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MODIFIED FRECHET DISTRIBUTIONS AND THEIR GENERALIZED FAMILIES

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ABSTRACT
The Frechet distribution is used for modeling extreme events. There are different approaches to developing statistical distributions which include the use of translation methods, system of differential equations, quantile methods among others. Existing statistical distributions are also modified or generalized to accommodate other different types of data and improve goodness of fit to data. Addition of extra parameter(s) is one approach used for generalizing existing distributions such that the base distributions are embedded in the new generalized distributions. Some methods of parameter induction were used to obtained families of generalized distributions. Parameter(s) were also introduced into the probability distributions of the Frechet distribution to derive functions of its modified versions belonging to each of the generalized families derived. Further study is recommended on some of the modified Frechet distributions and their generalized families.

Keywords: Modified Frechet Distributions; Generalized families; Lehmann Alternatives; Marshall-Olkin Method; α-Power Transformed Method.

INTRODUCTION

The Frechet distribution also known as Extreme Value Distribution or Inverse Weibull Distribution is useful in extreme value theory for describing extreme data. Extended Frechet distributions having additional parameter(s) have been introduced by different authors with the aim of introducing flexibility. There are different families of generalized distributions, among which are the exponentiated family, beta family, Weibull family, Alpha Transformed family, Gamma family, Kumaraswamy-G family, transformer-transformer family, exponentiated generalized transformer-transformer family and Marshall-Olkin family.

Nadarajah and Kotz (2003) introduced the Exponentiated Frechet Distribution, a proportional hazard model by using Lehmann Alternative II Method (LA2). The Probability Density Function (PDF), Cumulative Distribution Function (CDF), and Survival Function (SF) of a continuous random variable, Y, belonging to the family of generalized distribution obtained through the use of LA2 given any base random variable,

X ($X \in \Re$), and any additional shape parameter c ϵ (0, ∞) are respectively given as follows;

$$f_Y(x) = cf(x)(\bar{F}(x))^{c-1} \quad (1a)$$

$$F_Y(x) = 1 - (\bar{F}(x))^c \quad (1b)$$

$$\bar{F}_Y(x) = (\bar{F}(x))^c \quad (1c)$$

Similarly, corresponding functions of the exponentiated family of distributions obtained from the application of Lehmann Alternative 1 Method (LA1) are as follows;

$$f_Y(x) = cf(x)(F(x))^{c-1} \quad (1d)$$

$$F_Y(x) = (F(x))^c \quad (1e)$$

$$\bar{F}_Y(x) = 1 - (F(x))^c \quad (1f)$$

Krishna et al. (2013) studied the Marshall-Olkin Frechet distribution and applied distribution to a real data set on failure times of air-conditioning systems in jet planes. The Marshall-Olkin Method (MOM) was introduced by Marshall and Olkin (1997) as a method for adding extra parameter. PDF, CDF, and SF of this family of generalized distributions for a continuous random variable, Y, with an additional shape parameter (c > 0) are respectively

$$f(y;c) = \frac{cf(x)}{[1-(1-c)(1-F(x))]^2} = \frac{cf(x)}{[c+(1-c)F(x)]^2} \quad (1g)$$

$$F(y;c) = \frac{F(x)}{1-(1-c)(1-F(x))} = \frac{F(x)}{c+(1-c)F(x)}$$
(1h)

$$\bar{F}(y;c) = \frac{c\bar{F}(x)}{1-(1-c)(1-F(x))} = \frac{c\bar{F}(x)}{c+(1-c)F(x)} \quad (1i)$$

$f(x)$, $F(x)$, and $\bar{F}(x)$ are respectively the PDF, CDF, and SF of the base distribution for a continuous random variable, X.

Nasiru et al. (2019) applied the α-Power Transformation Method (APT) to obtain the alpha power transformed Frechet distribution. The α-Power transformations of the CDF (F(x)) and PDF (f(x)) of a continuous variable X for $X \in \Re$ are defined as follows:

$$f_Y(x) = \begin{cases} \frac{\log c}{c-1} f(x) c^{F(x)}, & c \neq 1 \\ f(x), & c = 1 \end{cases} \quad (1j)$$

$$F_Y(x) = \begin{cases} \frac{c^{F(x)}-1}{c-1}, & c \neq 1 \\ F(x), & c = 1 \end{cases} \quad (1k)$$

$$\bar{F}_Y(x) = \begin{cases} 1 - \frac{c^{F(x)}-1}{c-1} = \frac{c-c^{F(x)}}{c-1}, & c \neq 1 \\ \bar{F}(x), & c = 1 \end{cases} \quad (1l)$$

Equations 1j, 1k, and 1l are respectively the PDF, CDF, and SF of a continuous random variable, Y, whose distribution is a member of the α-Power transformed family of generalized distributions with additional shape parameter (c > 0). Nassar et al. (2019) extended the α-power transformation using MOM to obtain the Marshall-Olkin alpha power family of distributions.

Gupta and Kundu (2009) discussed the Power Transformed Method (PTM). Given X to be a non-negative random variable, considering a new random variable Y with an additional parameter (c > 0) such that $Y=X^{1/c}$, the PDF, CDF, and SF of the generalized distribution derived from PTM are as follows;

$$f_Y(x) = cx^{c-1} f_X(x^c) \quad (1m)$$
$$F_Y(x) = F_X(x^c) \quad (1n)$$
$$\bar{F}_Y(x) = 1 - F_X(x^c) \quad (1o)$$

There also exist extended Frechet distributions with more than one additional parameter obtained through the use of more than one parameter induction method. The exponentiated generalized Frechet distribution by Abd-Elfattah et al. (2016) with two additional parameters is a member of the exponentiated generalized class of distributions introduced by Cordeiro et al. (2013). The technique of generalization that produced the Kum-generalized (KW-G) distribution family proposed by Cordeiro and Castro (2011) was employed by Diab and Elbatal (2016) and Mansour et al. (2018) to derive the Kumaraswamy exponentiated Frechet distribution, an extension of the exponentiated Frechet distribution by Nadarajah and Kotz (2003).

The KW-G family was also employed by Mead and Abd-Eltawab (2014) to obtain the Kumarawamy Frechet distribution using Frechet distribution as base distribution. Teamah et al. (2020) introduced the Frechet -Weilbull distribution which belongs to the T-X generalized family of distributions. Pillai and Moolath (2019) proposed exponential transmuted Frechet distribution, a member of the T- transmuted X family by Jayakuma and Babu (2017). A six-parameter Frechet distribution, a generalization of the transmuted Marshall-Olkin Frechet distribution (Afify et al. 2015) was introduced by Yousef et al. (2016) using Kumarawamy generalized method. Another six-parameter Frechet model called Beta Generalized Exponentiated Frechet (BGEF) was studied by Badr (2019). BGEF is a beta generalized distribution.
Modified Kies generator (MK-G) by Al-Babtain et al. (2020) was employed by Al Sobhi (2021) to generate the Modified Kies Frechet distribution, an extension of the Frechet distribution. Riffi et al. (2019) also extended the Frechet distribution proposing the generalized transmuted Frechet distribution, a quadratic rank transmuted distribution belonging to the generalized transmuted family. Almetwally and Muhammed (2019) obtained the Bivariate Frechet distribution; FGM bivariate Frechet (FGMBF) and AMH Bivariate Frechet (AMHBF) using Farlie-Gumbel Morgenstern (FGM), Ali-Mikhail-Jaq (AMH) Copulas and Frechet distribution. The odd Frechet family was generalized by Marganpoor et al. (2020) to generate generalized odd Frechet family of distributions. Nasiru (2018) introduced the extended odd Frechet- G family of distributions using the transformed-transformer method. The beta Frechet by Barreto-Souza et al. (2011), transmuted Frechet (Mahmoud and Maudoun 2013), gamma extended Frechet distribution (Da Silva et al. 2013), Weibull Frechet distribution (Afify et al. 2016), and Burr X exponentiated Frechet (Zayed and Butt 2017) are some other extensions of the Frechet distribution. Application of some of these extended frechet distributions by authors showed their ability to improve goodness of fit to data.
This study provides details on how generalized families of distributions and respective members (generalized distributions) may be generated using some of the aforementioned parameter induction methods and the Frechet distribution.

MATERIALS AND METHODS

Lehmann Alternatives 1 and II Methods, PTM, APT, and MOM were used to introduce extra parameter(s) for the purpose of generalization in order to improve flexibility of distributions. LA1 and PTM were first used to show how the Frechet distribution may be generalized to produced modified frechet distributions with an additional shape parameter. MOM and LA2 were then used to illustrate how some functions of generalized families of distributions having two additional parameters and those of their corresponding modified Frechet distributions may be obtained. Subsequently, some functions of other modified Frechet distributions and those of their generalized families with two additional parameters similarly obtained were given.

RESULTS AND DISCUSSION

Let $f(x)$, $F(x)$, $s(y) = \bar{F}(x)$, $h(y)$, and $r(x)$ be respective denotations of the PDF, CDF, SF, hazard function (HF), and Reverse Hazard Function (RHF) of random variable, X. The PDF, CDF, SF, HF, RHF of a continuous random variable, X, that follows the Frechet distribution are respectively given as;

$$f(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right) \quad (3a)$$
$$F(x) = exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right) \quad (3b)$$
$$\bar{F}(x) = 1 - exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right) \quad (3c)$$
$$h(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\left(1 - exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right)^{-1} \quad (3d)$$
$$r(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}} \quad (3e)$$

$x > 0,\ \alpha, \beta > 0$. α and β are the shape and scale parameters respectively.

Modifed Frechet distributions with an extra shape parameter

Modified Frechet distribution with an additional shape parameter (c>0) may be obtained for modeling another continuous variable, Y, by applying any parameter induction method. Two examples using PTM and LA1 are shown below.

Application of PTM

To derive the PDF and CDF of a modified Frechet distribution for modeling Y using this method, use *f(x)* in 3a in equation *1m* and then *F(x)* in 3b in equation *1n* to obtain

$$f(y) = cy^{c-1}\frac{\alpha\beta^\alpha}{y^{c(\alpha+1)}} exp\left(-\left(\frac{\beta}{y^c}\right)^\alpha\right) \quad (3.11a)$$
$$F(y) = exp\left(-\left(\frac{\beta}{y^c}\right)^\alpha\right) \quad (3.11b)$$

Application of LA1

To derive the PDF of a modified Frechet distribution for Y using LA1, replace f(x) and F(x) in 1d with f(x) in 3a and F(x) in 3b respectively to obtain

$$f(y) = c\frac{\alpha\beta^\alpha}{y^{\alpha+1}} exp\left(-\left(\frac{\beta}{y}\right)^\alpha\right)\left(exp\left(-\left(\frac{\beta}{y}\right)^\alpha\right)\right)^{c-1} = \frac{c\alpha\beta^\alpha}{y^{\alpha+1}}\left(exp\left(-\left(\frac{\beta}{y}\right)^\alpha\right)\right)^c \quad (3.12a)$$

Similarly, F(x) in 1e is replaced with F(x) in 3b to derive the CDF of Y given below using LA1.

$$F(y) = \left(exp\left(-\left(\frac{\beta}{y}\right)^\alpha\right)\right)^c \quad (3.12b)$$

Modified Frechet distributions and respective generalized families

Generalized families of distributions and respective modified Frechet distributions with two additional shape parameters (c>0, t>0) may be obtained for modeling another continuous variable, Z, by sequentially applying any two parameter induction methods. The generalized families reduce to the base distribution when both c ad t assume the value of 1. This is done in the following either by sequentially using MOM and LA2 or LA2 and MOM.

Generalized family 1 (MOM and LA2)

This generalized family is as a result of first applying MOM to introduce an extra shape parameter $c > 0$ to the probability distribution of X which produces a continuous variable Y, a member of the Marshall Olkin family of generalized distributions. Subsequently, LA2 is then applied to introduce another shape parameter $t > 0$ to the probability distribution of Y which generalizes the Marshall-Olkin family. The functions of the new generalized family having two additional shape parameters c and t are obtained thus;

Replace c in equations 1a,1b, and 1c with t since t is the shape parameter introduced for the second application of LA2. Substituting $f(y;c)$ and $\bar{F}(y;c)$ in 1g and 1i respectively for f(x) and $\bar{F}(x)$ in 1a, the PDF for this generalized distribution becomes

$$f_Z(x) = \frac{c^t t\left(\bar{F}(x)\right)^{t-1} f(x)}{[1-(1-c)(1-F(x))]^{t+1}} = \frac{c^t t\left(\bar{F}(x)\right)^{t-1} f(x)}{[1-(1-c)(\bar{F}(x))]^{t+1}} \quad (3.21a)$$

Using $\bar{F}(y;c)$ in 1i for $\bar{F}(x)$ in 1b and 1c, the CDF and SF become

$$F_Z(x) = 1 - \left(\frac{c\bar{F}(x)}{1-(1-c)\left(1-F(x)\right)}\right)^t \quad (3.21b)$$

$$\bar{F}_Z(x) = \left(\frac{c\bar{F}(x)}{1-(1-c)(1-F(x))}\right)^t \quad (3.21c)$$

The HF and RHF are therefore given below;

$$h_Z(x) = \frac{f_z(x)}{\bar{F}_z(x)} = \frac{tf(x)}{\bar{F}(x)(1-(1-c)(1-F(x)))} = \frac{th(x)}{1-(1-c)(1-F(x))} \quad (3.21d)$$

$$r_Z(x) = \frac{f_z(x)}{F_{z(x)}} = \frac{c^t t\left(\bar{F}(x)\right)^{t-1} f(x)}{[1-(1-c)(1-F(x))][(1-(1-c)(1-F(x)))^t-(c\bar{F}(x))^t]} \quad (3.21e)$$

Jayakuma and Mathew (2008) introduced this family of generalized distributions as a generalization of the Marshall-Olkin family.

Modified Frechet distribution 1

Given the Frechet distribution as the base distribution for a continuous random variable, X, the PDF and CDF of modified Frechet distribution belonging to generalized family 1 are obtained as follows;

Replacing $f(x)$ and $\bar{F}(x)$ in 3.21a respectively with $f(x)$ and $\bar{F}(x)$ in 3a and 3c

$$f(z) = \frac{c^t t\alpha\beta^\alpha\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{t-1} exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}{z^{\alpha+1}\left[1-(1-c)\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)\right]^{t+1}} \quad (3.211a)$$

Replacing $F(x)$ and $\bar{F}(x)$ in 3.21b respectively with $F(x)$ and $\bar{F}(x)$ in 3b and 3c

$$F(z) = 1 - \left(\frac{c\left[1-exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right]}{1-(1-c)\left[1-exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right]}\right)^t \quad (3.211b)$$

Generalized family 2 (LA2 and MOM)

This generalized family is as a result of first applying LA2 to introduce an extra shape parameter $c > 0$ to the probability distribution of X which produces a continuous variable Y, a member of the exponentiated family of generalized distributions. Subsequently, MOM is then applied to introduce another shape parameter $t > 0$ to the probability distribution of Y which generalizes the exponetiated family (LA2). The functions of the new generalized family having two additional shape parameters c and t are obtained thus;

Replace c in equations 1g,1h, and 1i with t since t is the shape parameter introduced for the second application of MOM. Substituting $f_Y(x)$ and $F_Y(x)$ in 1a and 1b respectively for f(x) and $F(x)$ in 1g, the PDF for this generalized distribution becomes

$$f_Z(x) = \frac{ct\left(\bar{F}(x)\right)^{c-1} f(x)}{\left[1-\left(\bar{F}(x)\right)^c+t(\bar{F}(x))^c\right]^2} = \frac{ct\left(\bar{F}(x)\right)^{c-1} f(x)}{[1-(1-t)(\bar{F}(x))^c]^2} \quad (3.22a)$$

Using $F_Y(x)$ in 1b for $F(x)$ in 1h, the CDF becomes

$$F_Z(x) = \frac{1-\left(\bar{F}(x)\right)^c}{1-\left(\bar{F}(x)\right)^c+t(\bar{F}(x))^c} = \frac{1-\left(\bar{F}(x)\right)^c}{1-(1-t)(\bar{F}(x))^c} \quad (3.22b)$$

Using $F_Y(x)$ and $\bar{F}_Y(x)$ in 1b and 1c respectively for F(x) and $\bar{F}(x)$ in 1i, the SF becomes

$$\bar{F}_Z(x) = \frac{t\left(\bar{F}(x)\right)^c}{1-\left(\bar{F}(x)\right)^c+t(\bar{F}(x))^c} = \frac{t\left(\bar{F}(x)\right)^c}{1-(1-t)(\bar{F}(x))^c} \quad (3.22c)$$

The HF and RHF are therefore given below;

$$h_Z(x) = \frac{f_z(x)}{\bar{F}_z(x)} = \frac{c\left(\bar{F}(x)\right)^{-1} f(x)}{1-\left(\bar{F}(x)\right)^c+t(\bar{F}(x))^c} = \frac{c\left(\bar{F}(x)\right)^{-1} f(x)}{1-(1-t)(\bar{F}(x))^c} \quad (3.22d)$$

$$r_Z(x) = \frac{f_z(x)}{F_{z(x)}} = \frac{ct\left(\bar{F}(x)\right)^{c-1} f(x)(1-(\bar{F}(x))^c)^{-1}}{1-\left(\bar{F}(x)\right)^c+t(\bar{F}(x))^c} \quad (3.22e)$$

Modified Frechet distribution 2

Given the Frechet distribution as the base distribution for a continuous random variable, X, the PDF and CDF of modified Frechet distribution belonging to generalized family 2 are obtained as follows;

Replacing $f(x)$ and $\bar{F}(x)$ in 3.22a respectively with $f(x)$ and $\bar{F}(x)$ in 3a and 3c

$$f(z) = \frac{ct\alpha\beta^\alpha\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{c-1} exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}{z^{\alpha+1}\left[1-(1-t)\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c\right]^2} \quad (3.221a)$$

Replacing $\bar{F}(x)$ in 3.22b with $\bar{F}(x)$ in 3b

$$F(z) = \frac{1-\left[1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^c}{1-(1-t)\left[1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^c} \quad (3.221b)$$

PDF and CDF of some other generalized families and those of corresponding modified frechet distributions can be similarly derived and are given in the following sub-sections.

Generalized family 3 (LA1 and LA2)

$$f_Z(x) = ctf(x)\big(F(x)\big)^{c-1}(1-(F(x))^c)^{t-1} \qquad (3.23a)$$

$$F_Z(x) = 1-\Big(1-\big(F(x)\big)^c\Big)^t \qquad (3.23b)$$

This family of generalized distributions is the (KW-G) family proposed by Cordeiro and Castro (2011).

Modified Frechet distribution 3

$$f(z) = \frac{ct\alpha\beta^\alpha}{z^{\alpha+1}}\left[exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^c\left[1-\left[exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^c\right]^{t-1} \qquad (3.231a)$$

$$F(z) = 1-\left[1-\left[exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^c\right]^t \qquad (3.231b)$$

This modified Frechet distribution is Kumaraswamy Frechet distribution introduced by Mead and Abd-Eltawab (2014).

Generalized family 4 (LA2 and LA1)

$$f_Z(x) = ctf(x)\big(\bar{F}(x)\big)^{c-1}\Big(1-\big(\bar{F}(x)\big)^c\Big)^{t-1} \qquad (3.24a)$$

$$F_Z(x) = (1-(\bar{F}(x))^c)^t \qquad (3.24b)$$

This family of generalized distributions is the exponentiated generalized-G family proposed Cordeiro et al. (2013).

Modified Frechet distribution 4

$$f(z) = \frac{ct\alpha\beta^\alpha\left[1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^{c-1}\left[1-\left[1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^c\right]^{t-1}}{z^{\alpha+1}\left[exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right]^{-1}}$$

(3.241a)

$$F(z) = \left[1-\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c\right]^t \qquad (3.241b)$$

Generalized family 5 (MOM and LA1)

$$f_Z(x) = \frac{ctf(x)\big(F(x)\big)^{t-1}}{[1-(1-c)(1-F(x))]^{t+1}} = \frac{ctf(x)\big(F(x)\big)^{t-1}}{[1-(1-c)(\bar{F}(x))]^{t+1}} \qquad (3.25a)$$

$$F_Z(x) = \left(\frac{F(x)}{1-(1-c)(1-F(x))}\right)^t \qquad (3.25b)$$

Tahir and Nadarajah (2015) introduced this family of generalized distributions as a generalization of the Marshall-Olkin family.

Modified Frechet distribution 5

$$f(z) = \frac{ct\alpha\beta^\alpha\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{t-1}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}{z^{\alpha+1}\left[1-(1-c)\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)\right]^{t+1}} =$$

$$\frac{ct\alpha\beta^\alpha\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^t}{z^{\alpha+1}\left[1-(1-c)\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)\right]^{t+1}} \qquad (3.251a)$$

$$F(z) = \left(\frac{exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)}{1-(1-c)\left(1-exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right)}\right)^t \qquad (3.251b)$$

Generalized family 6 (LA1 and MOM)

$$f_Z(x) = \frac{ctf(x)\big(F(x)\big)^{c-1}}{\left[\big(F(x)\big)^c+t(1-(F(x))^c)\right]^2} = \frac{ctf(x)\big(F(x)\big)^{c-1}}{[t+(1-t)(F(x))^c]^2} \qquad (3.26a)$$

$$F_Z(x) = \frac{\big(F(x)\big)^c}{\big(F(x)\big)^c+t(1-(F(x))^c)} = \frac{\big(F(x)\big)^c}{t+(1-t)(F(x))^c} \qquad (3.26b)$$

Modified Frechet distribution 6

$$f(z) = \frac{ct\alpha\beta^\alpha\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c}{z^{\alpha+1}\left[t+(1-t)\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c\right]^2} \qquad (3.261a)$$

$$F(z) = \frac{\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c}{t+(1-t)\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c} \qquad (3.261b)$$

Generalized family 7 (LA2 and APT)

$$f_Z(x) = \begin{cases} \frac{\log t}{t-1}c\big(\bar{F}(x)\big)^{c-1}f(x)t^{1-(\bar{F}(x))^c}, \ t \neq 1 \\ c\big(\bar{F}(x)\big)^{c-1}f(x), t = 1 \end{cases} \qquad (3.271a)$$

$$F_Z(x) = \begin{cases} \frac{t^{1-(\bar{F}(x))^c}-1}{t-1}, t \neq 1 \\ 1-\big(\bar{F}(x)\big)^c, t = 1 \end{cases} \qquad (3.271b)$$

Modified Frechet distribution 7

$$f(z) = \begin{cases} \frac{\log t}{t-1}c\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{c-1}\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)t^{1-\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c}, \ t \neq 1 \\ c\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{c-1}\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right), t = 1 \end{cases}$$

(3.271a)

$$F(z) = \begin{cases} \frac{t^{1-\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c}-1}{t-1}, t \neq 1 \\ 1-\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c, t = 1 \end{cases} \qquad (3.271b)$$

Modifed Frechet Distribution is the Alpha Power Exponentiated Frechet distribution recently introduced by Baharith L.A. (2022).

Generalized family 8 (APT and LA2)

$$f_Z(x) = \begin{cases} t\left(\frac{c-c^{F(x)}}{c-1}\right)^{t-1}\frac{\log c}{c-1}f(x)c^{F(x)}, \ c \neq 1 \\ t\big(\bar{F}(x)\big)^{t-1}f(x), c = 1 \end{cases} \qquad (3.28a)$$

$$F_Z(x) = \begin{cases} 1-\left(\frac{c-c^{F(x)}}{c-1}\right)^t, c \neq 1 \\ 1-\big(\bar{F}(x)\big)^t, c = 1 \end{cases} \qquad (3.28b)$$

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$$\bar{F}_Z(x) = \begin{cases} \left(\frac{c-c^{F(x)}}{c-1}\right)^t, c \neq 1 \\ \left(\bar{F}(x)\right)^t, c = 1 \end{cases} \quad (3.28c)$$

The RHF is

$$r_Z(x) = \begin{cases} \dfrac{t\left(\frac{c-c^{F(x)}}{c-1}\right)^{t-1}\frac{\log c}{c-1}f(x)c^{F(x)}}{1-\left(\frac{c-c^{F(x)}}{c-1}\right)^t}, c \neq 1 \\ t\left(\bar{F}(x)\right)^{t-1}f(x)\left(1-\left(\bar{F}(x)\right)^t\right)^{-1}, c = 1 \end{cases} \quad (3.28d)$$

Modified Frechet distribution 8

$$f(z) = \begin{cases} t\left(\dfrac{c-c^{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}}{c-1}\right)^{t-1}\left(\frac{\log c}{c-1}\right)\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)c^{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}, c \neq 1 \\ t\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{t-1}\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right),\ c = 1 \end{cases}$$

(3.281a)

$$F(z) = \begin{cases} 1-\left(\dfrac{c-c^{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}}{c-1}\right)^t, c \neq 1 \\ 1-\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^t, c = 1 \end{cases} \quad (3.281b)$$

Generalized family 9 (LA1 and APT)

$$f_Z(x) = \begin{cases} \frac{\log t}{t-1}c\left(F(x)\right)^{c-1}f(x)t^{(F(x))^c},\ t \neq 1 \\ c\left(F(x)\right)^{c-1}f(x), t = 1 \end{cases} \quad (3.29a)$$

$$F_Z(x) = \begin{cases} \frac{t^{(F(x))^c}-1}{t-1}, t \neq 1 \\ \left(F(x)\right)^c, t = 1 \end{cases} \quad (3.29b)$$

The SF of this generalized distribution is given as

$$\bar{F}(x) = \begin{cases} \frac{t-t^{(F(x))^c}}{t-1}, t \neq 1 \\ 1-\left(F(x)\right)^c, t = 1 \end{cases} \quad (3.29c)$$

Modified Frechet distribution 9

$$f(z) = \begin{cases} \frac{\log t}{t-1}c\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{c-1}\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)t^{\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c},\ t \neq 1 \\ c\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{c-1}\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right), t = 1 \end{cases}$$

(3.291a)

$$F(z) = \begin{cases} \dfrac{t^{\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c}-1}{t-1}, t \neq 1 \\ \left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^c, t = 1 \end{cases} \quad (3.291b)$$

Generalized family 10 (APT and LA1)

$$f_Z(x) = \begin{cases} t\left(\frac{\log c}{c-1}\right)f(x)c^{F(x)}\left(\frac{c^{F(x)}-1}{c-1}\right)^{t-1},\ c \neq 1 \\ t\left(F(x)\right)^{t-1}f(x), c = 1 \end{cases} \quad (3.30a)$$

$$F_Z(x) = \begin{cases} \left(\frac{c^{F(x)}-1}{c-1}\right)^t, c \neq 1 \\ \left(F(x)\right)^t, c = 1 \end{cases} \quad (3.30b)$$

The SF is given as

$$\bar{F}_Z(x) = \begin{cases} 1-\left(\frac{c^{F(x)}-1}{c-1}\right)^t, c \neq 1 \\ 1-\left(F(x)\right)^t, c = 1 \end{cases} \quad (3.30c)$$

The HF of this generalized family is given as

$$h_Z(x) = \begin{cases} \dfrac{t\left(\frac{\log c}{c-1}\right)f(x)c^{F(x)}\left(\frac{c^{F(x)}-1}{c-1}\right)^{t-1}}{1-\left(\frac{c^{F(x)}-1}{c-1}\right)^t}, c \neq 1 \\ t\left(F(x)\right)^{t-1}f(x)\left(1-\left(F(x)\right)^t\right)^{-1}, c = 1 \end{cases} \quad (3.30d)$$

Modified Frechet distribution 10

$$f(z) = \begin{cases} t\left(\frac{\log c}{c-1}\right)\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)c^{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}\left(\frac{c^{exp(-(\beta/z)^\alpha)}-1}{c-1}\right)^{t-1}, c \neq 1 \\ t\frac{\alpha\beta^\alpha}{z^{\alpha+1}}exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\left(exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)^{t-1}, c = 1 \end{cases}$$

(3.301a)

$$F(z) = \begin{cases} \left(\frac{c^{exp(-(\beta/z)^\alpha)}-1}{c-1}\right)^t, c \neq 1 \\ \left(exp(-(\beta/z)^\alpha)\right)^t, c = 1 \end{cases} \quad (3.301b)$$

Generalized family 11 (MOM and APT)

$$f_Z(x) = \begin{cases} \frac{c\log(t)f(x)t^{\frac{F(x)}{1-(1-c)(1-F(x))}}}{(t-1)[1-(1-c)(1-F(x))]^2}, t \neq 1 \\ \frac{cf(x)}{[1-(1-c)(1-F(x))]^2}, t = 1 \end{cases}$$

(3.31a)

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$$F_Z(x) = \begin{cases} \dfrac{t^{\frac{F(x)}{1-(1-c)(1-F(x))}}-1}{(t-1)}, & t \neq 1 \\ \dfrac{F(x)}{1-(1-c)(1-F(x))}, & t = 1 \end{cases} \quad (3.31b)$$

The SF is given as

$$\bar{F}_Z(x) = \begin{cases} \dfrac{t-t^{\frac{F(x)}{1-(1-c)(1-F(x))}}}{(t-1)}, & t \neq 1 \\ \dfrac{c\bar{F}(x)}{1-(1-c)(1-F(x))}, & t = 1 \end{cases} \quad (3.31c)$$

The HF as

$$h_Z(x) = \begin{cases} \dfrac{clog(t)f(x)t^{\frac{F(x)}{1-(1-c)(1-F(x))}}}{[1-(1-c)(1-F(x))]^2\left(t-t^{\frac{F(x)}{1-(1-c)(1-F(x))}}\right)}, & t \neq 1 \\ \dfrac{h(x)}{1-(1-c)(1-F(x))}, & t = 1 \end{cases}$$

(3.31d)

and the RHF as

$$r_Z(x) = \begin{cases} \dfrac{clog(t)f(x)t^{\frac{F(x)}{1-(1-c)(1-F(x))}}}{[1-(1-c)(1-F(x))]^2\left(t^{\frac{F(x)}{1-(1-c)(1-F(x))}}-1\right)}, & t \neq 1 \\ \dfrac{cr(x)}{1-(1-c)(1-F(x))}, & t = 1 \end{cases}$$

(3.31e)

Modified Frechet distribution 11 (MOM and APT)

$$f_Z(x) = \begin{cases} \dfrac{clog(t)\alpha\beta^\alpha exp(-(\beta/z)^\alpha)t^{\frac{exp(-(\beta/z)^\alpha)}{1-(1-c)(1-exp(-(\beta/z)^\alpha))}}}{z^{\alpha+1}(t-1)\left[1-(1-c)\left(1-exp(-(\beta/z)^\alpha)\right)\right]^2}, & t \neq 1 \\ \dfrac{c\alpha\beta^\alpha exp(-(\beta/z)^\alpha)}{z^{\alpha+1}[1-(1-c)(1-exp(-(\beta/z)^\alpha))]^2}, & t = 1 \end{cases}$$

(3.311a)

$$F_Z(x) = \begin{cases} \dfrac{t^{\frac{exp(-(\beta/z)^\alpha)}{1-(1-c)(1-exp(-(\beta/z)^\alpha))}}-1}{(t-1)}, & t \neq 1 \\ \dfrac{exp(-(\beta/z)^\alpha)}{1-(1-c)(1-exp(-(\beta/z)^\alpha))}, & t = 1 \end{cases}$$

(3.311b)

Generalized family 12 (APT and MOM)

$$f_Z(x) = \begin{cases} \dfrac{tlog(c)f(x)c^{F(x)}}{(c-1)\left[\left(\frac{c^{F(x)}-1}{c-1}\right)+t\left(\frac{c-c^{F(x)}}{c-1}\right)\right]^2} = \dfrac{tlog(c)f(x)c^{F(x)}}{(c-1)\left[1-(1-t)(\frac{c-c^{F(x)}}{c-1})\right]^2}, & c \neq 1 \\ \dfrac{tf(x)}{\left[1-(1-t)\left(1-F(x)\right)\right]^2}, & c = 1 \end{cases}$$

(3.32a)

$$F_Z(x) = \begin{cases} \dfrac{c^{F(x)}-1}{(c-1)\left[\left(\frac{c^{F(x)}-1}{c-1}\right)+t\left(\frac{c-c^{F(x)}}{c-1}\right)\right]} = \dfrac{c^{F(x)}-1}{(c-1)\left[1-(1-t)(\frac{c-c^{F(x)}}{c-1})\right]}, & c \neq 1 \\ \dfrac{F(x)}{\left[1-(1-t)\left(1-F(x)\right)\right]}, & c = 1 \end{cases}$$

(3.32b)

The SF is given as

$$\bar{F}_Z(x) = \begin{cases} \dfrac{t\left(c-c^{F(x)}\right)}{(c-1)\left[\left(\frac{c^{F(x)}-1}{c-1}\right)+t\left(\frac{c-c^{F(x)}}{c-1}\right)\right]} = \dfrac{t\left(c-c^{F(x)}\right)}{(c-1)\left[1-(1-t)(\frac{c-c^{F(x)}}{c-1})\right]}, & c \neq 1 \\ \dfrac{t\bar{F}(x)}{\left[1-(1-t)\left(1-F(x)\right)\right]}, & c = 1 \end{cases}$$

(3.32c)

Modified Frechet distribution 12 (APT and MOM)

$$f_Z(x) = \begin{cases} \dfrac{tlog(c)\alpha\beta^\alpha exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)c^{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}}{z^{\alpha+1}(c-1)\left[1-(1-t)\left(\frac{c-c^{exp(-(\beta/z)^\alpha)}}{c-1}\right)\right]^2}, & c \neq 1 \\ \dfrac{t\alpha\beta^\alpha exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}{z^{\alpha+1}\left[1-(1-t)\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)\right]^2}, & c = 1 \end{cases}$$

(3.321a)

$$F_Z(x) = \begin{cases} \dfrac{c^{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}-1}{(c-1)\left[1-(1-t)\left(\frac{c-c^{exp(-(\beta/z)^\alpha)}}{c-1}\right)\right]}, & c \neq 1 \\ \dfrac{exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)}{\left[1-(1-t)\left(1-exp\left(-\left(\frac{\beta}{z}\right)^\alpha\right)\right)\right]}, & c = 1 \end{cases}$$

(3.321b)

Generalized family 13 (PTM and LA2)

$$f_z(x) = ctx^{c-1}f(x^c)\left(\bar{F}(x^c)\right)^{t-1} \quad (3.33a)$$

$$F_z(x) = 1 - \left(\bar{F}(x^c)\right)^t \quad (3.33b)$$

Modified Frechet distribution 13 (PTM and LA2)

$$f(z) = \frac{ctz^{c-1}\alpha\beta^\alpha}{z^{c(\alpha+1)}} exp\left(-\left(\frac{\beta}{z^c}\right)^\alpha\right)\Big(1 -$$

$$exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right)\Bigg)^{t-1}$$

(3.331a)

$$F_z(x) = 1 - \left(1 - exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right)\right)^t \qquad (3.331b)$$

Generalized family 14 (PTM and LA1)

$$f_z(x) = ctx^{c-1}f(x^c)\left(F(x^c)\right)^{t-1} \qquad (3.34a)$$
$$F_z(x) = (F(x^c))^t \qquad (3.34b)$$

Modified Frechet distribution 14 (PTM and LA1)

$$f(z) = \frac{ctz^{c-1}\alpha\beta^{\alpha}}{z^{c(\alpha+1)}} exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right)\left(exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right)\right)^{t-1}$$

(3.341a)

$$F_z(x) = \left(exp(-(\beta/z^c)^{\alpha})\right)^t \qquad (3.341b)$$

Generalized family 15 (PTM and APT)

$$f_Z(x) = \begin{cases} \frac{\log(t)}{t-1} cx^{c-1}f(x^c)t^{F(x^c)}, \ t \neq 1 \\ cx^{c-1}f(x^c), t = 1 \end{cases} \qquad (3.35a)$$

$$F_Z(x) = \begin{cases} \frac{t^{F(x^c)}-1}{t-1}, t \neq 1 \\ F(x^c), t = 1 \end{cases} \qquad (3.35b)$$

Modified Frechet distribution 15

$$f(z) = \begin{cases} \left(\frac{\log(t)}{t-1}\right) \frac{cz^{c-1}\alpha\beta^{\alpha}}{z^{c(\alpha+1)}} exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right) t^{exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right)}, \ t \neq 1 \\ cz^{c-1}\frac{\alpha\beta^{\alpha}}{z^{c(\alpha+1)}} exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right), t = 1 \end{cases}$$

(3.351a)

$$F(z) = \begin{cases} \frac{t^{exp\left(-(\beta/z^c)^{\alpha}\right)}-1}{t-1}, t \neq 1 \\ exp\left(-\left(\frac{\beta}{z^c}\right)^{\alpha}\right), t = 1 \end{cases} \qquad (3.351b)$$

Generalized family 16 (PTM and MOM)

$$f_z(x) = \frac{ctx^{c-1}f(x^c)}{[1-(1-t)(1-F(x^c)]^2} \qquad (3.36a)$$

$$F_z(x) = \frac{F(x^c)}{1-(1-t)(1-F(x^c))} \qquad (3.36b)$$

Modified Frechet distribution 16 (PTM and MOM)

$$f(z) = \frac{ctz^{c-1}\alpha\beta^{\alpha}exp(-(\beta/z^c)^{\alpha})}{z^{c(\alpha+1)}[1-(1-t)(1-exp(-(\beta/z^c)^{\alpha}))]^2} \qquad (3.361a)$$

$$F(z) = \frac{exp(-(\beta/z^c)^{\alpha})}{1-(1-t)(1-exp(-(\beta/z^c)^{\alpha}))} \qquad (3.361b)$$

Conclusion

Generalized family of distributions as was used in the study comprises of generalized distributions generated using any base distribution of choice but the same generalization method. These generalized distributions may have one or more additional parameter(s) which at value(s) of 1 reduce(s) to the base distribution. The method of parameter induction was employed for generalization and sequentially applying any two parameter induction methods per time illustrated how several generalized families of distributions having two additional parameters may be obtained. The Frechet distribution was used as base distribution producing modified Frechet distributions belonging to the generalized families obtained. Generalized Families of distributions and modified Frechet distributions not in literature are recommended for future studies.

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