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A JOURNEY THROUGH SIMPLICIAL COMPLEXES

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ABSTRACT
This paper reviews some aspects of simplicial complexes. The aspects reviewed in this paper are very key in building undoubtedly strong concept as reviewed on simplicial complexes. The aspects reviewed take into a consideration: how simplices are used to construct simplicial complexes. Key areas covered at the preliminary part of this paper are simplices, face of a simplex, more on facet of a simplex, orientation of simplices, boundary of simplices, interior of a simplex etc. Geometric objects such as points, edges, triangles and tetrahedrons were the materials used in constructing simplicial complexes. We also looked at the geometric realization theorem, skeleton of simplicial complexes, maximal elements and free face, theorems under simplicial complexes and creation of simplicial complexes. At the results and discussion part, Betti numbers and Euler characteristic of simplicial complexes were computed and chimed in with real life examples. The paper was finalized by chipping in some applications of simplicial complexes. In all, the paper has opened up more and interesting study into simplicial complexes and algebraic topology at large.

Keywords: Convex combination, Convex hull, Euclidean space, Fundamental group, Topological property.

1. INTRODUCTION

1.1 Preamble

Topology is concerned with the study of shapes and their properties which are free to undergo continuous deformation. When ideas and techniques from abstract algebra are employed to study topological spaces, the study turns to algebraic topology. At the basic level, algebraic topology divides naturally into two broad concepts known as homology and homotopy. The fundamental group records accurate information relating to the basic shapes or holes of topological space. It is the easiest and first homotopy group. The fundamental group $\Pi_1(X)$ is mostly useful when studying low-dimensions. The higher dimensional analogs are groups $\Pi_n(X)$ where $n \geq 2$. However, it is not easy to compute higher-dimensional homotopy groups in general.

Luckily, we have more computable option to homotopy groups: which is homology group of $H_n(X)$ (Hatcher, 2001). Homology is therefore the way of connecting sequence of algebraic objects (such as modules, abelian groups) to other mathematical objects such as the topological spaces (Eilenberg and Cartan, 1956). A type of homology called simplicial homology is defined for simplicial complexes (Hatcher, 2001).

1.2 Preliminary Definition

1.2.1 Simplex

Simplices or simplexes, the plural form of simplex are generalizations of triangles to arbitrary dimensions. Figure 1 shows geometric examples of simplices.

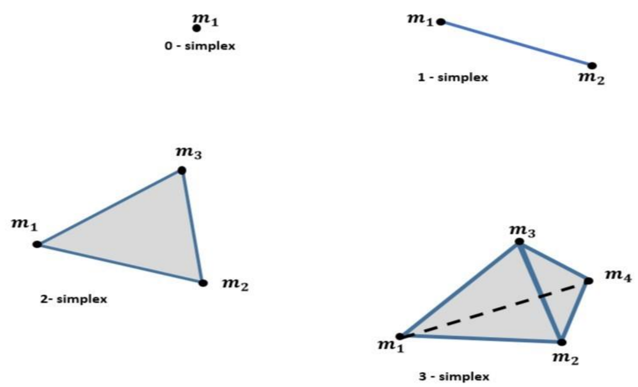

Figure 1: simplices

1.2.2 Face of Simplex

A face of a simplex *S* is a sub-simplex $P \subseteq S$ whose vertices are also vertices of *S*. A face *P* of *S* is said to be proper face if $P \neq S$ (Wildberger, 2012). Every simplex is a face of itself and all other sub-simplices of that simplex are faces of the simplex. For instance, all the faces of a 3−dimensional simplex are shown in figure 2.

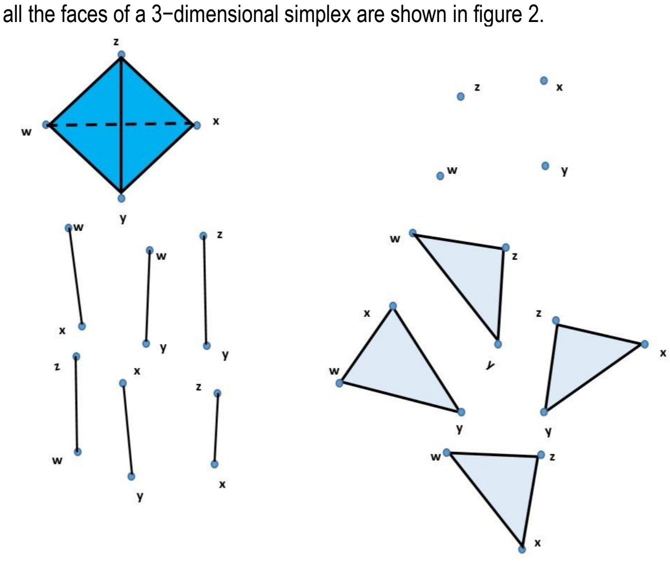

Figure 2: Faces of a 3-simplex

1.2.3 Facet of Simplex

If a simplex(S) has a dimension *k*, thus k-*S*, then its facets are $(k-1)$ simplices or sub-simplices of *S*. For instance, a 2-*S* has 1-simplices as its facets. Suppose a 2-simplex has (c_0, c_1, c_2) vertices, then the edges (c_0, c_1), (c_0, c_2) and (c_2, c_3) are its facets.

1.2.3 Orientation of Simplices
Orientation of simplex refers to giving the simplex a direction. This is done by ordering the vertices of a simplex. The two possible ways one can orient a simplex is through clockwise and counterclockwise directions.

1.2.4 Boundary of a Simplex
The boundary(b) of a simplex(S) denoted as b(S) is the union of all its facets. For instance, the boundary of 2-simplex (c_0, c_1, c_2) is given by:

$$b(S^2) = (c_1, c_2) + (c_2, c_0) + (c_0, c_1)$$

$$= (c_1, c_2) - (c_0, c_2) + (c_0, c_1)$$

$$= \sum_{i=0}^{2}(-1)^i(c_0, \cdots \check{c}_i, \cdots, c_2)$$

Where the symbol check over c_i means omit c_i

1.2.5 Interior of a simplex
The interior of a simplex *S* denoted by $Int(S) = S - b(S)$ of S. Where $b(S)$ represents the boundary of the simplex. The interior of *S* is made up of all the nodes of *S* which are not members of any proper face of *S*.

1.2.7 Ordered n-tuple
Suppose *n* is a positive integer, then an ordered *n*−tuple is a sequence of *n* real numbers ($y_1, y_2, \cdots, y_n$).

1.2.8 *n*−Euclidean space
The set of all ordered *n*−tuples is referred to as *n*−Euclidean space and is represented by R^n, where *n* is a natural number dimension of the space ($y_1, y_2, \cdots, y_n$).

1.2.9 Invariant
An invariant is a topological property or a map which ensures that the same objects are assigned to space of the same topological type (Whitehead, 2002).

1.2.10 Topology and Topological Space
Given that M is a set and Z is a collection of the subsets of M, which is contained in power set of M, then Z is classified as a topology on M provided Z satisfies the conditions below.

1. Each union of the members of Z belongs to Z.
2. Each finite intersection of the members of Z must belong to Z.
3. An empty set and set M itself must belong to Z.

Then the set Z is called a *topology* on M and the pair (M, Z) that is the set together with the topology on M is called *topological space.*

1.2.11 An *n*- simplex
Let $\{m_0, m_1, \cdots, m_k\}$ be a set of vertices in R^n. An *n*−simplex spanned by $m_0, \cdots, m_k$ denotes the set of all points *P* in R^n such that $P = \sum_{i=0}^{n} b_i m_k : b_i \in [0,n]$ for $i = 0,1,\cdots,n$ and $\sum_{i=0}^{n} b_i = 1$ where the non-negative integer *n* denotes the dimension of the simplex. The real numbers $b_0, b_1, \cdots, b_n$ which are uniquely determined by *P* are called *barycentric coordinates* of the point

1.2.12 Geometric Simplicial Complex
Geometric simplicial complex *K* is a body of finite simplices, possibly with varied dimensions in R^n such that these two conditions are guaranteed.

1. If $g_1 \in K$ is a simplex of *K*, then all faces of g_1 are equally faces of *K* as well.
   For instance, from figure 3, the triangle (c_0, c_2, c_3), edges (c_0, c_2), (c_0, c_3), (c_2, c_3) and the points (c_1), (c_2) which are faces of g_1, are also faces of *K*.
2. If $g_1, g_2 \in K$ are simplices in *K*, then either $g_1 \cap g_2$ is empty or $g_1 \cap g_1$ is a face shared by both g_1 and g_2. Also from the figure 3, the edge (c_0, c_3) is a face of both g_1 and g_2. And both edges (c_0, c_1) and (c_2, c_3) belong to *K*, however their intersection is empty.

1.2.13 Abstract Simplicial Complex
Let *G* be a finite set of elements, an abstract simplicial complex *K* with vertex set *G* is a set of subset of *G* such that:

1. For all $g \in G$, $g \in K$
2. If $f \in K$ and $d \subseteq f$, then $d \in K$

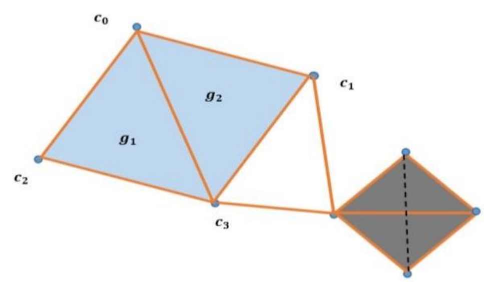

Figure 3: simplicial complex

1.2.14 Sub-complex
Suppose *K* is a complex in R^n, *J* is called a sub-complex of *K* if *g* is a simplex of *J and* then faces belonging to *g* also belong to *K*.

2. MATERIALS AND METHODS
2.1 Simplicial Complexes
Basically, simplicial complex (*K*) is a family of simplices which is seen as one group such that individual simplices are either separated, or gummed together by sharing a common face. The dimension of a simplicial complex is determined by the highest dimension of any of its simplices. A subset of a simplicial complex which is equally a complex is termed as sub-complex.
Simplicial complexes can be viewed or defined from two different perspectives: the geometric and abstract viewpoints. The geometric simplicial complex is related to the Euclidean space; it focuses on embedding of *K* in a Euclidean space. An example of geometric simplicial complex and two non-examples are shown in Figure 4 below.

Whereas in abstract simplicial complex, each simplex is represented by the set of its vertex set. An abstract simplicial complex is much concerned with how individual simplices are gummed together in order to concentrate on the combinatorial structure.

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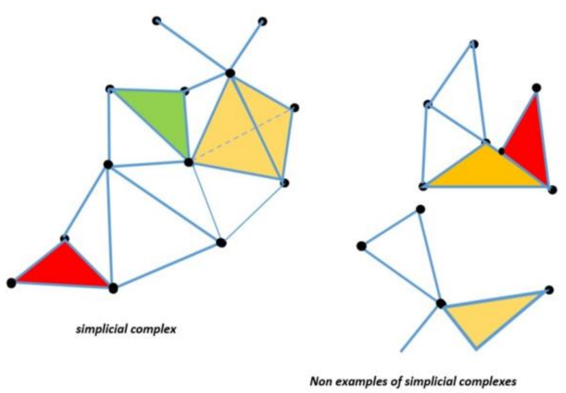

Figure 4: simplicial complex and non-simplicial complexes

Simplicial complex K is said to be pure provided each simplex in K is a face of n-simplex. The highest dimension of K determines the dimension of the K. For instance, in Figure 4 the dimension of K is 3 since the 3-simplex (tetrahedron) is the simplex with the highest dimension.

An abstract simplicial complex J can be obtained from a given geometric simplicial complex K. This can be done by ignoring all the simplices and retaining their geometric realization. J is called a vertex scheme of K and K is a geometric realization of J.

2.2 Geometric Realization Theorem

An abstract simplicial complex of dimension k has a geometric realization in R^{2k+1}.

Proof (Edelsbrunner and Harer, 2010)

We begin by embedding the set of vertices vert$K$ in R^{2k+1} as a set of $2k$+2 or fewer number of points are independent. Now, if $m_0, \cdots, m_n$ span an n-simplex in K, we construct the corresponding geometric simplex. The collection of them is K. Now to really show that K is a simplicial complex, we need to show for $g_1, g_2 \in K$, $g_1 \cap g_2$ is either an empty face for both. To show that, pick any 2 simplices g_1 and g_2. Note that the union of either vertices has cardinality Card $(g_1 \cup g_2)$ = Card (g_1) + Card (g_2) − Card $(g_1 \cap g_2)$ $\leq 2k + 2$.

Hence vertices V^0 in $g_1 \cup g_2$ are affinely independent. Hence $g_1 \cap g_2$ is either empty or lying in the convex combination of some subset. This is because since all vertices in V^0 are linearly independent, and any point p in convex hull (V^0) has a unique linear combination of vertices from V^0. If $x \in$ convex hull (V^0) and $p \in$ convex hull (vert (g_2)), then x must be a linear combination of only vertices in V^1= vert (g_1) ∩ vert (g_2), thus lying in the convex hull of V^1. Hence, they intersect along a face of g_1 and g_2.

2.3 Skeleton of Simplicial Complex

The τ-skeleton of a simplicial complex K which is represented by $K^{(\tau)}$ is the collection of the simplices in K of dimension τ or less. For instance, 0−skeleton of K refers to a set of 0-dimensional simplices, 1−skeleton of K consists of the set of 0-dimensional simplices and 1- dimensional simplices. 2−skeleton of K is a set of 0-dimensional simplices, 1- dimensional simplices as well as 2-dimensional simplices in that order, as shown in figure 5.

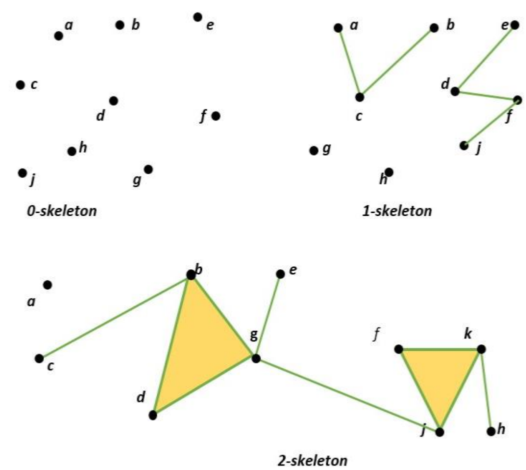

Figure 5: Skeleton of simplicial complex

It can be deduced that

1. For all τ, $K^{\tau} \subset K^{\tau+1}$
2. If $\tau = dim(K)$, $K^{\tau} = K$

2.4 Maximal Element and Free Face

If K is a simplicial complex, then any face (f) of K is known as maximal element, if f is a face of itself but not a face of any other simplex of K. Example, in Figure 6(1), the simplicial complex has six maximal elements, namely: the four edges (a, b), (a, c), (b, c), and (a, d), the triangle (a, e, f) and the tetrahedron (a, g, l, h).

Suppose m is a maximal element, and n is a face of m, then m is called a free face, if $n \neq m$ and n does not belong to any of the simplices of K.

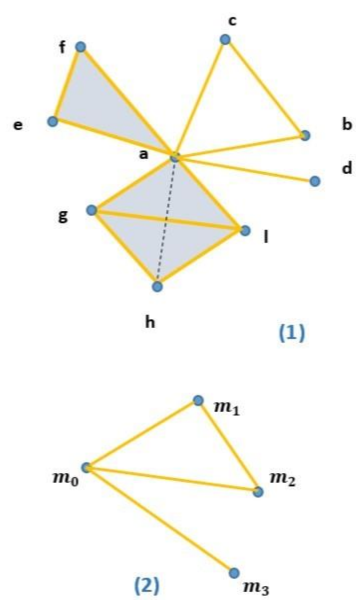

Figure 6: Skeleton of simplicial complex

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2.5 Theorem
Suppose *S* is a finite body of simplices belonging to certain R^n space, and |*S*| is a union of the entire simplices in *S*. Then *S* is called a simplicial complex (having the polyhedron |*S*|) provided the following conditions are assured.

1. *S* contains all the faces of its simplices.
2. Each point of |*S*| is an element of the interior of a unique simplex (Intx) of *S*.

Proof (Wilkins, 1988)
Suppose that *S* is a simplicial complex, then *S* contains the face of its simplices. To prove that each point of |*S*| is a member of the Intx of *S*. If $p \in |S|$, then *P* is a member of the interior of a face g_1 of certain simplex of *S* (due to the fact that each point of a simplex is an element of the interior of certain faces). But $g \in S$, because *S* contains the faces of all its simplices. Hence, *p* is a member of the interior of not less than one simplex of *S*.
If *P* is an element of the interior of any two different simplices g_1 as well as g_2 of *S*. It implies that *p* would belong to a certain common face $g_1 \cap g_2$ of g_1 together with g_2 (because *S* is a simplicial complex). But either g_1 or g_2 would be a proper face of the common face (because $g_1 \neq g_2$), denying the fact that *p* is an element of the interior of g_1 as well as g_2. Then we can say that g_1 of *S* which contains *p* in its interior is uniquely determined as demanded. On the contrary, we need to show that each collection of simplices that satisfies the given conditions is simplicial complex. In that, *S* carries the faces of the simplices in it, then we only have to ensure that if g_1 together with g_2 are any two simplices *S* whose intersection is not empty, then it implies that $g_1 \cap g_2$ is a face which is shared by g_1 as well as g_2.
If $p \in g_1 \cap g_2$, then *p* is an element of the Intx g_3 of *S*. Meanwhile, each point of g_1 or g_2 is an element of the Intx of that simplex, and every face of g_1 as well as g_2 is an element of *S*. It implies that g_3 is a face shared by g_1 together with g_2, hence, vertices belonging to g_3 also belong to g_1 as well as g_2.
Finally, it can be deduced that g_1 and g_2 share common vertices, therefore any point $g_1 \cap g_2$ belongs to the common face x^0 of g_1 and g_2 spanned by these common vertices. But this implies that $g_1 \cap g_2 = x$, as a result $g_1 \cap g_2$ is a face shared by g_1 as well as g_2 as demanded.

2.6 Creating Simplicial Complexes
Simplicial complexes can be created with zero-simplices(vertices), one-simplices (line segments), two-simplices(triangles), three-simplices as well as higher-dimensional simplices (Isabel, 2013).

2.6.1 Steps to create a simplicial complex:

1. Create 0-dimensional vertices.

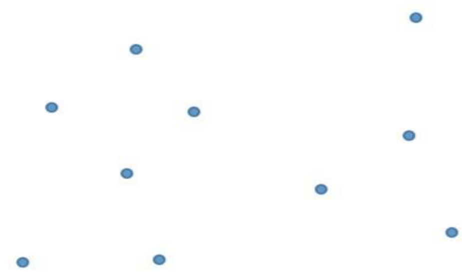

Figure 7: 0-dimensional simplicial complex
2. Connect 2 vertices to create 1-dimensional edges.

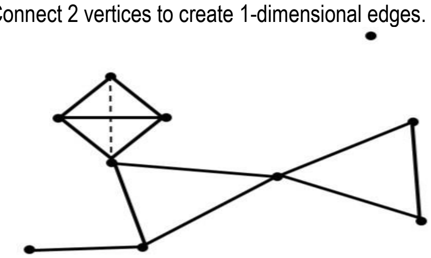

Figure 8: 1-dimensional simplicial complex
3. Create 2-dimensional triangles by filling in the boundaries of the triangles with 2dimensional triangular faces.

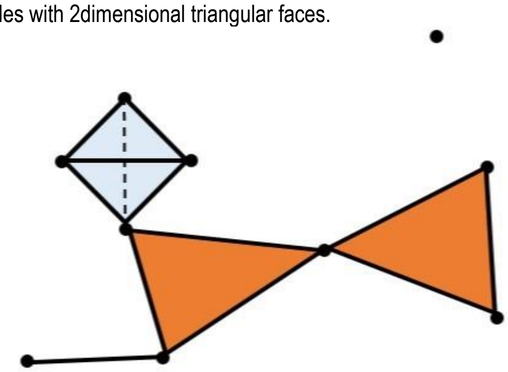

Figure 9: 2-dimensional simplicial complex
2. Create 3-dimensional tetrahedron by filling in the boundary of the tetrahedron.

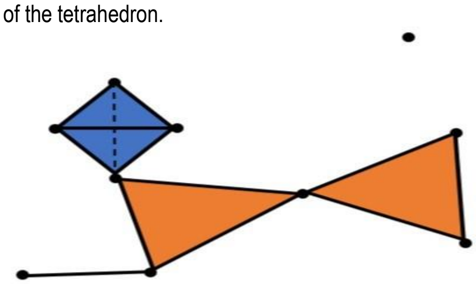

Figure 10: 3-dimensional simplicial complex

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3. RESULTS AND DISCUSSION

3.1 Euler Characteristic and Betti numbers

The Euler characteristic and Betti numbers are key topological invariants use to describe simplicial complexes.
In simple form, the Euler characteristic (X) is given by
$X = |V| - |E| + |F|$
where $|V|$ is the number of vertices, $|E|$ is the number of edges and $|F|$ is the number of faces.

3.1.1 Euler Characteristic of Simplicial Complex

Suppose K is a simplicial complex, K_j is the set of j-dimensional simplices in K and $|K_j|$ represents the number of elements in K. Then the Euler characteristic (X) for a simplicial complex (K) is given by
$X(K) = |K_0| - |K_1| + |K_2| - |K_3| + \cdots$
$X(K) = \sum_{i=0}^{dimk}(-1)^j|K_j|$

Example 1. The Euler characteristic of the simplicial complex in Figure 11 can be computed as:

$X(K) = |K_0| - |K_1| + |K_2|$
$X(K) = 4 - 5 + 1$
$\therefore \quad X(K) = 0$

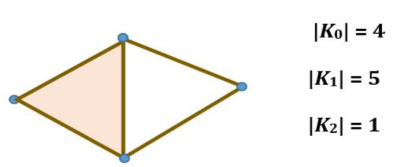

Figure: 11

Example 2. The Euler characteristic of the simplicial complex in Figure 12 can also be computed as:
$X(K) = |K_0| - |K_1| + |K_2| - |K_3|$
$X(K) = 7 - 10 + 5 - 1$
$\therefore \quad X(K) = 1$

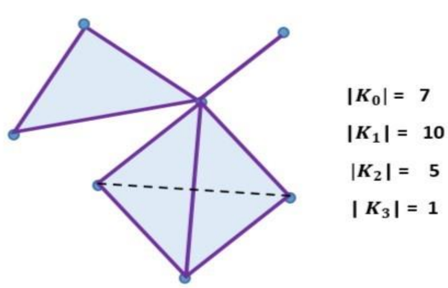

Figure: 12

3.1.2 Betti Numbers

Betti numbers refer to a sequence of numbers which indicate how many holes of each dimension an object has (Houston, 2017). The *nth* Betti number (B_n) calculates the number of n-dimensional holes in a simplicial complex(K). Figure 13a illustrates some elementary examples of Betti numbers of simplices.

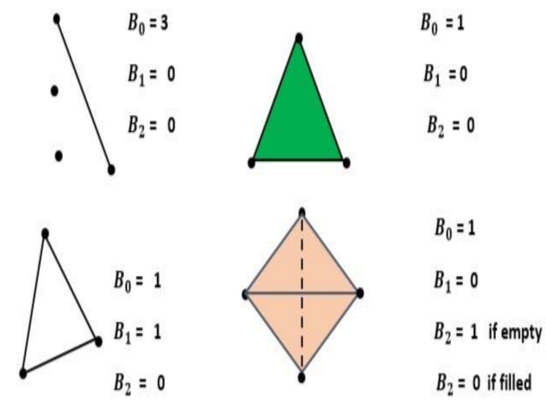

Figure:13a Betti numbers of simplices

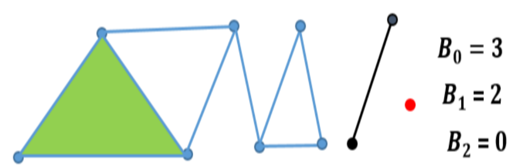

Figure: 13b

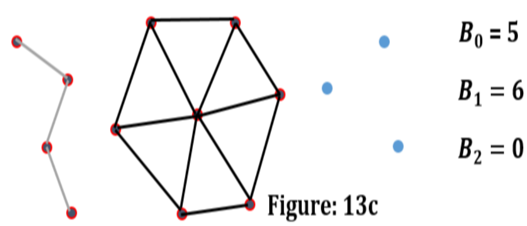

Figure: 13c

Example1. In Figure 13b, B_0=3 since there are three connected components: the red vertex is one component, the black edge is another connected component and the rest of the figure is also another connected component. B_1= 2 since there are two 1-dimensional holes in the figure: the space inside the two hollow triangles. Also B_2= 0 since there is no 2-dimensional hole in the simplicial complex.

Example2. The simplicial complex in Figure 13c has five different connected components: three different blue vertices, one grey connected edges and the remaining is also one connected component, hence $B_0 = 5$. $B_1 = 6$ since there are six 1-dimensional holes in the connected triangles. $B_2 = 0$ since there is no 2-dimensional holes in the simplicial complex.

4. APPLICATION OF SIMPLICIAL COMPLEXES

In Algebraic topology, simplicial complexes are used for the definition of homology group of simplicial complexes. Simplicial homology is defined for simplicial complex (Hatcher, 2001). It is very useful for concrete calculations. Networks can be realized as simplicial complexes (Estrada and Ross, 2018).

Simplicial complexes are also applied in real life situations. For example, there has been an introduction of a higher-order model of social contagion in which a social system is represented by a simplicial complex where contagion occurrences are viewed beyond the framework of pairwise interaction and take into account that contagion can also occur through group interaction (Lacopini *et al,* 2019)

Simplicial complexes have been used to model Network Geometry. Based on simplicial complexes, a model for dynamic epistemic logic has been introduced to study distributed task computability.

Moreover, it has also been used to represent social aggregation in human communication which allows several agents into a group rather than in pair. The brain can be viewed as higher dimensional simplicial complex as indicated below in Figures 14 and 15. In neuroscience, simplicial complexes are employed to study how the firing neurons are connected in the brain.

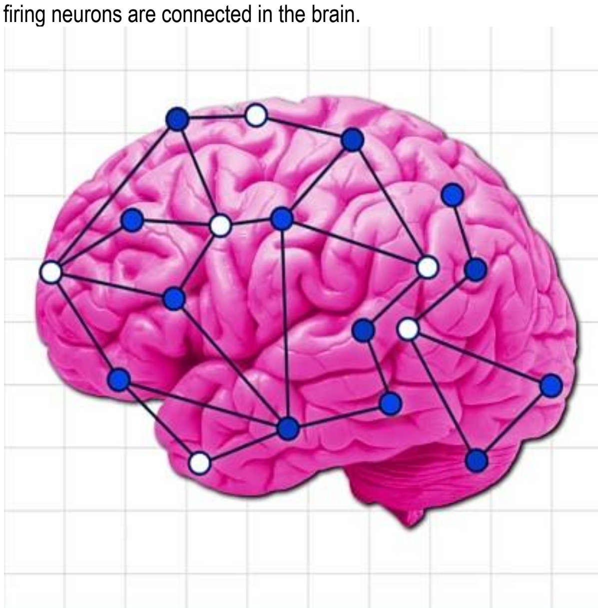

Figure 14: The brain as simplicial complex 1

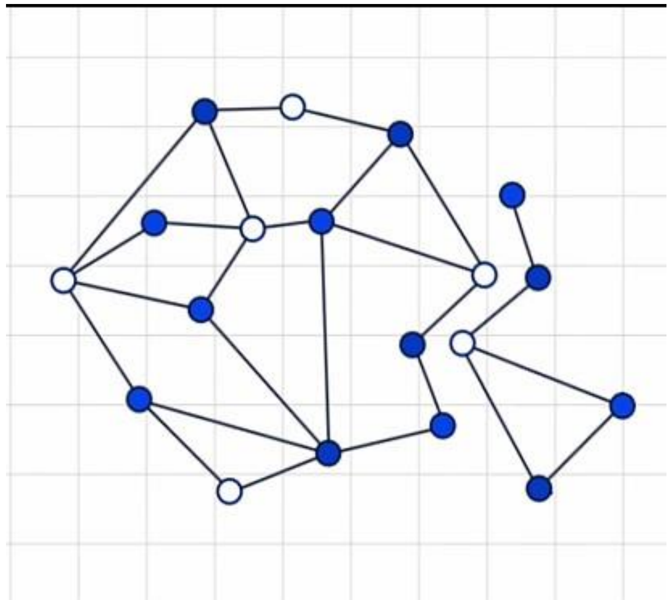

Figure 15: The brain as simplicial complex 2

5. CONCLUSION

This article was based on review of simplicial complexes. It also came out with real life examples. The Betti numbers and the Euler characteristic of certain simplices and simplicial complexes have been computed. Moreover, some applications of simplicial complexes were also highlighted.

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