+f_{n+v}\right) \quad \ldots (22)$$
$$3\sqrt{3}y_{n+v}=3\sqrt{3}y_{n+u}=h\left(-4f_{n+1}-f_{n+u}-f_{n+v}\right)$$

Differentiating (21) once and evaluating at $x=x_{n+2}, x=x_n, x=x_{n+\frac{1}{2}}$ yields the discrete formulae

$$\frac{6+3\sqrt{3}}{2}y_{n+u}-\frac{3\sqrt{3}-6}{2}y_{n+v}-6y_n=\frac{h}{2}\left(-2f_{n+2}+\left(10-\sqrt{3}\right)f_{n+u}+\left(10+\sqrt{3}\right)f_{n+v}\right)$$
$$-\left(\frac{6-3\sqrt{3}}{2}\right)y_{n+u}-\frac{3\sqrt{3}+6}{2}y_{n+v}+6y_n=\frac{h}{2}\left(-2f_n+\left(\sqrt{3}-2\right)f_{n+u}-\left(2+\sqrt{3}\right)f_{n+v}\right)$$
$$\left(\frac{3+3\sqrt{3}}{16}\right)y_{n+v}-\frac{3\sqrt{3}-3}{16}y_{n+u}-\frac{3}{8}y_n=\frac{h}{16}\left(-16f_{n+\frac{1}{2}}+\left(8-5\sqrt{3}\right)f_{n+u}+\left(8+5\sqrt{3}\right)f_{n+v}\right) \quad \ldots (23)$$

CASE III. Three collocation points

Expanding (10) for n=3 using the three Gaussian points (9) as collocation points and substituting into the polynomial (24) gives the expression (25) which produces the collocation matrix (26)

$$y(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6 \quad \ldots (24)$$

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$$\begin{aligned}
y_n &= a_0 + a_1 x_n + a_2 {x_n}^2 + a_3 {x_n}^3 + a_4 {x_n}^4 + a_5 x^5{}_n + a_6 x^6{}_n \\
y_{n+u} &= a_0 + a_1 x_{n+u} + a_2 x^2{}_{n+u} + a_3 x^3{}_{n+u} + a_4 x^4{}_{n+u} + a_5 x^5{}_{n+u} + a_6 x^6{}_{n+u} \\
y_{n+v} &= a_0 + a_1 x_{n+v} + a_2 x^2{}_{n+v} + a_3 x^3{}_{n+v} + a_4 x^4{}_{n+v} + a_5 x^5{}_{n+v} + a_6 x^6{}_{n+v} \\
y_{n+w} &= a_0 + a_1 x_{n+w} + a_2 x^2{}_{n+w} + a_3 x^3{}_{n+w} + a_4 x^4{}_{n+w} + a_5 x^5{}_{n+w} + a_6 x^6{}_{n+w} \\
f_{n+u} &= a_1 + 2a_2 x_{n+u} + 3a_3 x^2{}_{n+u} + 4a_4 x^3{}_{n+u} + 5a_5 x^4{}_{n+u} + 6a_6 x^5{}_{n+u} \\
f_{n+v} &= a_1 + 2a_2 x_{n+v} + 3a_3 x^2{}_{n+v} + 4a_4 x^3{}_{n+v} + 5a_5 x^4{}_{n+v} + 6a_6 x^5{}_{n+v} \\
f_{n+w} &= a_1 + 2a_2 x_{n+w} + 3a_3 x^2{}_{n+w} + 4a_4 x^3{}_{n+w} + 5a_5 x^4{}_{n+w} + 6a_6 x^5{}_{n+w}
\end{aligned} \qquad \ldots (25)$$

$$D = \begin{pmatrix}
1 & x_n & {x_n}^2 & {x_n}^3 & {x_n}^4 & {x_n}^5 & {x_n}^6 \\
1 & x_{n+r} & {x_{n+r}}^2 & {x_{n+r}}^3 & {x_{n+r}}^4 & {x_{n+r}}^5 & {x_{n+r}}^6 \\
1 & x_{n+s} & {x_{n+s}}^2 & {x_{n+s}}^3 & {x_{n+s}}^4 & {x_{n+s}}^5 & {x_{n+s}}^6 \\
1 & x_{n+t} & {x_{n+t}}^2 & {x_{n+t}}^3 & {x_{n+t}}^4 & {x_{n+t}}^5 & {x_{n+t}}^6 \\
0 & 1 & 2x_{n+r} & 3{x_{n+r}}^2 & 4{x_{n+r}}^3 & 5{x_{n+r}}^4 & 6{x_{n+r}}^5 \\
0 & 1 & 2x_{n+s} & 3{x_{n+s}}^2 & 4{x_{n+s}}^3 & 5{x_{n+s}}^4 & 6{x_{n+s}}^5 \\
0 & 1 & 2x_{n+t} & 3{x_{n+t}}^2 & 4{x_{n+t}}^3 & 5{x_{n+t}}^3 & 6{x_{n+t}}^5
\end{pmatrix} \qquad \ldots (26)$$

With the inversion of the matrix (26) and simplifying gives the continuous coefficients (27).

$$\alpha_0(x) = \frac{1}{h^6}\left\{ 4h^6 - 12h^5(x-x_n) + 51h^4(x-x_n)^2 - 95h^3(x-x_n)^3 + \frac{345}{4}h^2(x-x_n)^4 - \frac{150}{4}h(x-x_n)^5 + \frac{25}{4}(x-x_n) \right\}$$

$$\alpha_u(x) = \frac{1}{h^6}\left\{ h^5 \frac{8}{9}(x-x_n) - \frac{84}{9}h^4(x-x_n)^2 + \frac{280}{9}h^3(x-x_n)^3 - \frac{320}{9}h^2(x-x_n)^4 + \frac{150}{9}h(x-x_n)^5 - \frac{25}{9}(x-x_n)^6 \right\}$$

$$\alpha_v(x) = \frac{1}{h^6}\left\{ h^5\left(\frac{100+25\sqrt{15}}{18}\right)(x-x_n) - \left(\frac{750+175\sqrt{15}}{36}\right)h^4(x-x_n)^2 + \left(\frac{225\sqrt{15}+1150}{36}\right)h^3(x-x_n)^3 + \left(\frac{1825-250\sqrt{15}}{72}\right) h^2(x-x_n)^4 + \left(\frac{375+25\sqrt{15}}{36}\right)h(x-x_n)^5 - \frac{125}{7}(x-x_n)^6 \right\}$$

$$\left.\begin{array}{l}
\alpha_w(x) = \frac{1}{h^6}\left\{\dfrac{h^5\left(\frac{100-25\sqrt{15}}{18}\right)(x-x_n)+\left(\frac{175\sqrt{15}-750}{36}\right)h^4(x-x_n)^2-\left(\frac{225\sqrt{15}+1150}{36}\right)h^3(x-x_n)^3+\left(\frac{250\sqrt{15}-1825}{72}\right) h^2(x-x_n)^4+\left(\frac{375-25\sqrt{15}}{36}\right)h(x-x_n)^5-\frac{125}{72}(x-x_n)^6}{}\right\} \\
\beta_u(x) = \left\{\dfrac{-\frac{4}{9}h^5(x-x_n)+\frac{44}{9}h^4(x-x_n)^2-\frac{160}{9}h^3(x-x_n)^3+\frac{220}{9}h^2(x-x_n)^4-\frac{125}{9}h(x-x_n)^5+\frac{25}{9}(x-x_n)^6}{h^5}\right\} \\
\beta_v(x) = \frac{1}{h^5}\left\{\dfrac{-h^5\left(\frac{5\sqrt{15}+20}{18}\right)(x-x_n)+\left(\frac{305+75\sqrt{15}}{36}\right)h^4(x-x_n)^2-\left(\frac{715+170\sqrt{15}}{36}\right)h^3(x-x_n)^3+\left(\frac{1475+335\sqrt{15}}{72}\right) h^2(x-x_n)^4-\left(\frac{350+75\sqrt{15}}{36}\right)h(x-x_n)^5+\left(\frac{125+25\sqrt{15}}{72}\right)(x-x_n)^6}{}\right\} \\
\beta_w(x) = \frac{1}{h^5}\left\{\dfrac{h^5\left(\frac{-20+5\sqrt{15}}{18}\right)(x-x_n)+\left(\frac{305-75\sqrt{15}}{36}\right)h^4(x-x_n)^2-\left(\frac{715+170\sqrt{15}}{36}\right)h^3(x-x_n)^3+\left(\frac{1475-335\sqrt{15}}{72}\right) h^2(x-x_n)^4+\left(\frac{75\sqrt{15}-350}{36}\right)h(x-x_n)^5+\left(\frac{125-25\sqrt{15}}{72}\right)(x-x_n)^6}{}\right\}
\end{array}\right\} \quad \ldots (27)$$

Substituting (27) into $\bar{y}(x) = \alpha_0(x)y_n + \alpha_r(x)y_{n+r} + \alpha_s(x)y_{n+s} + \alpha_t(x)y_{n+t} + h\beta_r(x)f_{n+r} + \beta_s(x)f_{n+s} + \beta_t(x)f_{n+t}$ ... (28)
gives the continuous interpolant (29).

$$
\begin{aligned}
\bar{y}(x) = {} & \left[\frac{h^6 - 12h^5\left(x-x_n\right) + 51h^4(x-x_n)^2 - 95h^3(x-x_n)^3 + \frac{345}{4}h^2\left(x-x_n\right)^4 - \frac{150}{4}h\left(x-x_n\right)^5 + \frac{24}{4}\left(x-x_n\right)^6}{h^6}\right] y_n \\
& + \left[\frac{\frac{8}{9}h^5\left(x-x_n\right) - \frac{84}{9}h^4(x-x_n)^2 + \frac{280}{9}h^3(x-x_n)^3 - \frac{320}{9}h^2\left(x-x_n\right)^4 + \frac{150}{9}h\left(x-x_n\right)^5 - \frac{25}{9}\left(x-x_n\right)^6}{h^4}\right] y_{n+u} \\
& + \left[\frac{h^5\left(\frac{100+25\sqrt{15}}{18}\right)\left(x-x_n\right) - h^4\left(\frac{750+175\sqrt{15}}{36}\right)(x-x_n)^2 + h^3\left(\frac{1150+225\sqrt{15}}{36}\right)(x-x_n)^3 - h^2\left(\frac{1825+250\sqrt{15}}{72}\right)\left(x-x_n\right)^4 + h\left(\frac{375+25\sqrt{15}}{36}\right)\left(x-x_n\right)^5 - \frac{125}{72}\left(x-x_n\right)^6}{h^6}\right] y_{n+v} \\
& + \left[\frac{h^5\left(\frac{100-25\sqrt{15}}{18}\right)\left(x-x_n\right) - h^4\left(\frac{175\sqrt{15}-750}{36}\right)(x-x_n)^2 + h^3\left(\frac{-225\sqrt{15}+1150}{36}\right)(x-x_n)^3 + h^2\left(\frac{250\sqrt{15}-1825}{72}\right)\left(x-x_n\right)^4 + h\left(\frac{375-25\sqrt{15}}{36}\right)\left(x-x_n\right)^5 - \frac{125}{72}\left(x-x_n\right)^6}{h^6}\right] y_{n+w} \quad \ldots (29)
\end{aligned}
$$

$$+\left[\frac{-h^5\frac{4}{9}\left(x-x_n\right)+h^4\frac{44}{9}(x-x_n)^2-h^3\frac{160}{9}(x-x_n)^3+h^2\frac{220}{9}\left(x-x_n\right)^4-h\frac{125}{9}\left(x-x_n\right)^5+\frac{25}{9}\left(x-x_n\right)^6}{h^5}\right]f_{n+u}$$

$$+\left[\frac{h^5\left(\frac{-5\sqrt{15}-20}{18}\right)\left(x-x_n\right)+h^4\left(\frac{305+75\sqrt{15}}{36}\right)(x-x_n)^2-h^3\left(\frac{715+170\sqrt{15}}{36}\right)(x-x_n)^3+h^2\left(\frac{1475+335\sqrt{15}}{72}\right)\left(x-x_n\right)^4-h\left(\frac{350+75\sqrt{15}}{36}\right)\left(x-x_n\right)^5+\left(\frac{125+25\sqrt{15}}{72}\right)\left(x-x_n\right)^6}{h^5}\right]f_{n+v}$$

$$+\left[\frac{h^5\left(\frac{5\sqrt{15}-20}{18}\right)\left(x-x_n\right)+h^4\left(\frac{305+75\sqrt{15}}{36}\right)(x-x_n)^2+h^3\left(\frac{-715+170\sqrt{15}}{36}\right)(x-x_n)^3+4h^2\left(\frac{1475-335\sqrt{15}}{72}\right)\left(x-x_n\right)^4+h\left(\frac{-350+75\sqrt{15}}{36}\right)\left(x-x_n\right)^5+\left(\frac{125-25\sqrt{15}}{72}\right)\left(x-x_n\right)^6}{h^5}\right]f_{n+w}$$

Evaluating the continuous interpolant (29) at $x = x_{n+2}, x = x_{n+1}$ yields respectively the discrete schemes.

$$y_{n+2} = y_n + \frac{h}{9}\left(8f_{n+u} + 5f_{n+v} + 5f_{n+w}\right)$$

$$y_{n+\frac{1}{2}} = \frac{49}{256}y_n + \frac{145}{576}y_{n+u} + \left(\frac{1275+300\sqrt{15}}{4608}\right)y_{n+v} \qquad \dots \dots (30)$$

$$+\left(\frac{1275-300\sqrt{15}}{4608}\right)y_{n+w} + \frac{h}{4608}\left(-392f_{n+u} + \left(385+105\sqrt{15}\right)f_{n+v} + \left(385-105\sqrt{15}\right)f_{n+w}\right)$$

Differentiating (30) once and evaluating at $x = x_{n+2}, x = x_n, x = x_{n+\frac{1}{2}}$ yields the block method.

$$-\frac{259}{144}y_{n+r}+\left(\frac{1225+350\sqrt{15}}{1152}\right)y_{n+s}-\left(\frac{-1225+350\sqrt{15}}{1152}\right)y_{n+t}-\frac{42}{128}y_n=\frac{h}{1152}\left(-1152f_{n+\frac{1}{2}}-560f_{n+r}+\left(-5+5\sqrt{15}\right)f_{n+s}-\left(5+5\sqrt{15}\right)f_{n+t}\right) \quad \ldots (31)$$

$$-\frac{8}{9}y_{n+r}-\left(\frac{100+25\sqrt{15}}{18}\right)y_{n+s}-\left(\frac{-100-25\sqrt{15}}{18}\right)y_{n+t}+12y_n=\frac{h}{8}\left(-18f_n-8f_{n+r}-\left(20+5\sqrt{15}\right)f_{n+s}+\left(-20+5\sqrt{15}\right)f_{n+t}\right)$$

$$16y_{n+r}+\left(100+5\sqrt{15}\right)y_{n+s}+\left(100-5\sqrt{15}\right)y_{n+t}-216y_n=h\left(-18f_{n+2}+184f_{n+r}+\left(100+5\sqrt{15}\right)f_{n+s}+\left(100-5\sqrt{15}\right)f_{n+t}\right)$$

The three discrete schemes for the continuous interpolant (17) in case I are of uniform order 2 with error constants 0.33333, 0.0078125 and -0.235 respectively, while methods from its derivative are also of uniform order 2 and error constants 1/6, 5/6, 5/48 and 1/16 respectively.

Similarly, the discrete schemes for the continuous interpolant (22) and (23) for the case II (block methods 22 & 23) are of uniform order 4 with error constants 0.007407408184, 0.003703703, 0.096419708, 0.007407408 and 0.012037037 respectively.

The discrete schemes forming the block methods (30 and 31) for case III are of uniform order 6 with error constant 0.00006349, 0.014000000, 0.012999999, 0.014285706 and 0.00571428668 respectively.

Numerical experiments

The newly constructed methods are tested on the following stiff and non-stiff initial value problems.

(i) $y' = 2y + x + 1, 0 \le x \le 1, y(0) = 2, h = 0.1$

$$y(x) = \frac{11}{4}e^{2x} - \frac{1}{2}x - \frac{3}{4}$$

(ii) $y' = -60y + 10x, 0 \le x \le 1, h = 0.1, y(0) = \frac{1}{6}$

$$y(x) = \frac{1}{6}(x + e^{-60x}).$$

(iii) $y' = -1000y, 0 \le x \le 0.005, h = 0.005, y(0) = 1$

$$y(x) = e^{-1000x}.$$

The results of the experiments using the block methods 30 and 31 involving the new Gaussian points and the standard Gaussian points are displayed on Table 1 for problem (i) ,using the block methods 22 and 23 involving the new Gaussian points is displayed on Table 2 for problem (ii) and Table 3 for problem (iii).

TABLE 1. ERRORS OF PROBLEM I (NON STIFF) USING THE NEW BLOCK METHOD FOR CASE III (N=3) WITH THE NEW GAUSSIAN POINTS (7) AND THE STANDARD GAUSSIAN POINTS

X	Errors using the new Gaussian points	Errors using the standard Gaussian points
0.0	0.0	0.0
0.1	6.01 x10^{-6}	7.72 x10^{-1}
0.2	7.29 x10^{-6}	1.00 x10^{-2}
0.3	6.63 x10^{-6}	4.06 x10^{-1}
0.4	5.37 x10^{-6}	1.67 x10^{-2}
0.5	4.07 x10^{-6}	5.57 x10^{-1}
0.6	2.07 x10^{-6}	7.59 x10^{-2}
0.7	2.10 x10^{-6}	7.61 x10^{-1}
0.8	1.45 x10^{-6}	1.66 x10^{-1}
0.9	2.78 x10^{-6}	1.09 x10^{-1}
1.0	6.68 x10^{-6}	2.97 x10^{-1}

TABLE 2. ERRORS OF PROBLEM II (NON STIFF) USING THE NEW BLOCK METHOD FOR CASE II (N=2) WITH THE NEW GAUSSIAN POINTS (9) AND THE STANDARD GAUSSIAN POINTS

X	Errors using the new Gaussian points	Errors using the standard Gaussian points
0.0	0.0	0.0
0.1	9.67 x10^{-3}	8.46 x10^{-3}
0.2	1.97 x10^{-3}	8.17 x10^{-3}
0.3	3.04 x10^{-3}	9.40 x10^{-3}
0.4	3.19 x10^{-3}	1.25 x10^{-2}
0.5	3.27 x10^{-3}	1.53 x10^{-2}
0.6	3.34 x10^{-3}	1.82 x10^{-2}
0.7	3.41 x10^{-3}	2.11 x10^{-2}
0.8	3.48 x10^{-3}	2.39 x10^{-2}
0.9	3.56 x10^{-3}	2.68 x10^{-2}
1.0	3.63 x10^{-3}	2.97 x10^{-2}

TABLE 3. ERRORS OF PROBLEM III (STIFF) USING THE NEW BLOCK METHOD FOR CASE II (N=2) COMPARED TO THE EXACT SOLUTIONS

x	Errors using the new Gaussian points
0.0	0.0
0.1	6.01 x10^{-6}
0.2	7.29 x10^{-6}
0.3	6.63 x10^{-6}
0.4	5.37 x10^{-6}
0.5	4.07 x10^{-6}
0.6	2.96 x10^{-6}
0.7	2.10 x10^{-6}
0.8	1.45 x10^{-6}
0.9	9.91 x10^{-7}
1.0	6.68 x10^{-7}

Conclusions

In this paper, New Gaussian points have been constructed through polynomial transformation from the intervals $x \in [-1,1]$ *to* $x \in [x_n, x_{n+2}]$.The constructed methods using the new Gaussian points as collocation points yielded block methods for n=1,n=2 and n=3.

The method for n=2 tested on stiff initial value problems have proved to be very efficient as shown on table 3 and the method n=3 using the new Gaussian points used on non-stiff initial value problems performed better than those using the standard Gaussian points see (Tables 1 &2). Future work in this area will address issues relating to the construction of higher order methods using higher intervals of transformation.

REFERENCES

Ascher,U. & Bader, G. (1986). Stability of collocation at Gaussian points. *SIAM Journal of Numerical Analysis* 23:412-422

Burage, K. & Butcher, J. C. (1979). Stability criteria for implicit Runge-Kutta methods, *SIAM Journal of Numerical Analysis.*16:46-57

Chollom, J .P. & Onumanyi, P. (2004). Variable order A-Stable Adams Moulton type block hybrid methods for the solution of stiff first order ODE's *Abacus Journal of the Mathematical Association of Nigeria* 2B,31: 177-191

Chollom, J .P. & Donald, J. Z. (2009). L-Stable Block Hybrid Adams Moulton methods for the parallel solutions of Ordinary Differential Equations. *Global Journal of Mathematics and Statistics* 1:33-40

Hairer, E. & Wanner, G. (1981). Algebraically stable and implementable Runge-Kutta methods of high order, *SIAM Journal of Numerical Analysis.*18:1098-1108

Lie, I. & Norsett, S. (1986). Supper Convergence for Multi-step Collocation. *Mathematics of Computations.*52: 65-79

Onumanyi, P., Awoyemi, D. O., Jator, S. N., & Sirisena, U. W. (1994). New linear Multi-Step methods with continuous coefficients for the first order Ordinary Initial Value Problems. *Journal of .Nigerian Mathematical Society* 13:37-51

Yakubu, D.G. (2005). A more accurate set of collocation points for the Lobatto one step method. *Abacus the journal of the Mathematical Association of Nigeria* 2A 32:142-151.