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Full Length Research Article

GENERALIZATION OF FORMULAS FOR PARTITION FUNCTIONS

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ABSTRACT
The idea of formulas for partition functions dates back to Cayley and Macmahon to obtain traceable power series expressions, after which several extensions were made. In this paper, further extension was provided together with a generalized method of finding formulars for some partition functions. We also relate the technical results with their graphical interpretations through a novel use of Bar Charts and Scatter diagrams. We generalized a method of finding formulas for some partition functions.

Keywords: Formula, Integer Partitions, Partition Functions, Bar Chart, Scatter Diagram.

INTRODUCTION
Partition of a non-negative integer n is a non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda n$, that sum to n. We denote by $p_k(n)$ the number of partitions of n into at most k parts. Andrew and Erikson (2004) also stated that $p_k(n)$ equals the number of partitions of n into at most k parts. There is an extensive literature concerning the formula for $p_k(n)$, including contributions by Cayley (1856), Sylvester (1882), Glaisher (1909) and Gupta (1958). Ladan et al (2018) extended the idea of using elementary method for finding formulas for some partition functions dates back to Cayley and Macmahon (George E. Andrews and Kimmo Erikson 2004). For additional references and historical notes, see Gupta (1970). Furthermore, the theory of q-partial fractions and its formula was developed in Munagi (2007).

BASIC DEFINITIONS
Partition
Singh *et al.* (2012) defined a partition of a positive integer n as a sequence of positive integers whose sum is n.
The order of the summands is unimportant when writing the partitions of n. For example the partitions of $n = 4$ are given as

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

Partition function $p(n)$
Andrew and Erikson (2004) stated that the partition function $p(n)$ counts the number of unique partitions of the positive integer n. For example, there are five unique partitions of 4. Hence, $p(4) = 5$.
Intermediate function
The intermediate function denoted by $p_k(n)$ is defined such that it counts the partitions of n with the largest added being no smaller than k.

METHODOLOGY
In this section, we consider the formula $p_k(n)$, which was computed for $k = 1, 2, 3\ and\ 4$ by Andrew (2003). So our desire in this paper is to compute this formula for $k = 5, 6, \ldots, 11$. Furthermore, we implement this formula using computer program called maple software.
That is, Andrew (2003) deduced

$$p_1(n) = 1,$$
$$p_2(n) = \left\lfloor \frac{n+1}{2} \right\rfloor,$$
$$p_3(n) = \{(n+3)^2/12\}$$

Thereafter, Andrew and Erikson (2004) gave $p_k(n) = p(n,k)$ the following generating function

$$\sum_{n=0}^{\infty} p(n,k)q^n = \frac{1}{(1-q)(1-q^2)\ldots(1-q^k)} \quad (1)$$

Now, consider the case $k = 4$, we have maple calculate that

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$$\sum_{n=0}^{\infty} p(n,4)\, q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$$
$$= \frac{\frac{25}{144}}{(1-q)^2} + \frac{\frac{1}{8}}{288(1-q)^3}$$
$$+ \frac{\frac{1}{24}}{(1-q)^4} + \frac{\frac{1}{16}}{(1-q)^2}$$
$$+ \frac{\frac{1}{8}}{(1-q^2)(1-q)^2}$$
$$+ \frac{1}{q(q+2)}\left(\frac{1}{1-q^3}\right) + \frac{\frac{1}{4}}{(1-q^4)}$$
$$= \frac{\frac{1}{24}}{(1-q)^4} + \frac{\frac{1}{8}}{(1-q)^3} + \frac{\left(\frac{5}{12}\right)^2}{(1-q)^2}$$
$$+ \frac{\frac{1}{8}}{(1-q^2)^2} + \frac{\frac{1}{16}}{1-q^2} + \frac{\frac{(2+q)}{9}}{1-q^3}$$
$$+ \frac{\frac{1}{4}}{1-q^4}$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{24}\binom{n+3}{3} + \frac{1}{8}\binom{n+2}{2} + \left(\frac{5}{12}\right)^2 (n+1) \right) q^n$$
$$+ \left(\frac{1}{8}(n+1) + \frac{1}{16}\right) q^{2n}$$
$$+ \sum_{n=0}^{\infty} \left(-\frac{1}{16} q^{2n} + \frac{2}{9} q^{3n} + \frac{1}{9} q^{3n+1} + \frac{1}{4} q^{4n} \right)$$

Here, the coefficient of q^n is the formula for

$$p(n,4) = \frac{1}{24}\binom{n+3}{3} + \frac{1}{8}\binom{n+2}{2} + \frac{25}{144}(n+1) \qquad (2)$$

Case $k = 5$

$$\sum_{n=0}^{\infty} p(n,5)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^5)}$$
$$= \frac{\frac{1}{120}}{(1-q)^5} + \frac{\frac{1}{24}}{(1-q)^4} + \frac{\frac{31}{288}}{(1-q)^3}$$
$$+ \frac{\frac{11}{64}}{(1-q)^2} + \frac{\frac{11}{64}}{1-q^2} + \frac{\frac{1}{9}}{1-q^3}$$
$$+ \frac{\frac{1}{5}}{1-q^5}$$
$$= \left(\frac{1}{120}\binom{n+4}{4} + \frac{1}{24}\binom{n+3}{3} + \frac{31}{288}\binom{n+2}{2} + \frac{11}{64}\binom{n+1}{1} \right) q^n$$
$$+ \frac{11}{64}\sum_{n=0}^{\infty} q^{2n} + \frac{1}{9} q^{3n} + \frac{1}{5}\sum_{n=0}^{\infty} q^{5n}$$

$$p(n,5) = \frac{1}{120}\binom{n+4}{4} + \frac{1}{24}\binom{n+3}{3} + \frac{31}{288}\binom{n+2}{2} + \frac{11}{64}\binom{n+1}{1} \qquad (3)$$

Case $k = 6$

$$\sum_{n=0}^{\infty} p(n,6)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^6)}$$
$$= \frac{1}{720(1-q)^6} + \frac{1}{96(1-q)^5}$$
$$+ \frac{17}{432(1-q)^4} + \frac{331}{3456}\left(\frac{1}{(1-q)^3}\right)$$
$$+ \frac{777611}{518400}\left(\frac{1}{(1-q)^2}\right) + \frac{25}{256(1-q^2)}$$
$$= \frac{1}{720}\sum_{n=0}^{\infty}\binom{n+5}{5} q^n$$
$$+ \frac{1}{96}\sum_{n=0}^{\infty}\binom{n+4}{4} + \frac{17}{432}\sum_{n=0}^{\infty}\binom{n+3}{3}$$
$$+ \frac{331}{3456}\sum_{n=0}^{\infty}\binom{n+2}{2}$$
$$+ \frac{777611}{518400}\sum_{n=0}^{\infty}(n+1) + \frac{25}{256} q^{2n}$$

$$p(n,6) = \frac{1}{720}\binom{n+5}{5} + \frac{1}{96}\binom{n+4}{4} + \frac{17}{432}\binom{n+3}{3} + \frac{331}{3456}\binom{n+2}{2} + \frac{777611}{518400}\binom{n+1}{1} \quad (4)$$

Case $k = 7$

$$\sum_{n=0}^{\infty} p(n,7)\,q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^7)}$$
$$= \frac{1}{5040}\left(\frac{1}{(1-q)^7}\right) + \frac{1}{480(1-q)^6} + \frac{47}{4320}\left(\frac{1}{(1-q)^5}\right) + \frac{161}{4320}\left(\frac{1}{(1-q)^4}\right) + \frac{7913}{86400}\left(\frac{1}{(1-q)^3}\right) + \frac{3187}{20736}\left(\frac{1}{(1-q)^2}\right)$$
$$= \frac{1}{5040}\sum\binom{n+6}{6}q^n + \frac{1}{400}\sum\binom{n+5}{5}q^n + \frac{47}{4320}\sum\binom{n+4}{4}q^n + \frac{161}{4320}\sum\binom{n+3}{3}q^n + \frac{7713}{86400}\sum\binom{n+2}{2}q^n + \frac{3187}{20736}\sum\binom{n+1}{1}q^n$$
$$p(n,7) = \frac{1}{5040}\binom{n+6}{6} + \frac{1}{480}\binom{n+5}{5} + \frac{47}{4320}\binom{n+4}{4} + \frac{161}{4320}\binom{n+3}{3} + \frac{7913}{86400}\binom{n+2}{2} + \frac{3187}{20736}\binom{n+1}{1} \quad (5)$$

Case $k = 8$

$$\sum_{n=0}^{\infty} p(n,8)\,q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^8)}$$
$$= \sum_{n=0}^{\infty}\left\{\left(\frac{1}{40320}\right)\left(\frac{1}{(1-q)^8}\right) + \left(\frac{1}{2880}\right)\left(\frac{1}{(1-q)^7}\right) + \left(\frac{83}{34560}\right)\left(\frac{1}{(1-q)^6}\right) + \left(\frac{25}{2304}\right)\left(\frac{1}{(1-q)^5}\right) + \left(\frac{24523}{691200}\right)\left(\frac{1}{(1-q)^4}\right) + \left(\frac{139}{1600}\right)\left(\frac{1}{(1-q)^3}\right) + \frac{487033}{3175200}\binom{n+1}{1}\right\}$$
$$p(n,8) = \frac{1}{40320}\binom{n+7}{7} + \frac{1}{2880}\binom{n+6}{6} + \frac{83}{34560}\binom{n+5}{5} + \frac{25}{2304}\binom{n+4}{4} + \frac{24523}{691200}\binom{n+3}{3} + \frac{139}{1600}\binom{n+2}{2} + \frac{487033}{3175200}\binom{n+1}{1} \quad (6)$$

Case $k = 9$

$$\sum_{n=0}^{\infty} p(n,9)\,q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^9)} = \sum_{n=0}^{\infty}\left\{\frac{1}{326880(1-q)^9} + \frac{1}{20160(1-q)^8} + \frac{319}{725760(1-q)^7} + \frac{1843}{725760(1-q)^6} + \frac{929377}{87091200(1-q)^5} + \frac{239185}{6967296(1-q)^4} + \frac{8335199}{97542144(1-q)^3} + \frac{1821389}{11943936(1-q)^2}\right\}q^n = \frac{1}{326880}\binom{n+8}{8} + \frac{1}{20160}\binom{n+7}{7} + \frac{319}{725760}\binom{n+6}{6} + \frac{1843}{725760}\binom{n+5}{5} + \frac{929377}{87091200}\binom{n+4}{4} + \frac{239185}{6967296}\binom{n+3}{3} + \frac{8335199}{97542144}\binom{n+2}{2} + \frac{1821389}{11943936}\binom{n+1}{1} \quad (7)$$

Case $k = 10$

$$\sum_{n=0}^{\infty} p(n,10)\,q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^{10})}$$
$$= \sum\left\{\frac{1}{362880(1-q)^{10}} + \frac{1}{161280(1-q)^9} + \frac{199}{2903040(1-q)^8} + \frac{2867}{5806080(1-q)^7} + \frac{161111}{62208000(1-q)^6} + \frac{2600533}{248832000(1-q)^5} + \frac{1826428579}{54867456000(1-q)^4} + \frac{36618675691}{438939648000(1-q)^3} + \frac{40769204821}{268738560000(1-q)^2}\right\}q^n$$
$$p(n,10) = \frac{1}{326880}\binom{n+9}{9} + \frac{1}{161280}\binom{n+8}{8} + \frac{199}{2903040}\binom{n+7}{7} + \frac{2867}{5806080}\binom{n+6}{6} + \frac{161111}{62208000}\binom{n+5}{5} + \frac{2600533}{248832000}\binom{n+4}{4} + \frac{1826428579}{54867456000}\binom{n+3}{3} + \frac{36618675691}{438939648000}\binom{n+2}{2} + \frac{40769204821}{268738560000}\binom{n+1}{1} \quad (8)$$

Case $k = 11$

$$\sum_{n=0}^{\infty} p(n,11)\, q^n = \frac{1}{(1-q)(1-q^2)\dots\dots\dots\dots(1-q^{11})}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{39916800(1-q)^{11}} + \frac{1}{14151520(1-q)^{10}} + \frac{1}{107520(1-q)^{9}} + \frac{95}{1161216(1-q)^{8}} + \frac{114221}{217728000(1-q)^{7}} + \frac{2262473}{870912000(1-q)^{6}} + \frac{2250121541}{219469824000(1-q)^{5}} + \frac{14347821041}{438939648000(1-q)^{4}} + \frac{1099442945011}{13168189440000(1-q)^{3}} + \frac{17080331207}{107495424000(1-q)^{2}} \right\} q^n$$

$$p(n,11) = \frac{1}{39916800}\binom{n+10}{10} + \frac{1}{1415120}\binom{n+9}{9} + \frac{1}{107520}\binom{n+8}{8} + \frac{95}{1161216}\binom{n+7}{7} + \frac{114221}{217728000}\binom{n+6}{6} + \frac{2262473}{870912000}\binom{n+5}{5} + \frac{2250121541}{219469824000}\binom{n+4}{4} + \frac{14347821041}{438939648000}\binom{n+3}{3} + \frac{1099442945011}{13168189440000}\binom{n+2}{2} + \frac{17080331207}{107495424000}\binom{n+1}{1} \qquad (9)$$

Case $k = 12$

$$\sum_{n=0}^{\infty} p(n,12)\, q^n = \frac{1}{(1-q)(1-q^2)\dots\dots\dots\dots(1-q^{12})}$$

$$= \sum \left\{ \frac{2082349}{21555072000(1-q)^{8}} + \frac{147097}{10777536000(1-q)^{9}} + \frac{43}{28740096(1-q)^{10}} + \frac{689}{5748019200(1-q)^{11}} + \frac{1}{159667200(1-q)^{12}} + \frac{1768797250759}{9674588160000(1-q)^{2}} + \frac{15920836215203}{187300026777600(1-q)^{3}} + \frac{1}{6227020800(1-q)^{13}} + \frac{1133921227859}{3543489920000(1-q)^{4}} + \frac{352180748033}{35473489920000(1-q)^{5}} + \frac{126344147}{49268736000(1-q)^{6}} + \frac{7959273253}{14485008384000(1-q)^{7}} \right\} q^n$$

$$= p(n,12) = \frac{1}{479001600}\begin{bmatrix} n+11 \\ 11 \end{bmatrix} + \frac{1}{14515200}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} + \frac{583}{522547200}\begin{bmatrix} n+9 \\ 9 \end{bmatrix} + \frac{3077}{261273600}\begin{bmatrix} n+8 \\ 8 \end{bmatrix} + \frac{355619}{3919104000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix} + \frac{121219}{223948800}\begin{bmatrix} n+6 \\ 6 \end{bmatrix} + \frac{3404664443}{13168168944000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix} + \frac{368650483}{36578304000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix} + \frac{5106621148711}{158018273280000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix} + \frac{13435995840307}{158018273280000}\begin{bmatrix} n+2 \\ 2 \end{bmatrix} + \frac{41434452757831663}{229442532802560000}\begin{bmatrix} n+1 \\ 1 \end{bmatrix}$$

$$\sum_{n=0}^{\infty} p(n,13)\, q^n = \frac{1}{(1-q)(1-q^2)\dots\dots\dots\dots(1-q^{13})}$$

$$= \sum \left\{ \frac{2082349}{21555072000(1-q)^{8}} + \frac{147097}{10777536000(1-q)^{9}} + \frac{43}{28740096(1-q)^{10}} + \frac{689}{5748019200(1-q)^{11}} + \frac{1}{159667200(1-q)^{12}} + \frac{1768797250759}{9674588160000(1-q)^{2}} + \frac{1592083651203}{187300026777600(1-q)^{3}} + \frac{1}{6227020800(1-q)^{13}} + \frac{1133921227859}{35473489920000(1-q)^{4}} + \frac{352180748033}{35473489920000(1-q)^{5}} + \frac{126344147}{49268736000(1-q)^{6}} + \frac{7959273253}{14485008384000(1-q)^{7}} \right\} q^n \qquad (10)$$

Case $k = 13$

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$$= p(n,13) = \frac{1}{6227020800}\begin{bmatrix} n+12 \\ 12 \end{bmatrix} + \frac{1}{159667200}\begin{bmatrix} n+11 \\ 11 \end{bmatrix}$$
$$+ \frac{689}{5748019200}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} + \frac{43}{28740096}\begin{bmatrix} n+9 \\ 9 \end{bmatrix} + \frac{147097}{10777536000}\begin{bmatrix} n+8 \\ 8 \end{bmatrix}$$
$$+ \frac{2082349}{21555072000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix} + \frac{7959273253}{14485008384000}\begin{bmatrix} n+6 \\ 6 \end{bmatrix}$$
$$+ \frac{126344147}{49268736000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix}\frac{352180748033}{35473489920000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix} + \frac{1133921227859}{35473489920000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}$$
$$+ \frac{15920836215203}{187300026777600}\begin{bmatrix} n+2 \\ 2 \end{bmatrix} + \frac{1768797250759}{9674588160000}\begin{bmatrix} n+1 \\ 1 \end{bmatrix} \quad (11)$$

Case $k = 14$

$$\sum_{n=0}^{\infty} p(n,14)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots(1-q^{14})}$$

$$=\sum\left\{ \frac{11564717}{115221657600\,(1-q)^8} + \frac{1}{87178291200\,(1-q)^{14}} \right.$$
$$+ \frac{50955014259469609}{16060977296179200000\,(1-q)^4} + \frac{1606678949699}{163870801920000\,(1-q)^5}$$
$$+ \frac{89263}{49268736000\,(1-q)^{10}} + \frac{67}{5748019200\,(1-q)^{12}} + \frac{31}{182476800\,(1-q)^{11}}$$
$$+ \frac{879209865376557264601}{4750034035344998400000\,(1-q)^2} + \frac{1}{1916006400\,(1-q)^{13}}$$
$$+ \frac{273610829547325859}{3212195459235840000\,(1-q)^3} + \frac{2072197}{137952460800\,(1-q)^9}$$
$$\left. + \frac{61818946718537}{24334814085120000\,(1-q)^6} + \frac{700442723}{1267438233600\,(1-q)^7} \right\} q^n$$

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$$= p(n, 14) = \frac{1}{87178291200}\begin{bmatrix} n+13 \\ 13 \end{bmatrix} + \frac{1}{1916006400}\begin{bmatrix} n+12 \\ 12 \end{bmatrix}$$
$$+ \frac{67}{5748019200}\begin{bmatrix} n+11 \\ 11 \end{bmatrix} + \frac{31}{182476800}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} + \frac{89263}{49268736000}\begin{bmatrix} n+9 \\ 9 \end{bmatrix}$$
$$+ \frac{2072197}{137952460800}\begin{bmatrix} n+8 \\ 8 \end{bmatrix}\frac{11564717}{115221657600}\begin{bmatrix} n+7 \\ 7 \end{bmatrix} + \frac{700442723}{1267438233600}\begin{bmatrix} n+6 \\ 6 \end{bmatrix}$$
$$+ \frac{61818946718537}{24334814085120000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix} + \frac{1606678949699}{163870801920000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix}$$
$$+ \frac{50955014259469609}{1606097729617920000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix} + \frac{273610829547325859}{3212195459235840000}\begin{bmatrix} n+2 \\ 2 \end{bmatrix}$$
$$+ \frac{879209865376557264601}{4750034035344998400000}\begin{bmatrix} n+1 \\ 1 \end{bmatrix} \tag{12}$$

Case $k = 15$

$$\sum_{n=0}^{\infty} p(n, 15)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots\ldots(1-q^{15})}$$

$$= \sum \left\{ \frac{1}{1307674368000\,(1-q)^{15}} + \frac{29}{28021593600\,(1-q)^{13}} \right.$$
$$+ \frac{2807577628572357919}{14990245476433920000\,(1-q)^2} + \frac{48733677590539525966 79}{570004084241399808000000\,(1-q)^3}$$
$$+ \frac{437839}{2037934080000\,(1-q)^{11}} + \frac{131849294393001 8947}{41758540970065920000\,(1-q)^4} +$$
$$+ \frac{1}{24908083200\,(1-q)^{14}} + \frac{7811}{448345497600\,(1-q)^{12}}$$
$$+ \frac{23882897422147301}{949057749319680000 0\,(1-q)^6} + \frac{562899812680201 09}{57997973569536000 00\,(1-q)^5}$$
$$+ \frac{10576762909}{102979356480000\,(1-q)^8} + \frac{2623447721394037}{47452887465984000 00\,(1-q)^7}$$
$$\left. + \frac{92504921}{44834549760000\,(1-q)^{10}} + \frac{316517792681}{19772036444160000\,(1-q)^9} \right\} q^n$$

$$
\begin{aligned}
p(n,15) = {} & \frac{1}{1307674368000}\begin{bmatrix} n+14 \\ 14 \end{bmatrix} + \frac{1}{24908083200}\begin{bmatrix} n+13 \\ 13 \end{bmatrix} \\
& + \frac{29}{28021593600}\begin{bmatrix} n+12 \\ 12 \end{bmatrix} + \frac{7811}{448345497600}\begin{bmatrix} n+11 \\ 11 \end{bmatrix} \\
& + \frac{437839}{2037934080000}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} + \frac{92504921}{44834549760000}\begin{bmatrix} n+9 \\ 9 \end{bmatrix} \\
& + \frac{316517792681}{19772036444160000}\begin{bmatrix} n+8 \\ 8 \end{bmatrix} + \frac{10576762909}{102979356480000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix} \\
& + \frac{2623447721394037}{4745288746598400000}\begin{bmatrix} n+6 \\ 6 \end{bmatrix} + \frac{23882897422147301}{9490577493196800000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix} \\
& + \frac{56289981268020109}{5799797356953600000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix} \\
& + \frac{1318492943930018947}{41758540970065920000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix} \frac{487336775905395259 6679}{5700040842413998080000 0}\begin{bmatrix} n+2 \\ 2 \end{bmatrix} \\
& + \frac{2807577628572357919}{14990245476433920000}\begin{bmatrix} n+1 \\ 1 \end{bmatrix}
\end{aligned}
\tag{13}
$$

Case $k = 16$

$$\sum_{n=0}^{\infty} p(n,16)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots\ldots(1-q^{16})}$$

$$=\sum\left\{\frac{4087}{2510734786560\,(1-q)^{13}}+\frac{83894919671}{5021469573120000\,(1-q)^{9}}+\frac{24457481}{96566722560000\,(1-q)^{11}}+\frac{10361391871396789137774 13}{5472039208717438156800000\,(1-q)^{2}}+\frac{714590252921}{316352583106560000\,(1-q)^{10}}+\frac{2871985726408753501534 7}{9120065347862396928000 00\,(1-q)^{4}}+\frac{30926603324552653}{3212195459235840000\,(1-q)^{5}}+\frac{1}{20922789888000\,(1-q)^{16}}+\frac{1}{348713164800\,(1-q)^{15}}+\frac{493379}{21459271680000\,(1-q)^{12}}+\frac{107919943304835881729}{125620734819041280000 0\,(1-q)^{3}}+\frac{67764362867}{799065243648\,q+799065243648}+\frac{1061}{12553673932800\,(1-q)^{14}}+\frac{7902812357932477}{759246199455744000 00\,(1-q)^{8}}+\frac{654130379211637}{1186322186649600000\,(1-q)^{7}}+\frac{5723072334941039}{2294425328025600000\,(1-q)^{6}}\right\}q^{n}$$

$$=p(n,16)=\frac{1}{20922789888000}\begin{bmatrix}n+15\\15\end{bmatrix}+\frac{1}{348713164800}\begin{bmatrix}n+14\\14\end{bmatrix}+\frac{1061}{12553673932800}\begin{bmatrix}n+13\\13\end{bmatrix}+\frac{4087}{2510734786560}\begin{bmatrix}n+12\\12\end{bmatrix}+\frac{493379}{21459271680000}\begin{bmatrix}n+11\\11\end{bmatrix}+\frac{24457481}{96566722560000}\begin{bmatrix}n+10\\10\end{bmatrix}+\frac{714590252921}{316352583106560000}\begin{bmatrix}n+9\\9\end{bmatrix}+\frac{83894919671}{5021469573120000}\begin{bmatrix}n+8\\8\end{bmatrix}+\frac{7902812357932477}{75924619945574400000}\begin{bmatrix}n+7\\7\end{bmatrix}+\frac{654130379211637}{1186322186649600000}\begin{bmatrix}n+6\\6\end{bmatrix}+\frac{5723072334941039}{2294425328025600000}\begin{bmatrix}n+5\\5\end{bmatrix}+\frac{30926603324552653}{3212195459235840000}\begin{bmatrix}n+4\\4\end{bmatrix}+\frac{28719857264087535015347}{912006534786239692800000}\begin{bmatrix}n+3\\3\end{bmatrix}+\frac{107919943304835881729}{1256207348190412800000}\begin{bmatrix}n+2\\2\end{bmatrix}+\frac{1036139187139678913777413}{5472039208717438156800000}\begin{bmatrix}n+1\\1\end{bmatrix}\qquad(14)$$

Case $k = 17$

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$$\sum_{n=0}^{\infty} p(n,17)\, q^{n} = \frac{1}{(1-q)(1-q^{2})\ldots\ldots\ldots\ldots(1-q^{17})}$$

$$= \sum \Bigg\{ \frac{12737715926290759}{23199189427814400000\,(1-q)^{7}} + \frac{251060242614421}{101454181171200000\,(1-q)^{6}}$$
$$+ \frac{1}{5230697472000\,(1-q)^{16}} + \frac{576487999645850942483}{301489763565699072000 0\,(1-q)^{2}}$$
$$+ \frac{1}{355687428096000\,(1-q)^{17}} + \frac{110542124553061}{1054508610355200000\,(1-q)^{8}}$$
$$+ \frac{761928893273}{316352583106560000\,(1-q)^{10}} + \frac{14346133593451530052471}{456003267393119846400000\,(1-q)^{4}}$$
$$+ \frac{1006331}{35867639808000\,(1-q)^{12}} + \frac{9646262766082803190613}{111674269565662003200000\,(1-q)^{3}}$$
$$+ \frac{2921}{20922789888000\,(1-q)^{14}} + \frac{401}{62768369664000\,(1-q)^{15}}$$
$$+ \frac{8827}{3941498880000\,(1-q)^{13}} + \frac{186491377854443}{10846374277939200000\,(1-q)^{9}}$$
$$+ \frac{1745332121189459148503}{182401306957247938560000\,(1-q)^{5}} + \frac{90193055353}{316352583106560000\,(1-q)^{11}} \Bigg\} q^{n}$$

$$= p(n,17) = \frac{1}{355687428096000}\begin{bmatrix} n+16 \\ 16 \end{bmatrix} + \frac{1}{5230697472000}\begin{bmatrix} n+15 \\ 15 \end{bmatrix} \qquad (15)$$
$$+ \frac{401}{62768369664000}\begin{bmatrix} n+14 \\ 14 \end{bmatrix} + \frac{2921}{20922789888000}\begin{bmatrix} n+13 \\ 13 \end{bmatrix}$$
$$+ \frac{8827}{3941498880000}\begin{bmatrix} n+12 \\ 12 \end{bmatrix} + \frac{1006331}{35867639808000}\begin{bmatrix} n+11 \\ 11 \end{bmatrix}$$
$$+ \frac{90193055353}{316352583106560000}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} + \frac{761928893273}{316352583106560000}\begin{bmatrix} n+9 \\ 9 \end{bmatrix}$$
$$+ \frac{186491377854443}{10846374277939200000}\begin{bmatrix} n+8 \\ 8 \end{bmatrix} + \frac{110542124553061}{1054508610355200000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix}$$
$$+ \frac{12737715926290759}{23199189427814400000}\begin{bmatrix} n+6 \\ 6 \end{bmatrix} + \frac{251060242614421}{101454181171200000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix}$$
$$+ \frac{1745332121189459148503}{182401306957247938560000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix} + \frac{14346133593451530052471}{456003267393119846400000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}$$
$$+ \frac{9646262766082803190613}{111674269565662003200000}\begin{bmatrix} n+2 \\ 2 \end{bmatrix} + \frac{576487999645850942483}{301489763565699072000 0}\begin{bmatrix} n+1 \\ 1 \end{bmatrix}$$

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Case $k = 18$

$$\sum_{n=0}^{\infty} p(n,18)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots\ldots(1-q^{18})}$$

$$= \sum \left\{ \frac{511281961}{180772904632320000\,(1-q)^{13}} + \frac{1333}{120515269754880\,(1-q)^{15}} \right.$$
$$+ \frac{4133066267256490730829721}{1313289410092185157632000000\,(1-q)^{4}} + \frac{677}{1506440871936000\,(1-q)^{16}}$$
$$+ \frac{133367777481241489609}{542681574418258329600000\,(1-q)^{6}} + \frac{1723019120896040273879}{1808938581394194432000000\,(1-q)^{5}}$$
$$+ \frac{740052937051}{22777385983672320000\,(1-q)^{12}} + \frac{4721590695329}{15184923989114880000\,(1-q)^{11}}$$
$$+ \frac{22818492602611189605414673}{262657882018437031526400000\,(1-q)^{3}}$$
$$+ \frac{7258395714706370715819043933}{37640394001650397741056000000\,(1-q)^{2}} + \frac{1}{83691159552000\,(1-q)^{17}}$$
$$+ \frac{1}{6402373705728000\,(1-q)^{18}} + \frac{18078913}{90386452316160000\,(1-q)^{14}}$$
$$+ \frac{210756655661900129}{2004409966563164160000\,(1-q)^{8}} + \frac{10950860303292592453}{20044099665631641600000\,(1-q)^{7}}$$
$$\left. + \frac{63815539444039}{25308206648524800000\,(1-q)^{10}} + \frac{1140551376299033}{65078245667635200000\,(1-q)^{9}} \right\} q^n$$

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$$= p(n,18) = \frac{1}{6402373705728000}\begin{bmatrix} n+17 \\ 17 \end{bmatrix} + \frac{1}{83691159552000}\begin{bmatrix} n+16 \\ 16 \end{bmatrix}$$
$$+ \frac{677}{1506440871936000}\begin{bmatrix} n+15 \\ 15 \end{bmatrix} + \frac{1333}{120515269754880}\begin{bmatrix} n+14 \\ 14 \end{bmatrix}$$
$$+ \frac{18078913}{90386452316160000}\begin{bmatrix} n+13 \\ 13 \end{bmatrix} + \frac{511281961}{180772904632320000}\begin{bmatrix} n+12 \\ 12 \end{bmatrix}$$
$$+ \frac{740052937051}{22777385983672320000}\begin{bmatrix} n+11 \\ 11 \end{bmatrix} + \frac{4721590695329}{15184923989114880000}\begin{bmatrix} n+10 \\ 10 \end{bmatrix}$$
$$+ \frac{63815539444039}{25308206648524800000}\begin{bmatrix} n+9 \\ 9 \end{bmatrix} + \frac{1140551376299033}{65078245667635200000}\begin{bmatrix} n+8 \\ 8 \end{bmatrix}$$
$$+ \frac{210756655661900129}{2004409966563164160000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix} + \frac{1095086030329259245 3}{2004409966563164160000}\begin{bmatrix} n+6 \\ 6 \end{bmatrix}$$
$$+ \frac{13336777748124148960 9}{5426815744182583296000 0}\begin{bmatrix} n+5 \\ 5 \end{bmatrix} + \frac{1723019120896040273879}{1808938581394194432000 00}\begin{bmatrix} n+4 \\ 4 \end{bmatrix}$$
$$+ \frac{4133066267256490730829721}{1313289410092185157632000 00}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}$$
$$+ \frac{2281849260261118960541467 3}{2626578820184370315264000 00}\begin{bmatrix} n+2 \\ 2 \end{bmatrix}$$
$$+ \frac{725839571470637071581904393 3}{376403940016503977410560000 00}\begin{bmatrix} n+1 \\ 1 \end{bmatrix}$$

(16)

Case $k = 19$

$$\sum_{n=0}^{\infty} p(n,19)\, q^n = \frac{1}{(1-q)(1-q^2)\ldots\ldots\ldots\ldots(1-q^{19})}$$

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$$=\sum\left\{\frac{374558661097}{10325748312598118400\,(1-q)^{12}}+\frac{505830253243654626022245381}{260487155720763998208000000\,(1-q)^{2}}\right.$$
$$+\frac{39795340839895977970721713 5307}{455448767419969812666777600000 0\,(1-q)^{3}}$$
$$+\frac{35843893035560832029}{34074969431573790720 0000\,(1-q)^{8}}+\frac{1552437450615739}{5957162488037376000 00\,(1-q)^{10}}$$
$$+\frac{757}{25609494822912000\,(1-q)^{17}}+\frac{214008357129043}{645359269537382400000\,(1-q)^{11}}$$
$$+\frac{655553}{2508685207142400\,(1-q)^{14}}+\frac{25329149}{153656968937472000 0\,(1-q)^{15}}$$
$$+\frac{9091080786086480313211 51}{372098666192785794662400000\,(1-q)^{6}}+\frac{10439678527}{30977244937794355 20\,(1-q)^{13}}$$
$$+\frac{1}{121645100408832000\,(1-q)^{19}}+\frac{30337175448816008258760 1}{5581479992891786919936 00000\,(1-q)^{7}}$$
$$+\frac{41693}{51218989645824000\,(1-q)^{16}}+\frac{140708063894148230950618097}{44651839943134295359488 00000\,(1-q)^{4}}$$
$$+\frac{42394867965529316429132 77}{44651839943134295359488 0000\,(1-q)^{5}}$$
$$\left.+\frac{30234450441412695 29}{17037484715786895360 0000\,(1-q)^{9}}+\frac{1}{1422749712384000\,(1-q)^{18}}\right\}q^{n}$$

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$$= p(n,19) = \frac{1}{121645100408832000}\begin{bmatrix} n+18 \\ 18 \end{bmatrix} + \frac{1}{1422749712384000}\begin{bmatrix} n+17 \\ 17 \end{bmatrix}$$
$$+ \frac{757}{25609494822912000}\begin{bmatrix} n+16 \\ 16 \end{bmatrix} + \frac{41693}{51218989645824000}\begin{bmatrix} n+15 \\ 15 \end{bmatrix}$$
$$+ \frac{25329149}{1536569689374720000}\begin{bmatrix} n+14 \\ 14 \end{bmatrix} + \frac{655553}{2508685207142400}\begin{bmatrix} n+13 \\ 13 \end{bmatrix}$$
$$+ \frac{10439678527}{3097724493779435520}\begin{bmatrix} n+12 \\ 12 \end{bmatrix} + \frac{374558661097}{10325748312598118400}\begin{bmatrix} n+11 \\ 11 \end{bmatrix}$$
$$+ \frac{214008357129043}{645359269537382400000}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} + \frac{1552437450615739}{595716248803737600000}\begin{bmatrix} n+9 \\ 9 \end{bmatrix}$$
$$+ \frac{3023445044141269529}{1703748471578689536000000}\begin{bmatrix} n+8 \\ 8 \end{bmatrix} + \frac{35843893035560832029}{3407496943157379072000000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix}$$
$$+ \frac{303371754488160082587601}{5581479992891786919936000000}\begin{bmatrix} n+6 \\ 6 \end{bmatrix}$$
$$+ \frac{9091080786086480313211151}{37209866619278579466240000000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix}$$
$$+ \frac{4239486796552931642913277}{4465183994313429535948800000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix}$$
$$+ \frac{140708063894148230950618097}{44651839943134295359488000000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}$$
$$+ \frac{3979534083989597797072171353307}{455448767419969812666777600000000}\begin{bmatrix} n+2 \\ 2 \end{bmatrix}$$
$$+ \frac{5058302532436546260224538 1}{2604871557207639982080000000}\begin{bmatrix} n+1 \\ 1 \end{bmatrix} \quad (17)$$

Case $k = 20$

$$\sum_{n=0}^{\infty} p(n,20)\, q^n = \frac{1}{(1-q)(1-q^2) \ldots \ldots \ldots (1-q^{20})}$$

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$$
\begin{aligned}
=\sum \Bigg\{ & \frac{1}{2432902008176640000\,(1-q)^{20}} + \frac{1}{25609494822912000\,(1-q)^{19}} \\
& + \frac{1}{547796680704000\,(1-q)^{18}} + \frac{179}{3201186852864000\,(1-q)^{17}} \\
& + \frac{38737157}{30731393787494400000\,(1-q)^{16}} + \frac{3618794341036946924543099 3}{14883946647711431786496000000\,(1-q)^{6}} \\
& + \frac{6037179129760548259924817}{1116295998578357383987200000 0\,(1-q)^{7}} \\
& + \frac{2488672865717236514082167}{2626578820184370315264000 00\,(1-q)^{5}} + \frac{684264869}{30731393787494400000\,(1-q)^{15}} \\
& + \frac{51957159660965591}{1947141111037564518400 00\,(1-q)^{10}} + \frac{87063146604937247}{4867852775939112960000\,(1-q)^{9}} \\
& + \frac{1172824917023950391834917}{1116295998578357383987200000 0\,(1-q)^{8}} + \frac{681885394871}{17286409005465600000\,(1-q)^{12}} \\
& + \frac{11665866288593}{33525156859084800000\,(1-q)^{11}} + \frac{225562464523}{70402829404078080 0000\,(1-q)^{14}} \\
& + \frac{304528466243}{79023584024985600000\,(1-q)^{13}} \\
& + \frac{48537473048876042326331460924713780 71}{2485985116183769627875325509632000000 0\,(1-q)^{2}} \\
& + \frac{287622785502726592792472792104 3}{910897534839939625333555200000 00\,(1-q)^{4}} \\
& + \frac{800332176939083214376119641818 9}{910897534839939625333555200000 00\,(1-q)^{3}} \Bigg\} q^{n}
\end{aligned}
$$

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$$
\begin{aligned}
= p(n,20) = {} & \frac{1}{2432902008176640000}\begin{bmatrix} n+19 \\ 19 \end{bmatrix} + \frac{1}{25609494822912000}\begin{bmatrix} n+18 \\ 19 \end{bmatrix} \\
& + \frac{1}{547796680704000}\begin{bmatrix} n+17 \\ 17 \end{bmatrix} + \frac{179}{3201186852864000}\begin{bmatrix} n+16 \\ 16 \end{bmatrix} \\
& + \frac{38737157}{307313937874944000 00}\begin{bmatrix} n+15 \\ 15 \end{bmatrix} + \frac{684264869}{30731393787494400000}\begin{bmatrix} n+14 \\ 14 \end{bmatrix} \\
& + \frac{225562464523}{704028294040780800000}\begin{bmatrix} n+13 \\ 13 \end{bmatrix} + \frac{304528466243}{790235840249856 00000}\begin{bmatrix} n+12 \\ 12 \end{bmatrix} \\
& + \frac{681885394871}{1728640900546560 0000}\begin{bmatrix} n+11 \\ 11 \end{bmatrix} + \frac{11665866288593}{335251568590848 00000}\begin{bmatrix} n+10 \\ 10 \end{bmatrix} \\
& + \frac{51957159660965591}{194714111037564518 40000}\begin{bmatrix} n+9 \\ 9 \end{bmatrix} + \frac{87063146604937247}{486785277593911296 0000}\begin{bmatrix} n+8 \\ 8 \end{bmatrix} \\
& + \frac{117282491702395039183 4917}{111629599857835738398720 00000}\begin{bmatrix} n+7 \\ 7 \end{bmatrix} \\
& + \frac{6037179129760548259924817}{11162959985783573839872000000}\begin{bmatrix} n+6 \\ 6 \end{bmatrix} \\
& + \frac{3618794341036946924543099 3}{14883946647711431786496000000}\begin{bmatrix} n+5 \\ 5 \end{bmatrix} \\
& + \frac{2488672865717236514082167}{2626578820184370315264 00000}\begin{bmatrix} n+4 \\ 4 \end{bmatrix} \\
& + \frac{2876227855027265927924727921043}{9108975348399396253335552 0000000}\begin{bmatrix} n+3 \\ 3 \end{bmatrix} \\
& + \frac{8003321769390832143761196418189}{91089753483993962533355520000000}\begin{bmatrix} n+2 \\ 2 \end{bmatrix} \\
& + \frac{48537473048876042326331460924713780 71}{248598511618376962787532550963200 00000}\begin{bmatrix} n+1 \\ 1 \end{bmatrix}
\end{aligned}
\tag{18}
$$

.

$$
p(n,k) = C_1\binom{n+(k-1)}{k-1} + C_2\binom{n+(k-2)}{k-2} + C_3\binom{n+(k-3)}{k-3} + \ldots + C_{K-2}\binom{n+2}{2} + C_{K-1}\binom{n+1}{1} \tag{19}
$$

Example
We illustrate the above formulas for the case $n = 50, k = 10$
Substituting$n = 50\,, k = 10$.

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$$p\,(50{,}10) = \frac{1}{3628800}\binom{59}{9} + \frac{1}{161280}\binom{58}{8} + \frac{199}{2903040}\binom{57}{7} + \frac{2867}{5806080}\binom{56}{6} + \frac{161111}{62208000}\binom{55}{5} + \frac{2600533}{248832000}\binom{54}{4} + \frac{1826428579}{54867456000}\binom{53}{3} + \frac{36618675691}{438939648000}\binom{52}{2} + \frac{40769204821}{268738560000}\binom{51}{1}$$
$$= 16928$$

This result agrees with the one obtained with maple via inbuilt command "numbpart" from the combination package:

```
>with (combinat, numbpart);
>NumbPart(50, 10);
        16928
```

Table 1: The number of restricted partitions of n

(n,k)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	1																			
2	1	1																		
3	1	1	1																	
4	1	2	1	1																
5	1	2	2	1	1															
6	1	3	3	2	1	1														
7	1	3	4	3	2	1	1													
8	1	4	5	5	3	2	1	1												
9	1	4	7	6	5	3	2	1	1											
10	1	5	8	9	7	5	3	2	1	1										
11	1	5	10	11	10	7	5	3	2	1	1									
12	1	6	12	15	13	11	7	5	3	2	1	1								
13	1	6	14	18	18	14	11	7	5	3	2	1	1							
14	1	7	16	23	23	20	15	11	7	5	3	2	1	1						
15	1	7	19	27	30	26	21	15	11	7	5	3	2	1	1					
16	1	8	21	34	37	35	28	22	15	11	7	5	3	2	1	1				
17	1	8	24	39	47	44	38	29	22	15	11	7	5	3	2	1	1			
18	1	9	27	47	57	58	49	40	30	22	15	11	7	5	3	2	1	1		
19	1	9	30	54	70	71	65	52	41	30	22	15	11	7	5	3	2	1	1	
20	1	10	33	64	84	90	82	70	54	42	30	22	15	11	7	5	3	2	1	1

Table 2: p(n,4)

n	k	P(n,k)
4	**4**	1
5	**4**	1
6	**4**	2
7	**4**	3
8	**4**	5
9	**4**	6
10	**4**	9
11	**4**	11
12	**4**	15
13	**4**	18
14	**4**	23
15	**4**	27
16	**4**	34
17	**4**	39
18	**4**	47
19	**4**	54
20	**4**	64

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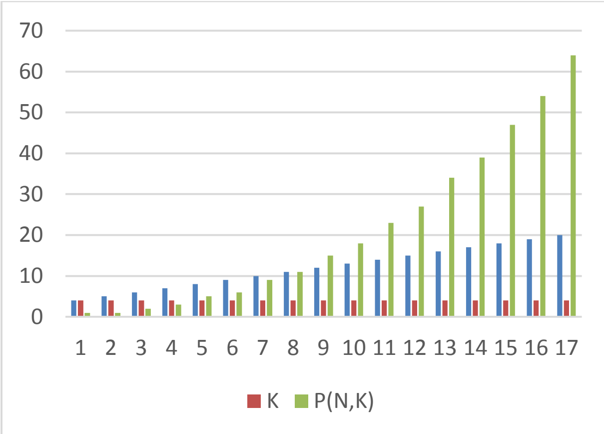

Fig. 1a: p(n,4)

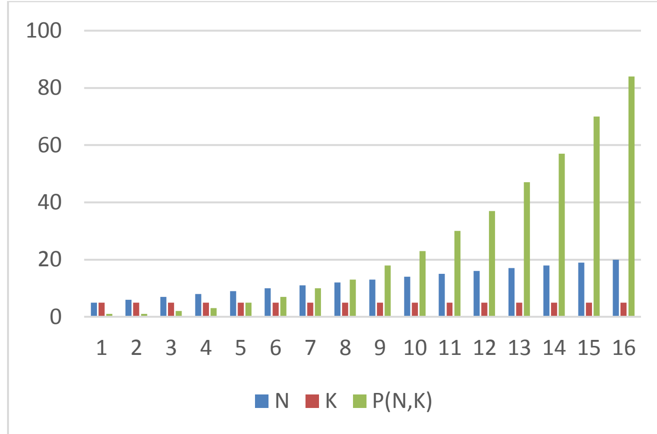

Fig. 2a: p(n,5)

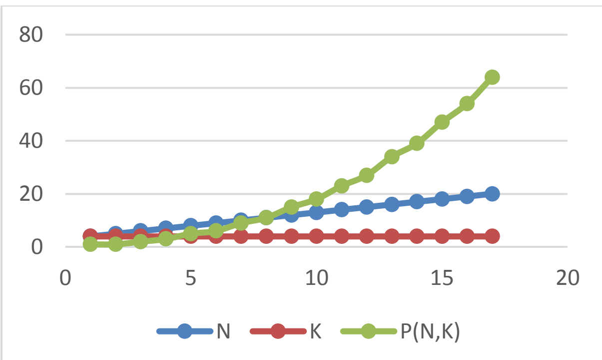

Fig. 1b: p(n,4)

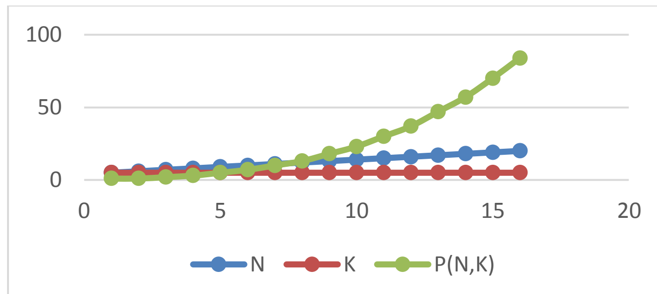

Fig. 2b: p(n,5)

Table 3: p(n,5)

n	k	P(n,k)
5	5	1
6	5	1
7	5	2
8	5	3
9	5	5
10	5	7
11	5	10
12	5	13
13	5	18
14	5	23
15	5	30
16	5	37
17	5	47
18	5	57
19	5	70
20	5	84

Table 4: p(n,6)

n	k	P(n,k)
6	6	1
7	6	1
8	6	2
9	6	3
10	6	5
11	6	7
12	6	11
13	6	14
14	6	20
15	6	26
16	6	35
17	6	44
18	6	58
19	6	71
20	6	90

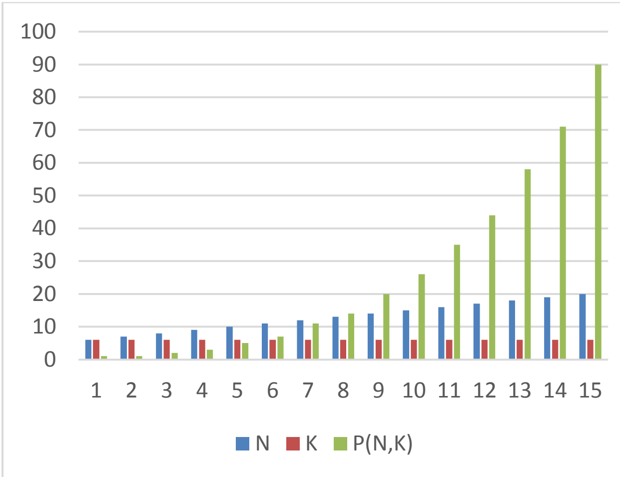

Fig. 3a: p(n,6)

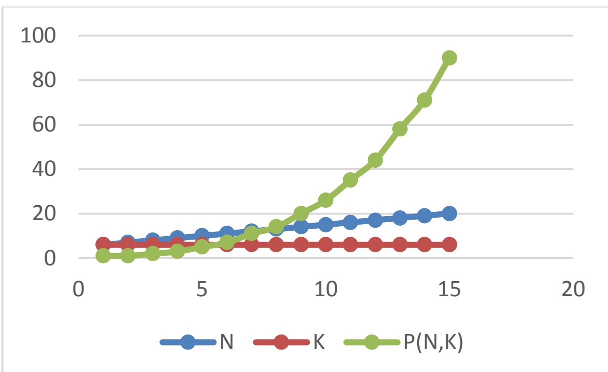

Fig. 3b: p(n,6)

Table 5: p(n,7)

n	k	P(n,k)
7	7	1
8	7	1
9	7	2
10	7	3
11	7	5
12	7	7
13	7	11
14	7	15
15	7	21
16	7	28
17	7	38
18	7	49
19	7	65
20	7	82

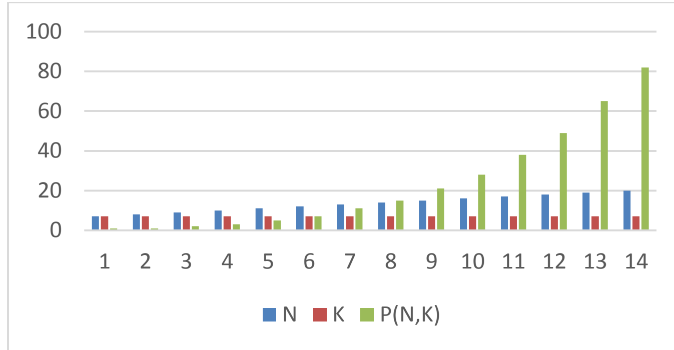

Fig.4a: p(n,7)

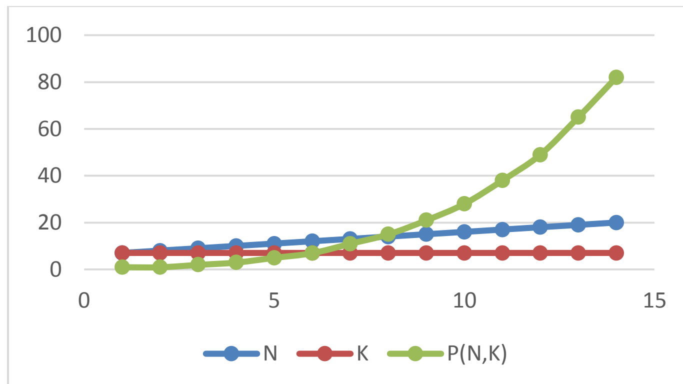

Fig.4b: p(n,7)

Table 6: p(n,8)

n	k	P(n,k)
8	8	1
9	8	1
10	8	2
11	8	3
12	8	5
13	8	7
14	8	11
15	8	15
16	8	22
17	8	29
18	8	40
19	8	52
20	8	70

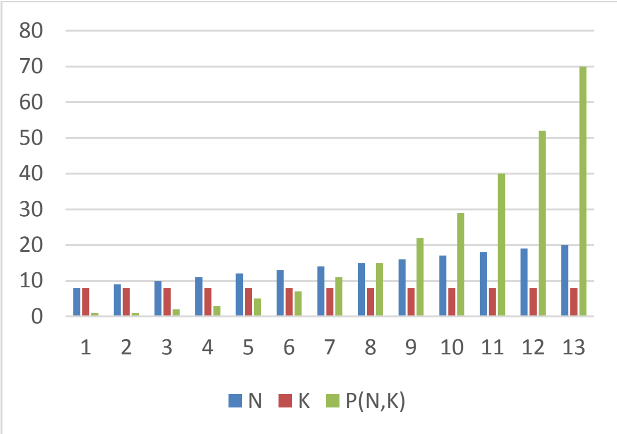

Fig. 5a: p(n,8)

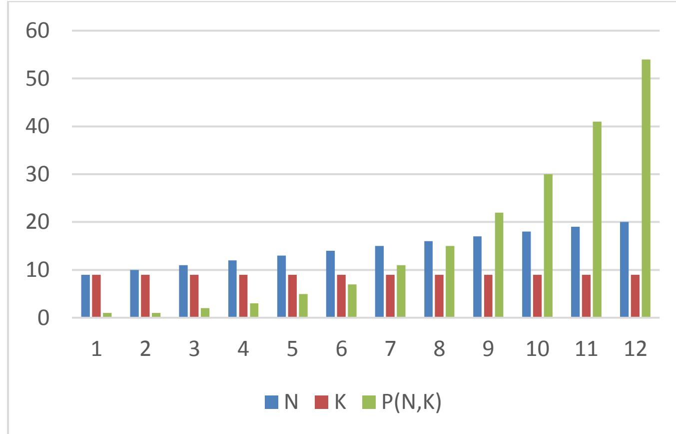

Fig. 6a: p(n,9)

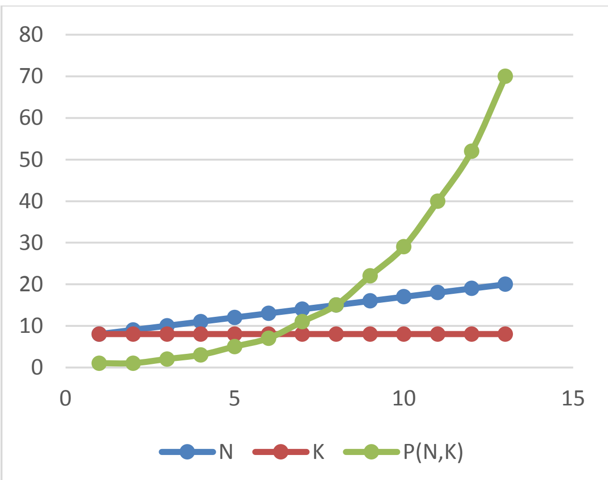

Fig. 5b: p(n,8)

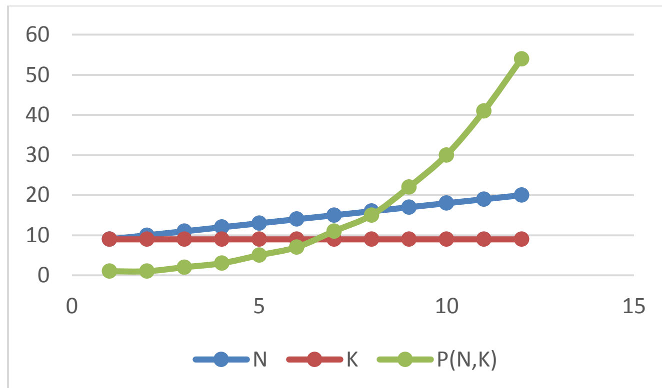

Fig.6b: p(n,9)

Table 7: p(n,9)

n	k	P(n,k)
9	9	1
10	9	1
11	9	2
12	9	3
13	9	5
14	9	7
15	9	11
16	9	15
17	9	22
18	9	30
19	9	41
20	9	54

Table 8: p(n,10)

n	k	P(n,k)
10	10	1
11	10	1
12	10	2
13	10	3
14	10	5
15	10	7
16	10	11
17	10	15
18	10	22
19	10	30
20	10	42

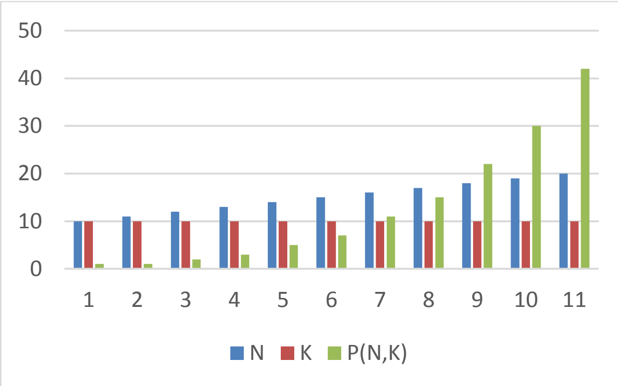

Fig. 7a: p(n,10)

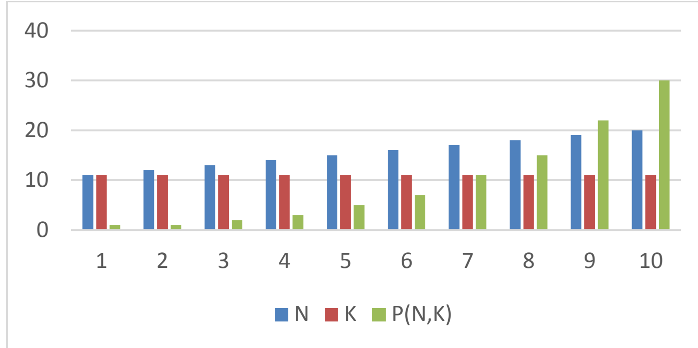

Fig. 8a: p(n,11)

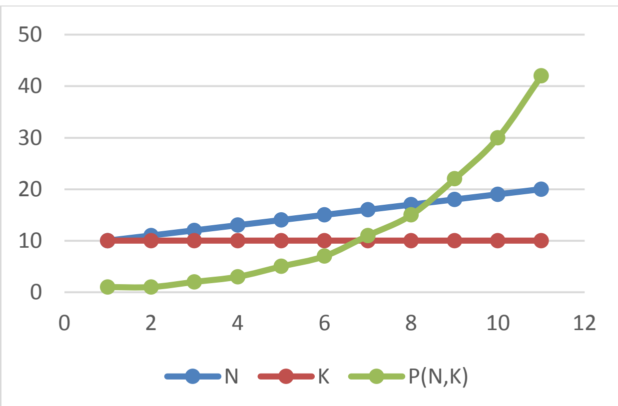

Fig.7b: p(n,10)

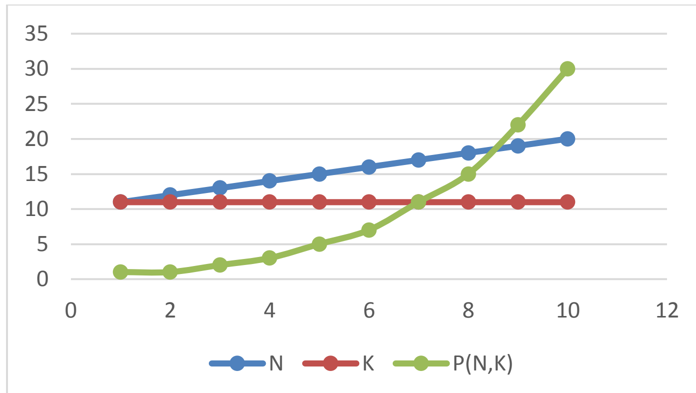

Fig. 8b: p(n,11)

Table 9: p(n,11)

n	k	P(n,k)
11	**11**	1
12	**11**	1
13	**11**	2
14	**11**	3
15	**11**	5
16	**11**	7
17	**11**	11
18	**11**	15
19	**11**	22
20	**11**	30

Table 10: p'(n,12)

n	k	P(n,k)
12	**12**	1
13	**12**	1
14	**12**	2
15	**12**	3
16	**12**	5
17	**12**	7
18	**12**	11
19	**12**	15
20	**12**	22

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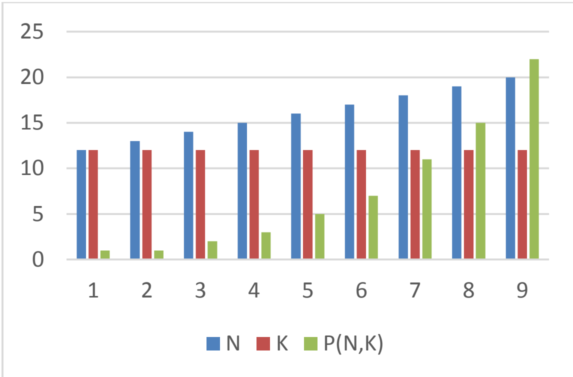

Fig. 9a: p(n,12)

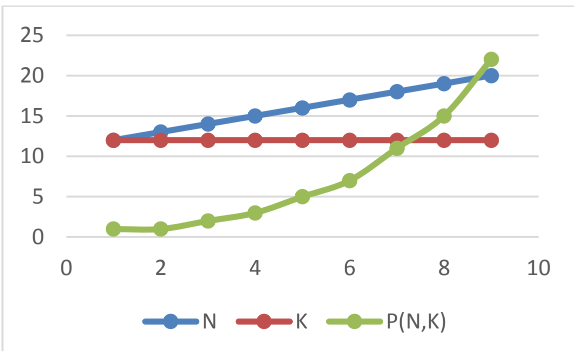

Fig.9b: p(n,12)

Table 11: p(n,13)

n	k	P(n,k)
13	13	1
14	13	1
15	13	2
16	13	3
17	13	5
18	13	7
19	13	11
20	13	15

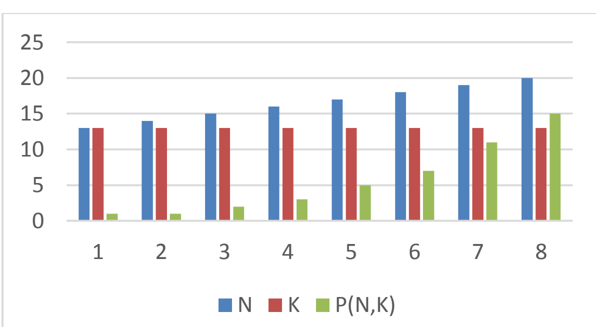

Fig.10a: p(n, 13)

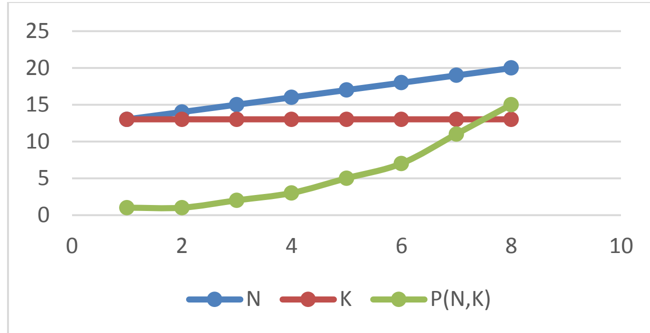

Fig.10b: p(n,13)

Table 12: p(n,14)

n	k	P(n,k)
14	14	1
15	14	1
16	14	2
17	14	3
18	14	5
19	14	7
20	14	11

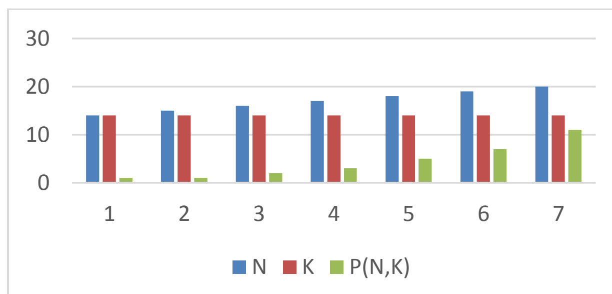

Fig. 11a: p(n,14)

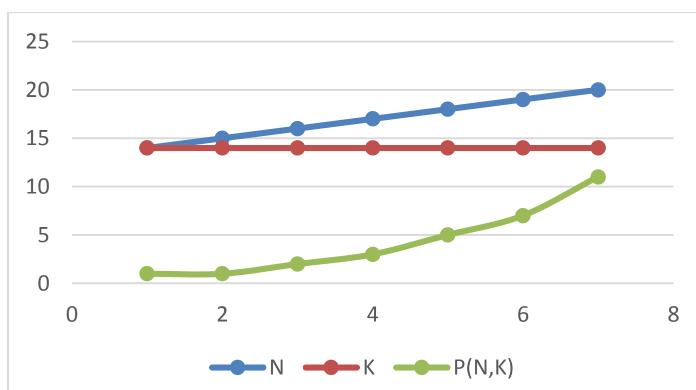

Fig. 11b: p(n,14)

Table 13: p(n,15)

n	k	P(n,k)
15	15	1
16	15	1
17	15	2
18	15	3
19	15	5
20	15	7

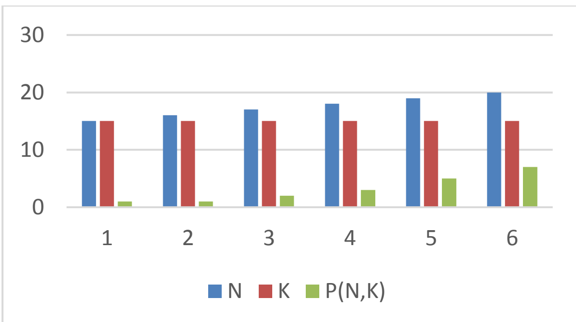

Fig.12a: p(n,15)

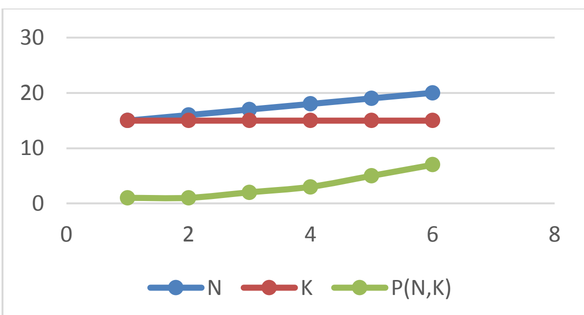

Fig.12b: p(n,15)

Table 14: p(n,16)

n	k	P(n,k)
16	16	1
17	16	1
18	16	2
19	16	3
20	16	5

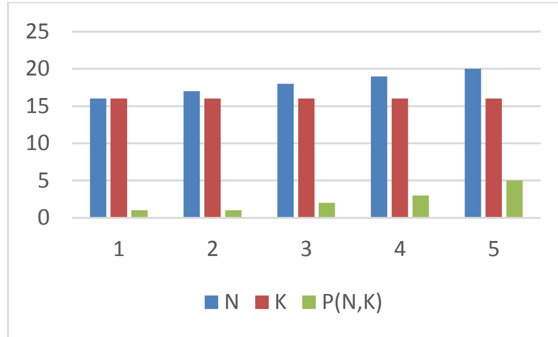

Fig. 13a: p(n,16)

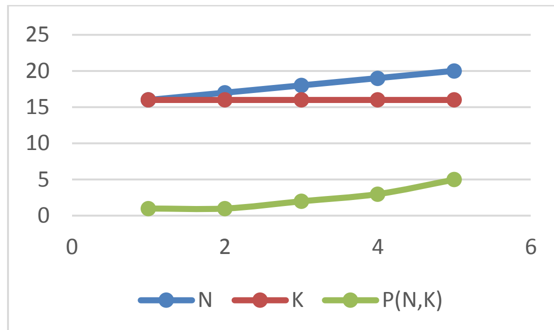

Fig.13b: p(n,16)

Table 15: p(n,17)

n	k	P(n,k)
17	17	1
18	17	1
19	17	2
20	17	3

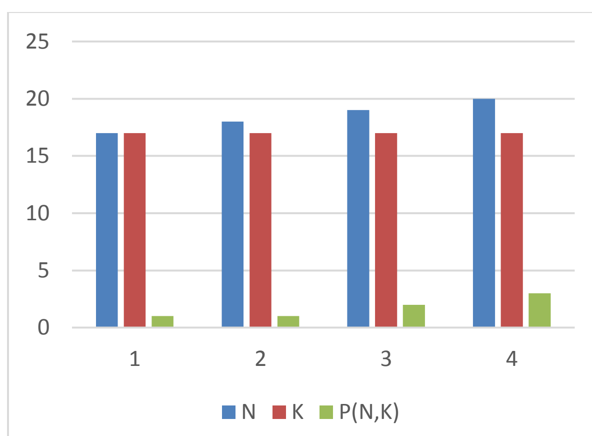

Fig.14a: p(n,17)

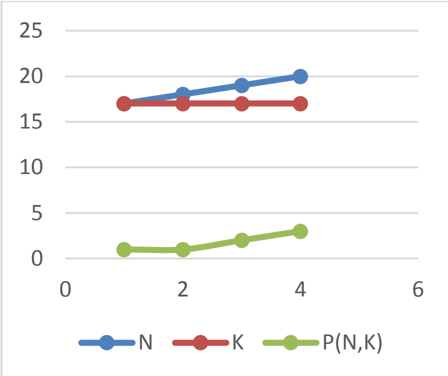

Fig. 14b: p(n,17)

Conclusion

In this paper, we have further extend the formular from $k = 11\ to\ k = 20$ with the aid of Marple software package, the original results were based on $k = 1,2, \ldots ,20.$, while we generalize it to include the set of all integers, $\{ k = 1,2, \ldots \}$, and relate the technical results with thier graphical interpretations through a novel use of Bar charts and Scatter diagrams, showing the trend at a restricted level of partitions.

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APPENDIX

Maple Code for Generating q-Partial Fraction and Iterative Transformation Algorithm

```
with(numtheory, cyclotomic, divisors, phi, invphi)
local a, t, i, h;
type/qfactors := proc(f, q::name)
a := primpart (f, q)*(1 + q^7)^2;
qseries:=table();
qseries[aqprod]:=proc(a,q,n)
: added else bit when n not an integer
localx,i:
if type(n,nonnegint) then
x:=1:
fori from 1 to n do
x := x * (1-a*q^(i-1)):
od:
else
x:=``(a,q)[n];
fi:
   RETURN(x):
end:
```

>Partitions Function:

```
pmn(3,n);
pmn:=proc(m, n) local a, gu, Gu, H, gu1, lu1, M, q, i, eqs, form,
unknowns, result, soln,j,k,l;
option remember;
if m = 1 then RETURN([[1]]) end if;
Gu :=mul(1/(1-q^i), i = 1 .. m);
gu := convert(Gu, parfrac);
H := [];
fori to m do
M := floor((m-i)/i);
gu1 := add(coeff(gu, numtheory:-cyclotomic(i, q),
 -j)/numtheory:-cyclotomic(i, q)^j, j = 1 .. floor(m/i));
lu1 := convert(taylor(gu1, q = 0, i*(1+M)), polynom);
unknowns := {seq(a[j], j = 0 .. -1+i*(1+M))};
eqs := {seq(seq(add((k*i+l)^j*a[j*i+l], j = 0 .. M)
    = coeff(lu1, q, k*i+l), k = 0 .. M), l = 0 .. i-1)};
form := [seq(add(a[j*i+l]*(1-q)^j, j = 0 .. M), l = 0 .. i-1)];
soln := solve(eqs, unknowns);
result := subs(soln, form);
result:=[op(2..nops(result),result),result[1]]:
H := [op(H), result]
od:
 H:
end:
```

>Polynomials of the form $\frac{1}{\left(1 - q^{n}\right)}$

```
with(numtheory, cyclotomic, divisors, ø, invphi)
type/qfactors := proc(f, q::name)
local a, t, i, h;
option
description   "Tests if f is a constant multiple of factors of
polynomials of the form 1-q^n";
if not type(f, polynom(anything, q ) ) then return false end if;
a := primpart (f, q)*(1 + q^7)^2;
if op(1,a) = -1 then a:= -a end if;
t :=0;
fori to nops(a) do
if type (op(i, a), { '^', 'I' } ) then
h :=op(1, op(i,a)^2); if abs(h) = abs(1 – q^degree (h, q))then t := t +
1 end if
else return false
end if
end do;
returnevalb(t = nops(a) – 1)
endproc
type/cyclotomics := proc(F, q::name)
Local subcyc, s, t, i;
```