

**FULL LENGHT RESEARCH ARTICLE**

**STATISTICAL ESTIMATION OF  
ABSOLUTE GRAVITY VALUES**

Aku, M. O.

Department of Physics  
Bayero University  
PMB 3011, Kano Nigeria  
[ugaku2003@yahoo.com](mailto:ugaku2003@yahoo.com)

**ABSTRACT**

Gravity measurements at stations in northwestern Nigeria were assumed to be random variables. Gravity data collected was used to illustrate the gravity network adjustment theories. Residuals of the network were inspected to detect gross errors by standardizing the residuals. Computed standard deviation for unit weight was used to determine the standard confidence interval which is one of the most important aspects of the least squares adjustment used for control surveys specifications, and for classifying surveys. The adjustment was carried out using the least-squares option to obtain estimated values for gravity for all the stations together with their accuracy estimates.

**Key words:** *Statistical-estimation, observation, variable, least squares, accuracy*

**INTRODUCTION**

Statistics is the scientific method of collecting, arranging, summarizing, presenting, analyzing and drawing valid conclusions from data. The method of least squares is the standard method used to obtain unique values for physical parameters from redundant measurements of those parameters. A least-square adjustment of survey observations is an important step in a gravimetric survey; properly used, it helps isolate blunders in the observations being adjusted and gives the accuracy and reliability of the gravity values being determined. The primary components of a least-square adjustment are the survey observations (in this case gravity differences) and the uncertainties associated. Due to measurement limitations of the surveying instruments and the influence of the operators, these observations include some level of error. These errors cause loops not to close perfectly and result in different computed values for the same station in the network (Pennington 1965).

The ultimate goal of a least-square adjustment is to produce a set of observations where all loops close perfectly and only one value can be computed for any point in the network. In order to accomplish this task, the observations going into the adjustment must be changed slightly, i.e. adjusted.. A successful adjustment is one where observations are changed as little as possible, and the amount of adjustment to any observation is within expected levels. Unfortunately there are a number of obstacles that can stand in the way of producing a successful adjustment. Primary on this list are blunders, errors in the observation due to equipment malfunction or operator error (incorrectly measured instrument height, insufficient data, wrong station identifier, etc.). Statistically based tools exist to assist in overcoming these

obstacles, both before and during the adjustment; as a result, it is very important that uncertainties (error estimates) are realistic. At times, these uncertainties may be little optimistic (too small) or pessimistic (too large). Methods exist to help identify when uncertainties are unrealistic and to help rectify this situation. Also, adjustment analysis tools cannot function properly without redundancy in the observations.

**Theory**

Consider the system of linear equations  $f(X, L) = 0$ , or in, matrix form:

$$AX = L \quad (1)$$

where  $X$  is the unknown vector,  $L$  is the constant vector, and  $A$  is the coefficient matrix or design matrix. Let's assume that the elements of  $L$  are the results of physical measurements,  $L$  is called the observation vector.

In the case where there are no redundant equations (minimum number of measurements),  $A$  is square and non-singular, and therefore has an inverse. The unique solution is then given by:

$$X = A^{-1}L, \text{ where } A^{-1} \text{ is the inverse of } A.$$

When there are redundant equations, the system is over determined:  $A$  is not square, but  $A^T A$  is (Mikhail & Ackerman 2000) and we have:

$$X = (A^T A)^{-1} A^T L$$

This is so if and only if the system is consistent. But if there are redundant measurements, they will be inconsistent because physical measurements are never perfect.

No unique solution will exist, and all we are able to do is make a unique estimate of the solution. The most commonly used criterion for the estimate to be unique is the least squares criterion; that the sum of the squares of the inconsistencies be minimum.

Using statistics, we are also usually able to establish the degree of reliability of the solution, and thus define the most probable unique solution. To cancel the inconsistencies, we add a vector to equation (1), which becomes:

$$AX - L = V \quad (2)$$

where  $V$  is usually called the residual vector (observation errors). The elements of  $V$  are not known and must be solved for. So we have to allow some of or all the elements of  $L$  change slightly while solving for  $X$ , or regard  $L$  as an appropriate value of some other value  $\hat{L}$  which yields the unique solution  $\hat{X}$ .

Now the least squares criterion states that the best estimate  $\hat{X}$  for  $X$  is the estimate, which will minimize the sum of the squares

of the residuals (discrepancies between observations and estimated values assigned to each observable), that is  $\hat{V}^T \hat{V}$  is minimum.

Often the physical measurements which make up the elements of  $L$  do not all have the same precision (they have been made using different instruments by different people, under different conditions etc.). This fact should be reflected in our least squares estimation process, so we assign to each measurement a known weight and call  $P$  the matrix whose elements are these weights, the weight matrix. We modify the criterion, which becomes  $\hat{V}^T P \hat{V}$  is minimum.

The resulting estimate is called the weighted least squares estimate, and is given by

$$\hat{X} = (A^T P A)^{-1} A^T P L \quad (3)$$

where  $A^T P A$  is the normal equation matrix, and must not be singular for the estimator to be unique.

It can be shown that the least squares unbiased estimator  $\hat{\sigma}_0^2$  of the variance factor  $\sigma_0^2$

is:  $\hat{\sigma}_0^2 = \frac{\hat{V}^T P \hat{V}}{n - u}$  where  $P = \sigma_0^2 \sum_L^{-1}$ , and that the least squares unbiased estimator of the covariance matrix of  $X$  is:

$$\hat{\Sigma}_{\hat{X}} = \hat{\sigma}_0^2 (A^T P A)^{-1} \quad (4)$$

**Data collection and reduction:** Gravity stations were occupied in February 2003 using LCR G446 gravimeter. The stations were tied to the control stations, which are part of the Nigerian gravity network, which is in turn tied to the IGS1971 (Osazuwa 1985). The measuring technique was such as to ensure the independence of the ties inside a loop by linking each station with its direct neighbours in a sequence. This procedure allows sometimes stabilization of instrument before starting precise measurements.

Gravity data reduction was performed using various steps (Aku 2005). Tidal, drift latitude, elevation (free air and Bouguer) corrections were carried out on the data. Tidal correction was computed and applied using regional data parameters as they can differ from a constant tidal factor by more than 10% (Poitevin & Ducarme 1980). For each loop and at each station, the mean value was converted to physical units using the instrument's constant value and corrected of the tidal effects. Instrument drift was calculated for different possible closures of the loop to compute the gravity difference for each tie. Terrain corrections were not applied since the area is relatively flat. The Bouguer anomaly values were reduced to the datum of the mean sea level using a uniform crustal density of 2.67 g/cm<sup>3</sup>.

**Computation of adjusted observations and residuals:**

Gravity models attempt to describe in detail the variations in the gravity field. The importance of this effort is related to the idea of leveling, thus the gravity differences can be adjusted as a leveling network. This is because the summation of gravity differences around a closed loop theoretically goes to zero, and this condition can then be used as the basis for the adjustment.

The stations with absolute gravity determination provide the anchoring point (fixed points) of the network, while the relative measurements provide the ties between the points. When the absolute and relative observations are made and assessed for accuracy, an adjustment can be carried out using a least-square adjustment technique. The adjustment results in the estimated values of gravity for all stations, together with their accuracy estimates. The adjustment procedure is practically identical with that of geodetic leveling (Poitevin & Ducarme 1980).

The gravity at point A is known and is constant (980100.000 mGal) (Osazuwa 1985) whereas gravity at B, C, D, E, F, and G are to be determined.

From the observation data, we have a number of observations  $n = 12$  and the number of unknowns  $u = 6$ . Therefore we have 6 redundant observations ( $n - u$ ) and 6 degrees of freedom.

**TABLE 1: THE GRAVITY NETWORK OBSERVATION USED.**

Gravity Difference (mGals)	$\Delta g_{AB}$	$\Delta g_{BA}$	$\Delta g_{BC}$	$\Delta g_{CD}$	$\Delta g_{DE}$	$\Delta g_{EF}$	$\Delta g_{FC}$	$\Delta g_{CF}$	$\Delta g_{FB}$	$\Delta g_{FA}$	$\Delta g_{FG}$	$\Delta g_{GF}$
	0.143	-0.143	2.370	1.437	-0.897	-0.414	0.880	-0.779	-0.591	-0.635	1.206	-1.201
Time between stations (hrs)	2	2	3	4	2	3	4	4	3	5	6	6

The 12 independent equations that can be generated are:

$$\begin{aligned}
 g_B &= g_A + 0.143 + V_1 \\
 g_A &= g_B - 0.143 + V_2 \\
 g_C &= g_B + 2.370 + V_3 \\
 g_D &= g_C + 1.437 + V_4 \\
 g_E &= g_D - 0.897 + V_5 \quad \dots\dots\dots(a) \\
 g_F &= g_E - 1.414 + V_6 \\
 g_C &= g_F + 0.880 + V_7 \\
 g_F &= g_C - 0.779 + V_8 \\
 g_B &= g_F - 1.591 + V_9 \\
 g_A &= g_F - 1.635 + V_{10} \\
 g_G &= g_F + 1.206 + V_{11} \\
 g_F &= g_G - 1.201 + V_{12}
 \end{aligned}$$

where  $g_A$  has a constant value and the weight of each observation is inversely proportional to the length of travel of the line. Putting equation (a) in matrix form, we have the following:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} g_B \\ g_C \\ g_D \\ g_E \\ g_F \\ g_G \end{pmatrix}, \quad L = \begin{pmatrix} 100 \quad .143 \\ -100 \quad .143 \\ 2.370 \\ 1.437 \\ -0.897 \\ -1.414 \\ 0.880 \\ -0.779 \\ -1.591 \\ -101 \quad .635 \\ 1.206 \\ -1.201 \end{pmatrix}$$

The observations are considered uncorrelated with variances proportional to the corresponding travel time between stations. Thus the weight matrix is given as the inverse of the variance-covariance matrix of the observations, that is:  $P = \text{diag} (1/2, 1/2, 1/3, 1/4, 1/4, 1/3, 1/4, 1/4, 1/3, 1/5, 1/6, 1/6)$

The normal equations are  $N \hat{X} = U$ , yielding the solution  $\hat{X} = N^{-1}U$

Spreadsheet and a Fortran program was used to compute  $\hat{X}$  and obtain the estimates for the gravity of points B, C, D, E, F, and G. We find:

$$\hat{X} = \begin{pmatrix} 100.133 \\ 102.515 \\ 103.971 \\ 103.083 \\ 101.684 \\ 102.887 \end{pmatrix} \text{ in mGals} \quad \text{and} \quad \hat{V} = A \hat{X} - L = \begin{pmatrix} -0.010 \\ 0.010 \\ 0.012 \\ 0.019 \\ 0.009 \\ 0.014 \\ -0.049 \\ -0.052 \\ 0.041 \\ -0.049 \\ -0.003 \\ -0.003 \end{pmatrix} \text{ in mGals}$$

Note: (980000 mGals to be added to  $\hat{X}$ )

The adjusted observations are  $\hat{L} = L + \hat{V}$

Even after one has acquired some experience in judging where weakness lies in survey configurations, one should inspect residuals of the adjusted network to detect gross errors because confidence regions do not reveal small pockets of distortion. If blunders are identified, they must be removed from the data set, and the adjustment must be rerun. Bear in mind that large residuals do not always indicate a blunder since least-squares adjustment tends to distribute the effects throughout the entire network.

In order to verify if the observations are normally distributed, their residuals obtained from the adjustment may be subjected to the  $\chi^2$  test of goodness of fit. The residuals have to be standardized since standardized random variable is normally distributed with a mean zero and variance 1. The standardization was carried out using the sample mean and sample variance in place of the population mean and variance. The standardized residuals take into account the fact that residuals generated by random errors are somewhat predictable statistically. This test allows verifying the normality of the residuals obtained after the adjustment.

The variance of unit weight monitor the relationship between the uncertainties assigned to the observations and the magnitude of the change required to each observation (residuals) in the adjustment. Changes to the observations should not be significantly greater than the associated uncertainties.

Since gravity accuracy at each point is related to the reference points that are treated as fixed (errorless) in the process of the network adjustment, the term absolute gravity confidence interval is applied. Variances and covariances of the analyzed network are a basis for the calculations of gravity errors. They allow the calculation of the standard deviations of gravity values (or differences in gravity values).

Good estimates of the standard deviations of the measurements are usually available, that's why a-prior standard deviation of unit weight is used to determine the standard confidence interval. However, if good estimates of the measurements are not available, because of inexperience or malfunctioning equipment, the standard deviation of unit weight computed the least-squares adjustment is used, provided there is sufficient redundancy to give a reasonable value.

For precise gravity surveys, it is both logical and prudent to perform the weighted adjustment rather than the equal weight adjustment. Often results of physical measurements do not have the same precision since they have been made using different instruments by different persons, under different conditions. This fact is reflected in the least squares estimation process. The parametric method of least squares remains a reliable tool that greatly assists in overcoming obstacles that can stand on the way of producing a successful adjustment.

#### REFERENCES

- Aku, M.O. 2005. Mathematical model for computation of gravity adjusted observations. *Journal of the Nigerian mathematical physics* 9:409-414
- Mikhail, E. M. & Ackerman, F. 2000. *Observations and Least Squares*. Donnelly Publisher, New York.
- Osazuwa, I. B. 1985. Primary gravity network of Nigeria. Ph.D. Thesis, Ahmadu Bello University, Zaria, Nigeria
- Pennington, R. H. 1965. *Introductory computer methods and numerical analysis*. The Macmillan Co., New York
- Poitevin, C. & Ducarme, B. (1980). Comparison of five LaCoste-Romberg gravity meters on the Belgium gravity network. *Bulletin Information Bureau of Gravity International*. 47:58-75