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APPLICATION OF ABOODH TRANSFORMS TO THE SOLUTION OF NTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

The study applied the Aboodh transform as a convenient and effective tool to provide exact solution to nth-order ordinary differential equation with constant coefficient based on the simplicity of the Aboodh transform and its fundamental properties. The Laplace transform is used to validate the exact solution provided by the Aboodh transform method.
Keywords: Aboodh transform, Laplace transform, Differential Equations, ODE.

INTRODUCTION

Ordinary linear differential equations with constant and variable coefficients are generally solved by different integral transform without finding their general solution. Aboodh transform is a derivative of the conventional Fourier integral. Aboodh (2013) initiated Aboodh transform to facilitate the process of solving differential equations. Mohammed E. B & Kacem B. (2019) investigate the Aboodh transform to solve first order constant coefficient complex equation. The method provides an effective and efficient way of solving range of linear operator. Aboodh *et al.*, (2018) executed the Aboodh transform to solve delay differential equation. It was revealed that the method converges quickly with logical solution.

The Aboodh transform is denoted by the operator A(.) and defined as

$A[y(x)] = K(v) = \frac{1}{v}\int_0^{\infty} y(x)e^{-vt}dx \quad t_1 \geq 0,\ k_1 \leq v \leq k_2$

Where set A is defined as $A = \{y(x): \exists M,\ k_1,\ k_2 > 0, |y(x)| < Me^{-vx}\}$

where M is finite constant, k_1, k_2 may be finite or infinite

Aboodh and Laplace transform of some important functions

Functions $[y(x)]$	*Aboodh Transform* $A[y(x)]$	*Laplace Transform* $L[y(x)]$
$[y(x^0)]$ or	$\frac{1}{v^2}$	$\frac{1}{S}$
$[y(x)]$	$\frac{1}{v^3}$	$\frac{1}{S^2}$
$[y(x^n)]$	$\frac{n!}{v^{n+2}}$	$\frac{n!}{S^{n+2}}$
$[y(e^{bx})]$	$\frac{1}{v^2 - bv}$	$\frac{1}{S - b}$
$[y(\sin bx)]$	$\frac{b}{v(v^2 + b^2)}$	$\frac{b}{S^2 + b^2}$
$[y(\cos bx)]$	$\frac{1}{v^2 + b^2}$	$\frac{S}{S^2 + b^2}$
$y[y'(x)]$	$vA(v) - \frac{y(0)}{v}$	$SL(y) - y(0)$
$y[y''(x)]$	$V^2A(v) - y'$ $y(0)$	$S^2L(y) - Sy(0) - y'($
$y[y^n(x)]$	$V^{(n)}A(v) - \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{V^{2-n+k}}$	$S^nL(y) - S^{n-1}y(0) - \dots - Sy'(0) - y''(\dots - y^{(n-1)}$

MATERIALS AND METHODS

Aboodh Transform of 1st, 2nd and nth order ordinary differential equations

(1) Consider the first order ODE given by

$$ay' + by = c, \qquad y(0) = k \tag{1}$$

where *a, b, c*, and *k* are constants. Taking Aboodh transform of equation (1) gives:

$A(ay^1) + A(by) = A(c)$

$a\left[vA(y) - \frac{y(0)}{v}\right] + bA(y) = A(c)$

$a\left[vA(y) - \frac{k}{v}\right] + bA(y) = A(c)$

$avA(y) - a\frac{k}{v} + bA(y) = A(c)$

$A(y)[av + b] = A(c) + \frac{ak}{v}$

$A(y)[av + b] = \frac{vA(c) + ak}{v}$

$$A(y) = \frac{vA(c) + ak}{v(av + b)} \tag{2}$$

By taking the inverse of Aboodh transform of (2), we arrived at

$y(x) = A^{-1}\left[\frac{vA(c) + ak}{v(av + b)}\right]$

(2) Let us consider this 2nd order ODE given by:
$ay'' + by' + cy = d, \quad y(0) = k, \quad y'(0) = p$ (3)
where *a, b, c, d, k* and *p* are constant
Taking Aboodh transform of equation (3), we get
$A(ay'') + A(by') + A(cy) = A(d)$
$a\left[v^2A(y) - \frac{y'(0)}{v} - y(0)\right] + b\left[vA(y) - \frac{y(0)}{v}\right] + cA(y) = A(d)$
Introducing the initial condition, we have
$av^2A(y) - \frac{aP}{v} - ak + bvA(y) - \frac{bk}{v} + cA(y) = A(d)$
$A(y)[av^2 + bv + c] = A(d) + \frac{aP}{v} + ak + \frac{bk}{v}$
$A(y)[av^2 + bv + c] = \frac{aP + akv + bk + vA(d)}{v}$
$A(y) = \frac{aP + bk + akv + vA(d)}{v[av^2+bv+c]}$ (4)
Simplifying and taking the inverse Aboodh transform of (4) gives the result,
$y(x) = A^{-1}\left[\frac{aP + bk + akv + vA(d)}{v[av^2+bv+c]}\right]$

(3) Given the third order ODE
$ay''' + by'' + cy' + dy = e, \quad y(0) = k, y'(0) = p$ and $y''(0) = m$ (5)
where *a, b, c, d, e, k, p* and *m* are constants
$A(ay''') + A(by'') + A(cy') + A(dy) = A(e)$
$a\left[v^3A(y) - \frac{y''(0)}{v} - y'(0) - vy(0)\right] + b\left[v^2A(y) - \frac{y'(0)}{v} - y(0)\right] + c\left[vA(y) - \frac{y(0)}{v}\right] + dA(y) = A(e),$

Applying the initial conditions, we have:
$av^3A(y) \quad -\frac{am}{V} - ap - aVk + bV^2A(y) - \frac{bp}{V} - bk + CVA(y) - \frac{Ck}{V} + dA(y) = A(e)$
$A(y)[av^3 + bv^2 + cv + d] = \frac{am}{v} + aP + avk + \frac{bP}{v} + bk + \frac{ck}{v} + A(e)$
$A(y)[av^3 + bv^2 + cv + d] = \frac{am + aPu + aku^2\ bP + bkv + Ck + VA(e)}{V}$
$A(y) = \frac{akv^2 + [aP + bk + A(e)]v + [am + bP + ck]}{v[av^3+bv^2+cv+d]}$ (6)
By taking the inverse Aboodh transform, we get the desired result.

(4) nth order ordinary differential equation
$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots a\frac{dy}{dx} + a_0y = b(x)$ (7)
$y(0) = k_0,\ y'(0) = k_1,\ y''(0) = k_2,\ y'''(0) = k_3, \ldots\ldots\ldots\ y^{n-2}(0) = k_{n-2},\quad y^{n-1}(0) = k_{n-1}$
Taking the Aboodh transform of (7)
$$A\left[a_n\frac{d^ny}{dx^n}\right] = a_nA\left[a_n\frac{d^ny}{dx^n}\right] = a_n\left[v^nA(v) - \frac{y(0)}{v^{2-n}} - \frac{y'(0)}{v^{3-n}} - \ldots . - \frac{y^{(n-1)}(0)}{v}\right] \quad (8)$$
$$A\left[a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}\right] = a_{n-1}A\left[\frac{d^{n-1}y}{dx^{n-1}}\right] = a_{n-1}\left[v^{n-1}K(v) - \frac{y(0)}{v^{3-n}} - \frac{y'(0)}{v^{4-n}} - \ldots . - \frac{y^{(n-2)}(0)}{v}\right] \quad (9)$$
$$A\left[a_{n-2}\frac{d^{n-2}y}{dx^{n-2}}\right] = a_{n-2}A\left[a_n\frac{d^{n-2}y}{dx^{n-2}}\right] = a_{n-2}\left[v^{n-2}A(v) - \frac{y(0)}{v^{4-n}} - \frac{y'(0)}{v^{5-n}} - \ldots . - \frac{y^{(n-3)}(0)}{v}\right] \quad (10)$$
.. ..
..
$$A\left[a\frac{dy}{dx}\right] = aA\left[\frac{dy}{dx}\right] = a\left[vA(v) - \frac{y(0)}{v}\right] \quad (11)$$
$$A[a_0y] = a_0A[y] = a_0A(v) \quad (12)$$
$$A[b(x)] = \frac{b(x)}{v^2} \quad (13)$$
if $b(x)$ is a constant, simplying equations (7) to (12) and applying the initial conditions gives:
$$[a_nv^n + a_{n-1}v^{n-1} + a_{n-2}v^{n-2} + \ldots\ldots av_1 + a_0v_0]\,A - \left[\frac{a_n}{v^{2-n}} + \frac{a_{n-1}}{v^{3-n}} + \frac{a_{n-2}}{v^{4-n}} + \cdots \frac{a_1}{v^1}\right] - \left[\frac{a_n}{v^{3-n}} + \frac{a_{n-1}}{v^{4-n}} + \frac{a_{n-2}}{v^{5-n}} + \cdots\right]k_1 - \ldots\ldots - \left[\frac{a_nk_{n-1}}{v} + \frac{a_{n-1}k_{n-2}}{v} + \frac{a_{n-2}k_{n-3}}{v^{5-n}} + \cdots\right] = A[b(x)] \quad (14)$$
Equation (14) is the general solution of the $\mathrm{n^{th}}$ order ordinary differential with constant coefficient by Aboodh transform.

Laplace transform of 1st, 2nd and nth order ordinary differential equations

(1)Consider the 1st order ordinary differential equation
$ay' + by = c, \quad y(0) = k$ (15)
where *a, b, c* and *k* are constant
Taking Laplace transform of equation (15), we have:
$L(ay') + L(by) = L(c)$
$a[S\overline{y} - y(0)] + b\overline{y} = L(c)$
$aS\overline{y} - ak + b\overline{y} = L(c)$
$aS\overline{y} + b\overline{y} = L(c) + ak$
$\overline{y}(aS + b) = L(c) + ak$
$\overline{y} = \frac{L(c)+ak}{aS+b}$ (16)
The inverse Laplace transform generate our desired result

(2) Given the 2nd order ordinary differential equation
$ay'' + by' + cy = d, \quad y(0) = k, \quad y'(0) = P$ (17)
where *a, b, c, d, e, k, p* and *m* are constant
Taking Laplace transform of the equation
$L(ay'') + L(by') + L(cy) = L(d)$
$a[S^2\overline{y} - S\overline{y}(0) - y'(0)] + b[S\overline{y} - y(0)] + c\overline{y} = L(d)$
$aS^2\overline{y} - aSk - aP + bS\overline{y} - bk + c\overline{y} = L(d)$
$\overline{y}[aS^2 + bS + c] = aSk + aP + bk + L(d)$
$\overline{y} = \frac{aSk+aP+bk+L(d)}{aS^2+bS+c}$ (18)
Simplifying and taking the inverse Laplace transform of equation (18) gives the desired result

RESULTS AND DISCUSSION

(1) Solve $y' + 2y = x, \quad y(0) = 1$ (19)

Using Aboodh transform of (1)
$A(y) = \frac{vA(c)+ak}{v(av+b)}$

a = 1, b=2, c= x and k=1
Substituting the above values, we have
$A(y) = \frac{1+v^2}{v^4+2v^3} = -\frac{1}{4v^2} + \frac{1}{2v^3} + \frac{5}{4(v^2+2v)}$
By taking inverse Aboodh transform of *A(y)*, we have
$y(x) = -\frac{1}{4} + \frac{1}{2}x + \frac{5}{4}e^{-2x}$
Using Laplace transform for example (1), we have

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$L(y) = \frac{L(c)+ak}{aS+b}$

a = 1, b =2, c = x and k =1

Substituting the above values, we have

$L(y) = \frac{1+S^2}{S^3+2S^2} = -\frac{1}{4S} + \frac{1}{2S^2} + \frac{5}{4(S+2)}$

The inverse Laplace transform of this equation is simply obtained as:

$y(x) = -\frac{1}{4} + \frac{1}{2}x + \frac{5}{4}e^{-2x}$

(2) Solve $y' + 2y = 5$, $y(0) = 1$

Using Aboodh transform

$A(y) = \frac{vA(c)+ak}{v(av+b)}$

a = 1, b=2, c = 5 and k =1

Substituting we have

$A(y) = \frac{5+v}{v^3+2v^2} = -\frac{3}{2(v^2+2v)} + \frac{5}{2v^2}$

The inverse Aboodh transform of *A(y)* leads to the solution

$y(x) = -\frac{3}{2}e^{-2x} + \frac{5}{2}$

Laplace transform of example (2)

$L(y) = \frac{L(c)+ak}{as+b}$

a = 1, b=2, c= 5 and k=1

Substituting the above values

$L(y) = \frac{5+s}{s^2+2s} = \frac{5}{2s} - \frac{3}{2(s+2)}$

The inverse Laplace transform of this equation yields

$y(x) = \frac{5}{2} - \frac{3}{2}e^{-2x}$

(3) solve $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 4$

By Aboodh transform,

$A(y) = \frac{ap+bk+akv+vA(d)}{v(av^2+bv+c)}$

a = 1, b= -3, c= 2, d = 0, k =1, and p = 4

By substitution, *A(y)* yields

$A(y) = \frac{v+1}{v^3-3v^2+2v} = \frac{3}{v^2-2v} - \frac{2}{v^2-v}$

The inverse Aboodh transform of this equation is simply obtained as

$y(x) = 3e^{2x} - 2e^{x}$

Using Laplace transform

Recall equation (17)

$L(y) = \frac{aks+ap+bk+L(d)}{as^2+bs+c}$

a = 1, b = -3, c = 2, d = 0, k=1 and P= 4

Substituting the above values gives,

$L(y) = \frac{s+1}{s^2-3s+2} = 3\left(\frac{1}{s-2}\right) - 2\left(\frac{1}{s-1}\right)$

The inverse Laplace transform of this equation is simply obtained as:

$y(x) = 3e^{2x} - 2e^{x}$

(4) solve $y'' - 3y' + 2y = 4e^{3x}$, $y(0) = -3$, $y'(0) = 5$

Using Aboodh transform

$A(y) = \frac{ap+bk+akv+vA(d)}{v(av^2+bv+c)}$

a = 1, b= -3, c= 2, d = $4e^{3x}$, k = -3, and p = 5

Substituting the above values gives

$A(y) = \frac{-3v^2+23v-38}{v^4-6v^3+11v^2-6v} = \frac{2}{v^2-3v} + \frac{4}{v^2-2v} - \frac{9}{v^2-v}$

By taking inverse Aboodh transform, result gives

$y(x) = 2e^{3x} + 4e^{2x} - 9e^{x}$

Using Laplace transform

$L(y) = \frac{aks+ap+bk+L(d)}{as^2+bs+c}$

a = 1, b= -3, c= 2, d = $4e^{3x}$*, k = -3, and p = 5*

Substituting the above values gives

$L(y) = \frac{-3s^2+23s-38}{s^3-6s^2+11s-6} = \frac{2}{s-3} + \frac{4}{s-2} - \frac{9}{s-1}$

By taking inverse Laplace transform, we have

$y(x) = 2e^{3x} + 4e^{2x} - 9e^{x}$

(5) solve $y'' + y = x$, $y(0) = 1$, $y'(0) = 0$

Using Aboodh transform and substituting the constant coefficients give

$A(y) = \frac{1+v^3}{v^5+v^3} = \frac{1}{v^3+v} + \frac{1}{v^3} + \frac{1}{v^2+v}$

The inverse Aboodh transform give the solution

$y(x) = -\sin x + x + \cos x$

Using Laplace transform gives

$L(y) = \frac{1+s^3}{s^4+s^2} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$

The inverse Laplace transform of this equation is simply obtained as

$y(x) = x + \cos x - \sin x$

$y''' + 2y' = \cos x$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$

Using Aboodh transform

Recall equation (3)

$A(y) = \frac{akv^2+[ap+bk+A(e)]v+(am+bp+ck)}{v(av^3+bv^2+cv+d)}$

a = 1, b = 0, c = 2, d = 0, e = cos*x*, k = 0, p = 1 and m = 0

Substituting the above values, we have

$A(y) = \frac{v^2+2}{v^5+3v^3+2v} = \frac{1}{v^3+v}$

Taking the inverse Aboodh transform, we have:

$y(x) = \sin x$

Using Laplace transform

Recall equation (6)

$L(y) = \frac{akS^2+apS+am+bkS+bp+ck+L(e)}{aS^3+bS^2+cS+d}$

a = 1, b = 0, c = 2, d = 0, e = cos*x*, k = 0, p = 1 and m = 0

Substituting the above values, we have

$L(y) = \frac{S^3+2S}{S^5+3S^3+2S} = \frac{1}{S^2+1}$

The inverse Laplace transform of this equation is simply obtained as

$y(x) = \sin x$

In the examples presented, the application of the fundamental and procedure of the Aboodh transform method in solving the differential equations give exact solutions to the ODEs.

Conclusion

The study established the Aboodh transform method to be effective and efficient in solving n^{th}- order differential equations. The simplicity and efficiency of the Aboodh Transform method in solving ordinary differential equations is revealed by the Laplace Transform method. The step-by-step procedure of the Aboodh method is responsible for its wide application in solving ordinary differential equations, partial differential equations and system of ordinary differential equations. Aboodh transform is an efficient method in solving differential equations with boundary values without finding the general solution and the arbitrary constants.

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