Science World Journal Vol. 18(No 1) 2023
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

Full Length Research Article

A MODIFIED LOGISTIC MODEL FOR POPULATION GROWTH OF FISHES IN AN ENVIRONMENT

[1]Ejakpovi S.U. and [2*]Siloko I.U.

[1]Department of Mathematics, College of Education, Warri, Delta State, Nigeria
[2]Department of Mathematics, Edo State University, Uzairue, Nigeria

*Corresponding Author Email Address: siloko.israel@edouniversity.edu.ng

ABSTRACT

This paper focuses on the modified logistic differential equation model for population growth of organisms in an environment. The situation considered was that of a fish population that were stocked in an environment, with the fact that the environment has a particular carrying capacity and the organisms are increasing as the other factors for their survival are kept constant. We incorporated a condition where the farmer is constrained to stock the same environment with additional fishes and another environment. These constraints led to the modification of the logistic differential equation model whose solution is based on the calculus approach. The results of the modified model demonstrate superiority over its classical counterparts in terms of carrying capacity with increase in the interval and step sizes.

Keywords: Logistic, Differential Equation, Calculus Approach.

INTRODUCTION

The models for fish population growth are based on the premise that the population grows at a rate proportional to its size. This assumption could also be applied to the population of bacteria or animals under real-life situation such as absence of adequate nutrition, unlimited environment, predators and diseases (Andire, 2003; Finley, 2011; Gebremedhin et al., 2021). The variables for this population model are identified as

$$\left.\begin{array}{l} t = \text{time, an independent variable} \\ P = \text{Population, a dependent variable denoted as } \beta \end{array}\right\} \quad (1)$$

The rate of the fish population growth which is proportional to the size can be modeled through different differential equations (Okuonghae, 2011; Shlyufman et al., 2018; Logofet, 2019). Considering the situational phenomenon in Equation (1), different differential equations can be modeled from it as follows

$$\left.\begin{array}{l} \frac{d\beta}{dt} = \alpha\beta \\ \frac{d\beta}{dt} = \alpha_1\beta + \alpha_2\beta^2 \\ \frac{d\beta}{dt} = f(\alpha(t)) + g(\alpha(t-\omega)) \end{array}\right\} \quad (2)$$

These formulated models of differential Equations in (2) depict the situations given in Equation (1) with the factor of the rate of change of the quantity β, which depends on the amount of the quantity. If $\beta > 0$, we have the exponential growth of the population parameter and when $\alpha < 0$ the situation becomes an exponential decay population parameter (Stewart, 1999; Il'in, 2007; Ikpotokin and Siloko, 2019). Thus, there is need to include more parameters in Equation (2) that makes it to be more robust and as such when the value of β is evaluated at time (t), the instantaneous rate of change at time (t) becomes a function of previous time called the delay differential equation (Murray, 2002; Abakumov and Izrailsky, 2022; Siloko and Siloko, 2023).

This paper modifies the logistic differential equation using the calculus approach by incorporating extra variables into the model. It presents a brief discussion of the logistic differential equation model for population growth of fishes. The analytic solution of the logistic differential equation model and the modified logistic differential equation model using the calculus approach via numerical simulation were obtained.

MATERIALS AND METHODS

The Logistic Differential Equation Model

The population of fishes grow exponentially in its early stage of development and later begins a level off which tends towards the capacity of the organisms' environment due to lack of resources. Thus, if $\beta(t)$ is the size of the fish population at time t, we can model this as

$$\frac{d\beta}{dt} \cong \partial\beta, \quad (3)$$

provided β is small while ∂ is the initial growth rate of the fish in stock. This implies that the population growth rate is almost a constant or uniform when the population size of the fish is small, but the population growth rate diminishes as the population β increases (Stewart, 1999). Another unique situation is when β becomes negative. The model for the population growth that incorporates these assumptions is the logistic differential equation given as

$$\frac{d\beta}{dt} = \partial\beta\left(1 - \frac{\beta}{\delta}\right) \quad (4)$$

where δ represents the capacity of the organism's environment (Stewart, 1999; Abakumov and Izrailsky, 2016; Ikpotokin et al., 2020). The constraints of the model in Equation (4) expressed as (t, β) implies that if $\beta > \delta$, then $1 - \beta/\delta$ is negative, so that $\frac{d\beta}{dt} < 0$ thereby leading to a decrease in the fish population. This makes Equation (4) the objective function model for determining the extra optimum stock of the fish variable in the model. Adopting the calculus approach that involves integration of Equation (4), we have

$$\int \frac{d\beta}{dt} = \int \partial\beta\left(1 - \frac{\beta}{\delta}\right) \quad (5)$$

Applying integration by partial fraction method on Equation (5), we have

$$\int \left(\frac{1}{\beta} + \frac{1}{\delta - \beta}\right) d\beta = \int \delta \, dt \quad (6)$$

If $A = \pm\exp(-\pi)$, where π is the constant of integration and letting $t = 0$ and $\beta = \beta_0$ called the initial population of the fish stock, then the solution of the logistic differential equation model for fish stock given in Equation (4), has its solution given as

$$\beta(t) = \frac{\delta}{1 + A exp(-\partial t)} = \frac{\delta}{1 + \left(\frac{\delta - \beta_0}{\beta_0}\right) \exp(-\partial t)} \quad (7)$$

The Modified Logistic Differential Equation Model

The proposed method is an extension of the logistic differential equation model by incorporating more parameters that depict the fish population growth rate. The Mathematical structure of this model is given as

$$\frac{d\beta}{dt} = \partial\beta\left(1 - \frac{\beta}{\delta}\right) - \alpha\vartheta, \quad (8)$$

where α is the maximum growth rate capacity, ϑ represents the extra optimum stock of the fish, δ is the carrying capacity of the organism (fish) environment, β is the population of the organism (fish) in stock, t is the time of instantaneous growth of the organism (fish). The constraints of the modified logistic differential equation model in Equation (8) is expressed as $\vartheta_{\max}(t, \beta)$ which is the objective function for determining the extra optimum stock of the fish variable in the proposed model. Applying the calculus approach by letting $\tau = \partial\beta\left(1 - \frac{\beta}{\delta}\right)$ on Equation (8), we have

$$\frac{d\beta}{dt} = \tau - \alpha\vartheta \quad (9)$$

Integrating both sides of equation (9), we have

$$\begin{cases} \int\int \frac{d\beta}{\tau - \alpha\vartheta} = \int dt \\ \vartheta = \frac{\tau}{\delta} - \frac{a^{\delta}}{\delta} exp(-\partial t) \end{cases} \quad (10)$$

Equation (10) shows that as $t \to \infty$, we have that $\frac{a^{\delta}}{\delta} exp(-\partial t) = 0$ which implies that $\vartheta = \frac{\tau}{\delta}$. Adopting the variable transformation techniques on Equation (10) by letting $\frac{d\tau}{dt} = \frac{dx}{dt}$ and $\frac{d\vartheta}{dt} = \frac{d^2x}{dt^2}$, we explicitly transform the modified logistic differential equation model into a second-order differential equation of the form

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = \tau \quad (11)$$

The solution of the non-homogeneous differential equation of (11) produces the maximum stock of fish in any environment in the modified logistic differential equation model (8). Hence, incorporating the solution of Equation (11) into the Euler's transformation, we have

$$\vartheta_{max}(t, \beta) = \frac{G(t, \beta)t}{\partial} - \frac{G(t, \beta)}{\partial^2}(1 - exp(-\partial t)) \quad (12)$$

RESULTS AND DISCUSSION

The efficiency of the proposed model depends on its potency to accommodate extra maximum fishes that could be stock in the environment with a fixed carrying capacity. In the implementation, we assumed $\delta = 100{,}000$ and the initial population of the fish stock denoted by $\beta_0 = 100$ which were substituted into Equations (4), (7) and (12) called the Logistic Differential Equation (LDE) model, Analytic Logistic Differential Equation (ALDE) model and Modified Logistic Differential Equation (MLDE) model using the Euler's method with varied step sizes (α_i) and periodic times(t) in weeks. The presentation in Table 1 and the graphical display are results of the models adopted in this study. The numerical computations were carried out at four varied growth rates with a comparison of results in Table 1.

Science World Journal Vol. 18(No 1) 2023
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

Table 1: Models Showing Extra Maximum Fish Stock at Fixed Carrying Capacity

i	Step (α_i)	Time (t)	$G(t,\beta)k = 0.02$			$G(t,\beta)k = 0.04$			$G(t,\beta)k = 0.06$			$G(t,\beta)k = 0.08$		
			ALDE	LDE	MLDE	ALDE	LDE	MLDE	ALDE	LDE	MLDE	ALDE	LDE	MLDE
1	2.0	0	100	108	0	100	108	0	100	112	0	100	116	0
2	1.9	5	111	108	1306	122	116	1358	134	125	1306	149	134	1306
3	1.8	10	122	112	5245	149	124	5450	182	139	5746	222	153	5961
4	1.7	15	135	116	11837	182	132	12277	246	153	13029	321	174	13626
5	1.6	20	149	120	21096	222	140	21816	331	168	23389	493	196	24558
6	1.5	25	165	124	33025	271	148	34029	447	183	36759	734	220	39027
7	1.4	30	182	128	47620	331	156	48866	602	193	51751	1091	245	57067
8	1.3	35	201	132	64873	404	164	66276	811	208	70631	1619	270	78502
9	1.2	40	222	136	84772	493	172	86204	1091	223	92341	2397	296	103635
10	1.1	45	246	140	107299	602	180	108596	1468	238	116831	3534	322	132187
11	1.0	50	271	144	132437	734	187	132692	1771	252	143485	5182	348	164121
12	0.9	55	300	148	160162	895	194	158935	2642	266	172670	7539	373	198872
13	0.8	60	331	152	190454	1091	200	186340	3534	279	203618	10844	397	236229
14	0.7	65	366	156	223287	1330	206	215563	4712	291	236053	15358	419	275329
15	0.6	70	404	160	258639	1619	211	245394	6258	302	269702	21303	439	315785
16	0.5	75	447	164	296483	1971	215	275440	8266	311	303321	28766	457	357208
17	0.4	80	493	168	336796	2397	218	305304	10844	318	336393	37595	472	398373
18	0.3	85	545	172	379554	2912	221	336110	14103	324	369548	47333	483	437803
19	0.2	90	602	176	424732	3534	223	366183	18142	328	401300	57279	491	475714
20	0.1	95	665	180	472306	4283	224	395132	23027	330	431140	66669	495	510507

The presentation of the results in Table 1 with different step sizes started with the Analytic Logistic Differential Equation (ALDE) model, Logistic Differential Equation (LDE) model and Modified Logistic Differential Equation (MLDE) model and from the results, the Modified Logistic Differential Equation (MLDE) model outperformed the other two models investigated.

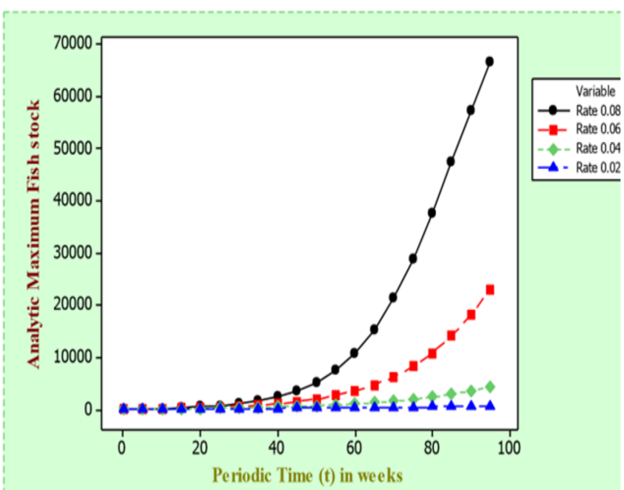

Figure 1: Graph of Maximum Fish Stock Using the Analytic Model

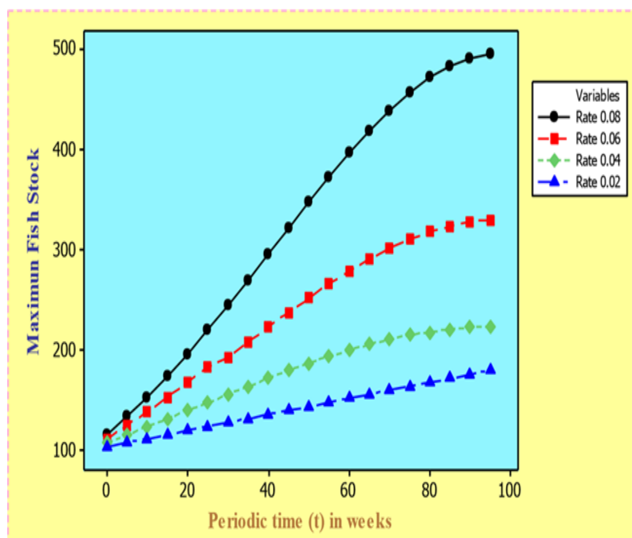

Figure 2: Graph of Maximum Fish Stock Using the Logistic Model

Science World Journal Vol. 18(No 1) 2023
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

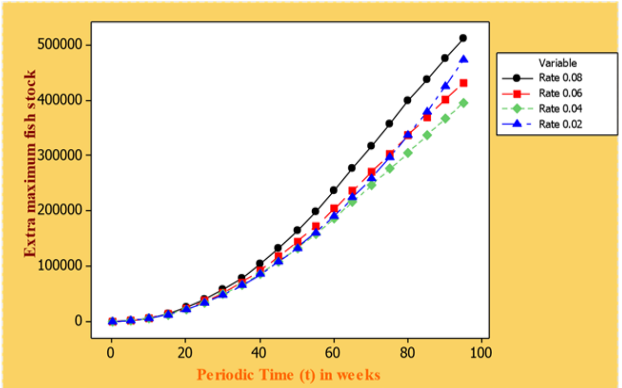

Figure 3: Graph of Maximum Fish Stock Using the Modified Model

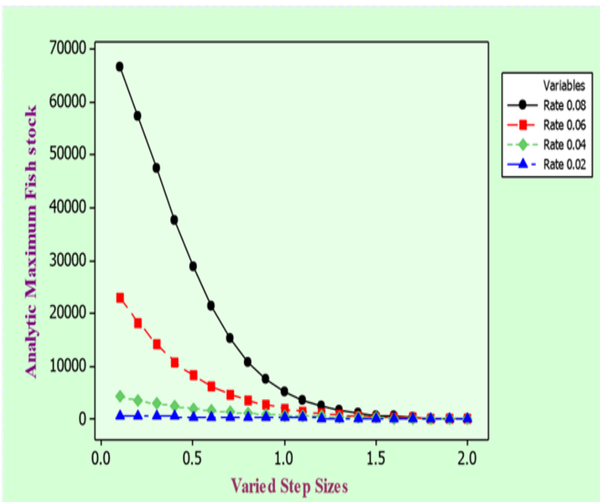

Figure 4: Graph of Maximum Fish Stock Using the Analytic Model

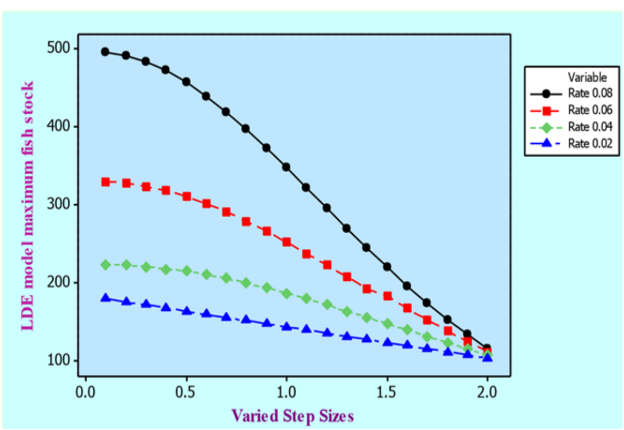

Figure 5: Graph of Maximum Fish Stock Using the Logistic Model

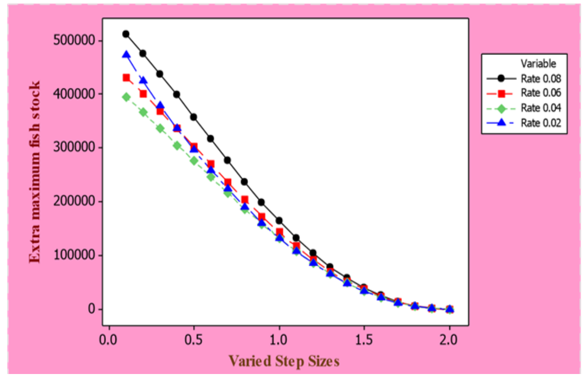

Figure 6: Graph of Maximum Fish Stock Using the Modified Model

The graphical results of the analytic logistic differential equation, logistic differential equation model and modified logistic differential equation model at varied periods in weeks are in Figures (1–3). Also, the graphs of the varied step sizes of the various models with the fish stock population that is increasing from the initial stock values in thousands are in Figures (4–6). The increased trend is also noticed as the sizes of the fishes increase from the first week of stocking to the period of maturity. As observed in Figures (1–3) and Figures (4–6), the modified logistic differential equation model outperformed the other existing models because it produces the maximum fish stock at varied periods in weeks and step sizes. The modified model is capable of accommodating more organisms (fishes) and this is due to the extra parameters incorporated into the model.

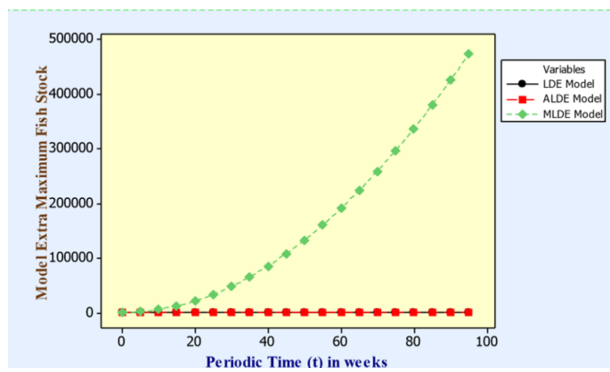

Figure 7: Graph of Model Comparison at the Rate of 0.02

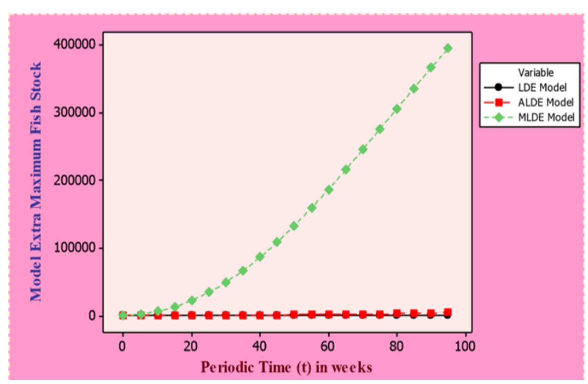

Figure 8: Graph of Model Comparison at the Rate of 0.04

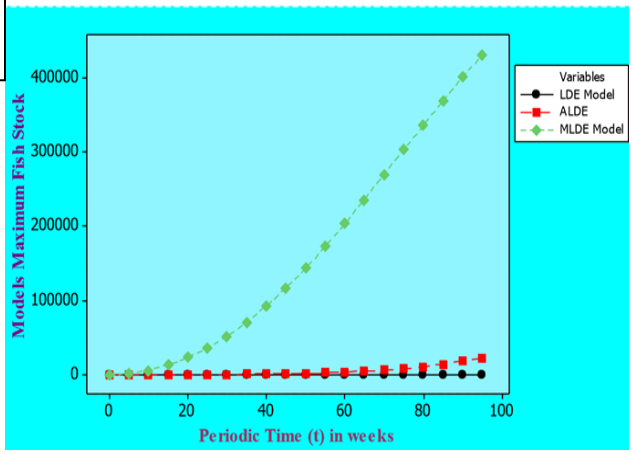

Figure 9: Graph of Model Comparison at the Rate of 0.06

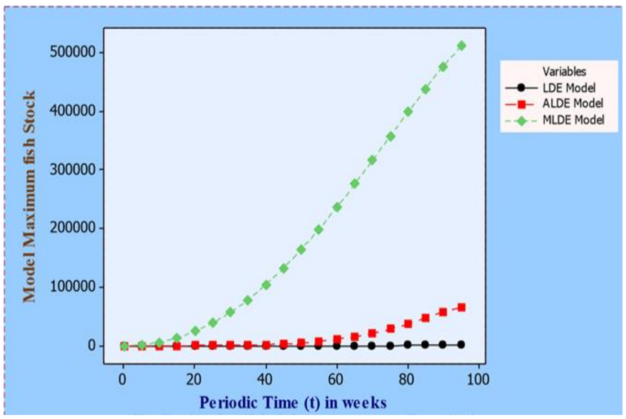

Figure 10: Graph of Model Comparison at the Rate of 0.08

The graphical comparison of models at varied rates are in Figures (7–10) and as seen from the graphs, the modified logistic differential equation model gives maximum amount of fish stock in the environment than the analytic logistic differential equation model and the logistic differential equation model. Again, at rates of higher values, the performance of the analytic logistic differential equation model tends to appreciate when compared with the logistic differential equation model as demonstrated in Figure 9 and Figure 10 respectively. Hence; it has been observed that higher rates are associated with better performance numerically and graphically.

CONCLUSIONS

It is evident from the graphical analysis and numerical results that the modified logistic differential equation model gives the maximum fish stock with a small interval of periods. This simply means that the model projected an increase when the size of the fish increases. Thus, fish farmers and agro-based industries should adopt the modified logistic differential equation model during fish stocking processes. This model is recommended in model selection due to the incorporation of the necessary parameters designed for its success.

ACKNOWLEDGEMENT

The authors are grateful to the anonymous reviewers for thoroughly going through the manuscript and making positive recommendations that improved the paper.

REFERENCES

Abakumov, A. I. and Izrailsky, Y. G. (2022): Optimal Harvest Problem for Fish Population–Structural Stabilization. Mathematics, 10, 986. https://doi.org/10.3390/math10060986

Abakumov, A. I. and Izrailsky, Y. G. (2016): Harvest impact on Fish Population. Matem. Biol. Bioinform, 11(2):191–204.

Andire, M. R. (2003): Modelling Population Dynamics. Institutes for Biodiversity and Ecosystem Dynamics, Population Biology Section, Amsterdam.

Finley, P. (2011): All the Fish in the Sea: Maximum Sustainable Yield and the Failure of Fisheries Management, Chicago: University Chicago Press.

Gebremedhin, S., Bruneel, S., Getahun, A., Anteneh, W. and Goethals, P. (2021): Scientific Methods to Understand Fish Population Dynamics and Support Sustainable Fisheries Management. Water, 13, 574. https://doi.org/10.3390/w1304057

Ikpotokin, O and Siloko, I. U. (2019). A Comparative Analysis of Bootstrap Multivariate Exponentially-Weighted Moving Average Control Limits. Industrial Engineering and Management Systems, 18(3):315–329.

Ikpotokin, O., Siloko, I. U. and Ehiemua, M. (2020). A Note on Computation of Multivariate Control Limits: The Bootstrap Approach. African Journal of Applied Statistics, 7 (2):1027–1038.

Il'in, O. I. (2007): Optimal Exploitation of Fish Populations with Age Structure. Sb. Zh. Industr. Mat., 10(3): 43–57.

Logofet, D. O. (2019): Does Averaging Overestimate Population Growth? It Depends. Ecology Modeling, 411, pp.108744.

Murray, J. D. (2002): Mathematical Biology: An introduction, Third Edition, Springer, New York.

Oghre, E. O. and Okonye, A. O. (2014): Dynamics of Predictor-Prey Model with Hunting of both Organisms. The Journal of the Mathematical Association of Nigeria, 41(2):164–170.

Okuoughae, D. U. (2011): Mathematical Modelling, Mindex Benin, Nigeria.

Shlyufman, K. V., Neverova, G. P. and Frisman, E. Y. (2018): Phase Multistability of Oscillatory Regimes of the Dynamics of the Rilker Model with a Periodically Varying Malthusian Parameter. Matem. Biol. Bioinform., 13(1):68–83.

Siloko, I. U. and Siloko, E. A. (2023). An Investigation on Interdependence Between Rainfall and Temperature in Ekpoma, Edo State, Nigeria, Iranian (Iranica) Journal of Energy and Environment, 14(3):197–204.

Stewart, S. (1999): Single Variable Calculus, Fourth Edition, Gary Ostedi, USA.