Science World Journal Vol. 16(No 4) 2021
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

ON THE STUDY OF CHAOTIC BEHAVIOR USING THE LOGISTIC FUNCTION

Patrick Anamuah Mensah[1], William Obeng-Denteh[2], Kwasi Baah Gyamfi[3], Mary-Ann Msuur Shior[4] Benedict Celestine Agbata[5]

[1,2,3] Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana
[4]Department of Mathematics/Computer Sciences, Benue State University, Makurdi, Nigeria.
[5]Department of Mathematics, University of Nigeria Nsukka, Nigeria

ABSTRACT
The study uses the idea of logistic function to illustrate the presence of chaos as the cause of the absence of periodic formation. The logistic function is used in demonstrating, proving, and explaining **Definition 6** and **Theorem 1** through examples, tables, and figures. A system in recurrent behavior is describable when it is stable. However, chaotic behavior is seen when the system moves beyond periodic making it difficult to predict or describe the nature of the system. The WolframAlpha computational knowledge engine was used in obtaining the tables and the figures for the study. The study shows that when the parameter of the logistic function is at exactly 4 there is an uncorrelated behavior of the system indicating a new regime called chaos. Finally, the study shows that after successive iterations of the system there is no recurrent formation which is due to the system showing un-periodic, unstable, and uncorrelated.

Keywords: logistic function, chaos, periodic, recurrent, dynamical system, formation, stable.

INTRODUCTION
As discussed by Nguyen et al, (1989) the dynamical system at any given time has a state that is a set of real numbers, a vector where a point is used to represent this state space. Changes in the numbers lead to a change in the state of the system. In dynamical system, the rule that describes the future from the current state is a fixed rule. This fixed rule is also called a cascade (Brown, 2007). This system is a mathematical formalization for any fixed rule which depends on time to describe the dependence of the position of a point in some state space. The fixed rule is deterministic in nature that is for a current state given a time interval, only one future state follows. Therefore the study of how systems change or evolve through time is termed as the dynamical system. The study of differential or difference equations explains dynamical systems. This process of time – evolution may be either non – linear or linear equation which gives us the notion dynamical system.

To Boeing (2016), the study of chaos theory/system is a type of dynamical system which is nonlinear. Chaos as a branch of mathematics in nonlinear dynamical system is that when the initial conditions are or is sensitive dependence. In the chaotic system, the deterministic system produces unpredictable behavior, fractal and divergent over some time as a result of sensitivity
Studies done by great scientists in all the areas of mathematics have shown that when the behavior of a system becomes unpredictable and uncorrelated the system is said to be in chaos. In most situations, the future behavior of the system is chaotic when there is a sensitive condition through the initial condition.
The state of recurrence as a system is always predictable and regular whether it's being stationary or periodic evolutions. The ends of these evolutions bring in a different evolution that is not regular and predictable, hence chaos (Henk & Floris, 2009).

PRELIMINARY DEFINITIONS

Definition 1: let (X,T)be a topological space, then a function $T: X \longrightarrow X$ is said to be chaos if;
a. The set of periodic points of T is dense in X
b. $\forall\ U, V$ open in X, $\exists\ x \in V$ and $n \in Z^+$ such that $T^n(x) \in V$

Definition 2: Let T be a rule/map and x_0 be an initial condition. The forward limit set of the orbit $\{T^n(x_0)\}$ is the set; $\omega(x_0) = \{x{:}\ \forall N\ and\ \varepsilon, \exists n > N\ such\ that\ |T^n(x_0) - x| < \varepsilon\}$**(Alligood et al, 1996)**

NOTE: It is also called the $\omega - limit$ set of the orbit. If $\omega(x_1) \in \omega(x_0)$, where x_1 is also another initial condition and $\omega(x_0)$ is a forward limit set of some orbit, then the orbit $\{T^n(x_1)\}$ is attracted to $\omega(x_0)$. The fixed points are limit sets, since $x_0 = \omega(x_0)$, likewise, the same rule holds for periodic orbits (Block & Franke, 1984)

Definition 3: Let $\{T^n(x_0)\}$ be a chaotic orbit, the $\omega(x_0)$ is called a chaotic set, if $x_0 \in \omega(x_0)$. The basin of attraction of the attractor is the set of the initial condition. A chaotic set that is also an attractor is called a chaotic attractor.

Definition 4: Sensitive Dependence on Initial Condition
The idea behind sensitivity to initial condition is that the initial condition of any system with a small uncertainty grows exponentially with time which turns out to be large enough and changes the core knowledge of the condition/state of the system (layek, 2015). This makes it difficult to predict the future behavior of the system.
Let $T: R \longrightarrow R$, that is T be a map on R itself. A point x_0 has sensitive dependence on the initial condition, if $\exists d > 0$ such that $\varepsilon > 0$ and x satisfying $|x - x_0| < d\ \ \forall\, n \geq 0$ such that $|T^n(x) - T^n(x_0)| \geq \varepsilon$. The point x_0 is called a sensitive point.

NOTE: when two orbits are not close to each other that is they move away eventually from the distance d for some large n, then x is sensitive.

Definition 5: 'Chaos' in Devaney sense
Devaney was the first to come out with a definition of deterministic chaos in his textbook "an introduction to chaotic dynamical system". His definition was in two strands in dynamical idea in terms of chaos. (Devaney, 2003)

Science World Journal Vol. 16(No 4) 2021
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

1. Regularity of element as provided in dynamics is a dense set of periodic orbits which simple
2. Irregularity of elements of topological transitivity are complicated dynamics

A function F is said to be Devaney Chaotic on T defined as; a map $F: \mathrm{T} \rightarrow \mathrm{T}, \mathrm{T} \subseteq R$ if;

i. F is topologically transitive
j. The periodic points are dense in T

RESULTS AND DISCUSSION

Major Definition and Theorem

Definition 6: If a compact topological dynamical system is of the form $(\boldsymbol{X}, \boldsymbol{T})$. Then $\boldsymbol{x} \in \boldsymbol{X}$ is a **periodic recurrent point** but transitioned to **chaos** if $\boldsymbol{T}$ is continuous such that;

$$\bigcup := \{\boldsymbol{n} \in \boldsymbol{R}: \boldsymbol{T}^{\boldsymbol{n}}(\boldsymbol{U}) \neq \emptyset \ and \ \boldsymbol{T}^{\boldsymbol{n}}(\boldsymbol{x}) \neq \boldsymbol{x}\}$$

Where $\boldsymbol{U}$ is the neighborhood of $\boldsymbol{x} \in \boldsymbol{X}, \boldsymbol{n} = \mathbf{1}, \mathbf{2}, \ldots$

Theorem 1: A chaotic system is **aperiodic** if a period-n recurrent point is absent. That is, if $(\boldsymbol{X}, \boldsymbol{T})$ is a topological dynamical system, $\boldsymbol{T}$ is continuous and compact then if there is **no attracting fixed point** and **periodic behavior** (which also means non-existence of recurrent), the system is aperiodic.

Logistic Map or Function: $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{a}\boldsymbol{x}(\mathbf{1} - \boldsymbol{x})$
Where $\boldsymbol{a}$ is the parameter which lies within 0 and 4 inclusive, i.e $0 \leq a \leq 4 \ or \ a \in [0,4] \ and \ x \in [0,1]$.

Sensitive Condition: Let $T: R \longrightarrow R$, that is T be a map on R itself. A point x_0 has sensitive dependence on the initial condition, if $\exists d > 0$ such that $\varepsilon > 0$ and x satisfying $|x - x_0| < d \ \forall \ n \geq 0$ such that $|T^n(x) - T^n(x_0)| \geq \varepsilon$. The point x_0 is called a sensitive point.

Note: The logistic function is sensitive to initial condition, hence the outcome of the function depends on the sensitive point. This indicates that the initial condition has a significant effect on the nature of the function.

FINDINGS AND DISCUSSIONS

Under this section we demonstrate the chaotic formation using the logistic function and the idea of sensitive condition as the absence of periodic orbits to explain and also prove **Definition 6** and **Theorem 1** respectively.

Now knowing that the logistic function is sensitive to initial condition (Klages, 2008).
We assume that when the parameter $\boldsymbol{a}$of the logistic function is $\mathbf{4}$, there is chaotic behavior.

Example 1: Given $f(X) = \alpha(X - X^2) \ and \ a = 4 \ implies \ f(x) = 4(x_n - x^2{}_n)$, at x_0 = 0.5

Table 1. Iteration of $(x) = 4(x_n - x^2{}_n)$, at $x_0 = 0.50000$,

n	0	1	2	3	4	5	6	7	8	9
x_n	0.50000	0.10000	0	0	0	0	0	0	0	0

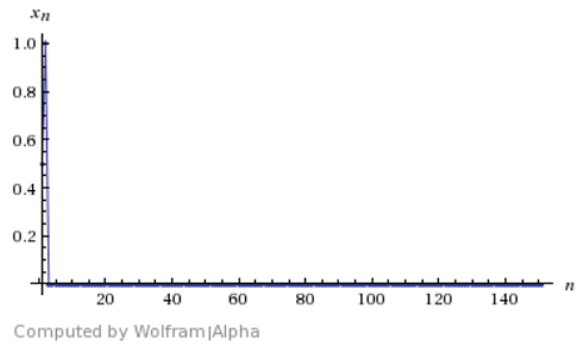

Figure 1: *Graph of* $f(x) = 4(x_n - x^2{}_n)$, at$x_0 = 0.50000$

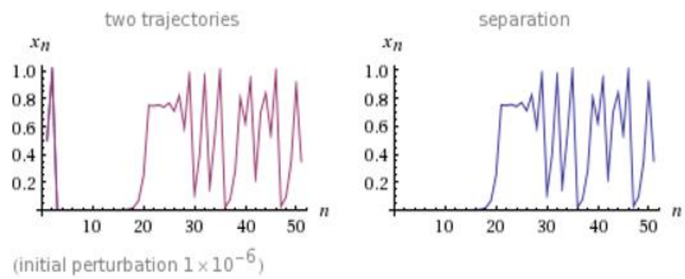

Figure 2: Graphical display of the logistic function: $f(x) = 4(x_n - x^2{}_n)$ $(\alpha = 4, x_0 = 0.50000)$ showing unstableness (chaos)

Table 2: The Unstableness of the Logistic Function at $\alpha = 4$ and $x_0 = 0.50000$

period	iterates	linear stability
1	0.	unstable
1	0.75	unstable
2	0.345492, 0.904508	unstable
3	0.116978, 0.413176, 0.969846	unstable
3	0.188255, 0.61126, 0.950484	unstable
4	0.0337639, 0.130496, 0.453866, 0.991487	unstable
4	0.0432273, 0.165435, 0.552264, 0.989074	unstable
4	0.277131, 0.801317, 0.636831, 0.925109	unstable

The unstableness of the system when $\alpha = 4$ and $x_0 = 0.50000$

The nature of **Table 1**, **Figure 1** and, **Figure 2** make it very difficult to describe them. This is as a result of the system becoming unstable as it keeps moving toward (approaches) infinity. It confirms the idea and the meaning of **Definition 6** that irrespective of the number of time one iterate the function its orbits are so uncorrelated having different trajectories. Hence, $\cup := \{\boldsymbol{n} \in \boldsymbol{R}: \boldsymbol{T}^{\boldsymbol{n}}(\boldsymbol{U}) \neq \emptyset \ and \ \boldsymbol{T}^{\boldsymbol{n}}(\boldsymbol{x}) \neq \boldsymbol{x}\}$ is proved.

Figure 1, **Figure 2,** and **Table 2** give the empirical explanation of **Theorem 1** that as the system shows no existence of fixed point and non-existence of periodic cycle (behavior) hence indicating the end of the formation of recurrence thereby showing aperiodic as its effect. Even though the tables and the graphs above give a clear picture of the uniformity equation to be flat and zero, but determining its nature will be difficult, **Figure 1** show that the behavior of the system is chaotic that is uncorrelated and unpredictable showing unstableness and aperiodic.

Clearly, **Table 2** shows that after successive iterations of the system there is no periodic behavior hence no recurrent formation which is due to the system showing un-periodic, unstable, and uncorrelated.

Convinsely, when a system is in a periodic state it prevents the formation of chaos, hence the absence of recurrent brings about chaos.

Conclusion

The behavior of a system in periodic – like recurrence changes or move to a different behavioral state called chaos when it is unpredictable. This behavior of the system as illustrated by the tables and figures goes to affirm the existence of chaos which is a result of the unstable nature or conditions of $\boldsymbol{4}(\boldsymbol{x_n} - \boldsymbol{x^2}_{\boldsymbol{n}})$ in both **Figure 2 and Table 2.** Since the logistic function is deterministic (in nature) as it evolves over and over, at a parameter 4, the outcome of the function appears to have no pattern making it difficult or impossible to make any valid prediction about the future events. This effect of the logistic function brings to an end the behavior of the recurrence formation through the periodic- cycle to a new state (chaotic state) which is more unrealistic in describing it or making predictions of its outcome.

Finally, at parameter **4**, the system transitioned totally without showing any forms or types of the periodic-cycle nature as a periodic–like recurrence but rather demonstrating aperiodic. Therefore, the sensitivity of the function at$\boldsymbol{a} = 4$ of the initial condition shows chaotic behavior.

REFERENCES

Alligood, K. T., Sauer, T. D.,& Yorke, J. A. (1996). Chaos: An introduction to Dynamical System. Springer – Verlarg. New York

Block, L & Franke, J. E. (1984). The Chain recurrent Set, attractors and explosions. Ergod. Th. & Dyuam. Sys. (1984), 5,321-327. Printed in Great Britain

Boeing, G. (2016). Visual Analysis of Nonlinear Dynamical Systems: Chaos, Fractals, Self- Similarity and the Limits of prediction. Dept. of City and Regional Planning. Univ. of California, Berkely, CA9472, USA. www.mdpi.com/journal/system

Brown, R. J. (2007). An Introduction to Dynamical System. John – Hopkins University (2007)

Devaney, K. L. (2003). Introduction to Chaotic Dynamical System. Boulder, Co: Westview Press.

Henk, B & Floris, T. (2009). Dynamical System and Chaos, Groningen, spring 2009

Klages, R. (2008), Introduction to Dynamical Systems. QMUL, Lecture Notes for MAS424/MTHMO2/, Version 1.2, 18/04/2008

Layek, G. C. (2015). An introduction to Dynamical System and Chaos. Springer New Delhi (India) Heidelberg. ISBN: 978 – 81 – 322 – 2555 – 3. www.springer.com

Nguyen, A. H., Fraser, S. J. (1989). Geometrical picture of reaction in enzyme kinetics. Journal of Medical Physics, (February), 186-193. Retrieved from http://jcp.aip.org/jcpsa6/v91/i1/p186s1