FULL LENGTH RESEARCH ARTICLE

COMMENTARIES ON HILBERT'S BASIS THEOREM

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ABSTRACT
The famous basis theorem of David Hilbert is an important theorem in commutative algebra. In particular the Hilbert's basis theorem is the most important source of Noetherian rings which are by far the most important class of rings in commutative algebra. In this paper we have used Hilbert's theorem to examine their unique properties which will help us to understand some of the characteristics of the Noetherian rings.

Keywords: Noetherian rings, Basis theorem

INTRODUCTION

In our study of Noetherian rings, we noticed that the rings characteristically reproduce themselves under various operations. We also noticed that the most important source of these classes of rings is the Hilbert's Basis theorem which states that: 'If R is a Noetherian ring then the polynomial ring $R[x]$ where x is an indeterminate is Noetherian.

Although some authors have discussed the theorem, the proof presented by Jacob (1969) is interesting. The commentaries in this paper are based on his proof. Here R denotes a commutative ring and $R[x]$ denotes a polynomial ring.

Preliminaries: The following definitions and prepositions are preliminaries in understanding the theorem in focus and the proof of it.

Definition 1: (Oscar & Pierre 1965)
Let R be a commutative ring and a non-empty sub-set I of R is said to be an ideal of R if:
(i): $a-b \in I$ for $a,b \in R$
(ii): $ra \in I$, for $a,b \in R$.

Definition 2: (Atiyah & Mcdonald 1969)
A finitely increasing sequence: $I_0 \subset I_1 ... I_n$, of ideals in a commutative ring R is called a chain of ideals.

Definition 3: (Atiyah & Mcdonald 1969)
A ring R in which the ascending chain: $I_1 \subseteq I_2 \subseteq I_3 ...$, of ideals of R is stationary is called a Noetherian ring.

Definition 4: (Jacob 1969**)**
If I is any ideal in $R[x]$, and the set $I_n = \{a \in R : a$ is the coefficient of x^n in some $f[x] \in I$ with $\deg f[x] \leq n\}$ then I_n is called the n^{th} associated ideal of I . If p is a primary ideal then its radical $r(p)$ is called the associated ideal of p and we say that p is a primary ideal belonging to the prime ideal $r(p)$ or simply that p is primary for $r(p)$.

Proposition 1: (Jacob 1969**)**
If I is an ideal of $R[x]$, then I is an ideal of R. Furthermore, $I_n \subseteq I_{n+1}$, for all n .

Proof: If $a,b \in I_n$, say $f(x) = a_0 + a_1x + ... + ax^n \in I$ and $g(x) = b_0 + b_1x + ... + bx^n \in I$,
then
$f(x) - g(x) = (a_0 - b_0) + (a_1 - b_1)x + ... + (a-b)x^n \in I$,
so $(a-b) \in I_n$.

Also if $r \in R$, then $rf(x) = ra_0 + ra_1x + ... + ra_nx^n \in I$, so $ra \in I_n$, from definition (2), I_n is an ideal.
To prove the second part, take the product $xf(x)$.
$xf(x) = (a_0 + a_1x + ... + ax^n)x = a_0x + a_1x + ... + ax^{n+1} \in I$
, so
$a \in I_{n+1}$, hence $I_n \subseteq I_{n+1}$ as $a \in I_n$.

Proposition 2: (Jacob 1969; Cohn 1977**)**
Let $C \subseteq D$, be ideals of $R[x]$.Then $C_n \subseteq D_n$ for all n . Furthermore if $C_n = D_n$, for all n then $C = D$.

Proof: We prove the first statement from the definition 4 by Jacob (1969). As C is an ideal in $R[x]$,
$C_n = \{c \in R : c$ is the coefficient of x^n in some $f(x) \in C\}$. D is an ideal of $R[x]$ implies that: $D = \{d \in R : d$ is the coefficient of x_n in some $f(x) \in D\}$ .Hence $C \subseteq D$, for all n.

To prove the second statement, we suppose that, $C_n = D_n$, for all n, and that

$$f(x) = d_0 + d_1 x + ... + d_n x^n \in D \qquad ... (1)$$

We wish to show that $f(x) \in C$. We prove that by induction on n.
If $n = 0$, then $f(x) = d_0 \in D_0 = C_0 \subseteq C$.
Thus we assume that the statement is true for all polynomials of $\deg \leq n-1$.
Since $f(x) \in D$, we have that $d_n \in D_n = C_n$. Thus there exists a polynomial $g(x)$ such that:

$$g(x) = c_0 + c_1 x + ... + c_{n-1} x^{n-1} + d_n x^n \in C \subseteq D - \quad ...(2)$$

From (1) and (2) we have: $f(x) - g(x) \in D$.
But $\deg(f(x) - g(x)) \leq n-1$, so by the induction hypothesis $f(x) - g(x) \in C$.
Since, $g(x) \in C$ we conclude that $f(x) \in C$. This completes the induction and hence $C_n = D_n$, then $C = D$.

We now restate the theorem and give the proof which is our main result.

The theorem:
If R is a commutative Noetherian ring, then $R[x]$ is Noetherian.

Proof:
Let I be an ideal of $R[x]$ and suppose that $I_0 \subseteq I_1 \subseteq ...$ is a chain of ideals of $R[x]$. Let $I_{i,j}$ denote the j^{th} associated ideal of I_i. We then have the following pattern of inclusions. Here $I_{3,1}$ means the first associated ideal of I_3.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\
I_{3,0} & \subseteq & I_{3,1} & \subseteq & I_{3,2} & \subseteq & I_{3,3} & \subseteq & ... \\
\cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\
I_{2,0} & \subseteq & I_{2,1} & \subseteq & I_{2,2} & \subseteq & I_{2,3} & \subseteq & ... \\
\cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\
I_{1,0} & \subseteq & I_{1,1} & \subseteq & I_{1,2} & \subseteq & I_{1,3} & \subseteq & ... \\
\cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\
I_{0,0} & \subseteq & I_{0,1} & \subseteq & I_{0,2} & \subseteq & I_{0,3} & \subseteq & ...
\end{array}$$

We can deduce a finite number of ascending chain of ideals of R thus:

i. From the horizontal chain;

$$I_{0,0} \subseteq I_{0,1} \subseteq I_{0,2} \subseteq I_{0,3} \subseteq ...$$
$$I_{1,0} \subseteq I_{1,1} \subseteq I_{1,2} \subseteq I_{1,3} \subseteq ...$$
$$I_{2,0} \subseteq I_{2,1} \subseteq I_{2,2} \subseteq I_{2,3} \subseteq ...$$
$$I_{3,0} \subseteq I_{3,1} \subseteq I_{3,2} \subseteq I_{3,3} \subseteq ...$$

and so on.

ii. From the vertical chains we have;

$$I_{0,0} \subseteq I_{1,0} \subseteq I_{2,0} \subseteq I_{3,0} \subseteq ...$$
$$I_{0,1} \subseteq I_{1,1} \subseteq I_{2,1} \subseteq I_{3,1} \subseteq ...$$
$$I_{0,2} \subseteq I_{1,2} \subseteq I_{2,2} \subseteq I_{3,2} \subseteq ...$$
$$I_{0,3} \subseteq I_{1,3} \subseteq I_{2,3} \subseteq I_{3,3} \subseteq ...$$

and so on.

iii. Diagonally, we have:

$$I_{1,0} \subseteq I_{2,1} \subseteq I_{3,2} \subseteq ...$$
$$I_{0,0} \subseteq I_{1,1} \subseteq I_{2,2} \subseteq ...$$
$$I_{0,1} \subseteq I_{1,2} \subseteq I_{2,3} \subseteq ...$$

as ascending chains.

Now since $I_{0,0} \subseteq I_{1,1} \subseteq I_{2,2} \subseteq I_{3,3} \subseteq ...$, is an ascending chain of ideals of R, there, exists an integer, k say such that: $I_{i,i} = I_{k,k}$ for, all $i \geq k$. Now we consider the k vertical chains $I_{0,j} \subseteq I_{1,j} \subseteq I_{2,j} \subseteq ...; j = 0,1,2,...,k-1$. There exists an integer n_j such that for all $t \geq n_j, I_{t,j} = I_{n_j,j}$.
Let $n = \max\{n_0, n_j, ..., n_{k-1}, k\}$. We claim that if $i \geq n$,

$I_i = I_n$. By the second proposition, it suffices to show that $I_{i,j} = I_{n,j}$ for all j.
If $0 \leq j \angle k$, then $i \geq n \geq n_j$, implies $I_{i,j} = I_{n_j,j} = I_{n,j}$. If $j \geq k$ and $i \geq n \geq k$, then: $I_{i,j} = I_{k,k} = I_{n,j}$, since $I_{r,s} = I_{k,k}$ whenever: $r,s \geq k$. This completes the proof.

Conclusion

This argument is applicable to all the ascending chains of ideals obtained in i, ii and iii above, hence $R[x]$ is Noetherian whenever R is a commutative Northerian ring. We conclude that:

1. The significance of this theorem is that it is very useful in the construction of Noetherian rings (Apine 2006).
2. The theorem also shows that Noetherian rings characteristically reproduce themselves.

3. The Hilbert Basis Theorem can be extended to the polynomial ring $k[x_1, x_2, ..., x_n]$ of finitely many indeterminates over a field k.

4. If we replace $R[x]$ or $k[x_1, ..., x_n]$, by $R[x_1, x_2, ...]$, the polynomial ring of infinitely many indeterminates, the theorem fails to hold since R is Noetherian.

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