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Full Length Research Article

STRUCTURE TO POLYNOMIAL FUNCTORS IN ORTHOGONAL CALCULUS II

Louis Osei[1], William Obeng-Denteh[1], Isaac Owusu-Mensah[2], David Delali Zigli[3]

[1]Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana
[2]Faculty of Science Education, Akenten Appiah-Menkah University of Skilled Training and Entrepreneurial Development, Mampong-Ashanti, Ghana
[3]Department of Mathematical Sciences, University of Mines and Technology, Tarkwa, Ghana

Authors Email Addresses: oseilouis59@ymail.com, wobengdenteh@gmail.com, isaacowusumensh@gmail.com, davidzigli@gmail.com

ABSTRACT
The orthogonal calculus of functors is a beautiful tool for calculating the homotopical properties of functors from the category of inner product spaces to pointed spaces or any space enriched over Top_*. It splits a functor *F* into a Taylor tower of fibrations, where our *n-th* fibrations will consist of maps from the *n*-polynomial approximation of *F* to the (*n* − 1)− polynomial approximation of *F*. The homotopy fiber or layer (the difference between *n*-polynomial and (*n* − 1)− polynomial approximation) of this map is then an *n*-homogeneous functor and is classified by an *O* (*n*)- spectrum up to homotopy which is usually denoted as D_nF .This structure is considered in this study

Keywords: Orthogonal Calculus; Homotopy Fiber; Homogeneous Functor; Homotopy; Spectrum; Approximation.

1. Introduction

There exist another brand of functors Calculus, which emerged after the papers of Goodwillie were published, which is known as the orthogonal calculus of functors, due to Weiss, and this theory is closely related to or he had his inspiration from the Goodwillie calculus of homotopy functor (Goodwillie, 1990), (Goodwillie, 1991), (Goodwillie, 2003). The orthogonal calculus of functor is a beautiful tool for calculating the homotopical properties of functors from the category of inner product space to pointed spaces or any space enriched over Top_*.

Interesting examples of such functors abound and include classical objects in algebraic and geometric topology:

1. $\Omega^V Y\left(S^V \wedge X\right)$
2. $BAut(V)$
3. $Emb(M \times N, N \times V)$
4. $BTop(V)$

In the first example X is a fixed based space, S^V is the one-point compactification of *V* and $\Omega^V Y$ denotes the space of continuous based maps from S^V to Y.

In the second example *BAut*(*V*) is *BO*(*V*) or *BO*(*U*). In the third example *M* and *N* are fixed (topological, smooth, etc.) manifolds with the dimension of *M* smaller than the dimension of *N*, and $Emb(-,-)$ stands for the space of (topological, smooth, etc.) embeddings. In the last example *Top* (*V*) is the group of homeomorphisms from V to itself (Arone, 2002), hence we can associate a homomorphism of groups such that compositions of maps yield compositions of homomorphisms of groups (Zigli *et al.,* 2017). Category of such functors from vector spaces to spaces and natural transformations between them will be call ξ_0. These functors satisfy an extrapolation condition, which allows one to identify the value at some vector space from the values at vector spaces of greater dimension (Barnes and Oman, 2013). Orthogonal calculus is based on the notion of n-polynomial functors (vector spaces at very high dimension), which are well-behaved functors in ξ_0 and which preserves weak equivalences as well. With these n-polynomial functors one can often infer the value at some vector spaces from the values at vector spaces of higher dimension.

In geometric sense, orthogonal calculus approximates a functor (locally around $\mathbb{R}^\infty$) via polynomial functors (approximate into sequence of simpler functors that are homotopy equivalent to the functor in question) and attempts to reconstruct the global functor from the associated 'infinitesimal ' information. The orthogonal calculus splits a functor F in ξ_0 into a Taylor tower of fibrations, where our n-th fibrations will consist of maps from the n.polynomial approximation of F to the (*n* − 1)−polynomial approximation of F. The homotopy ber or layer (the difference between npolynomial and (*n*−1)−polynomial approximation) of this map is then an nhomogeneous functor and is classied by an *O*(*n*)−spectrum up to homotopy which is usually denoted as D_nF (Barnes and Oman, 2013).

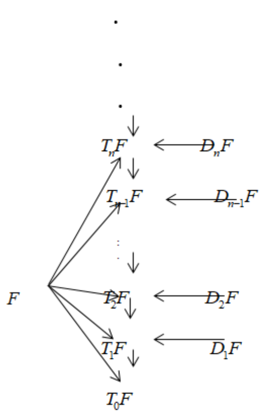
1.1. Continuous Functors

Let consider $\Im$ to be the category of vector space with an inner product and that is finite dimensional with linear maps to preserve the internal structure of the vector space. To see our category is finitely small let's assume our vector spaces belongs to some larger space $\mathbb{R}^\infty$, since orthogonal calculus is based on the notion of $n-$ polynomial functors (vector spaces at very high dimension), which are well-behaved functors in ξ_0 and which preserves weak equivalences as well (Barnes and Oman, 2013). With these n-polynomial functors one can often infer the value at some vector spaces from the values at some vector spaces of higher dimension.

Orthogonal calculus is concerned with covariant functors that are continuous i.e. E from $\mathfrak{J}$ to spaces. A functor been Continuous implies
(*V, W*) × *E* (*V*) → *E* (*W*) is continuous, for every $V, W \in \mathfrak{J}$ (Weiss, 1995). Some examples are $E(V) = BO(V)$, $E(V) = BTop(V), E(V) = BG(S(V))$
Suggesting that orthogonal groups are associated with classical spaces, like BO, BTop, BG equipped with a sophiscated filtration indexed by finite dimension linear subspaces V of $\mathbb{R}^{\infty}$.

1.2. The Tower of Classification

For a covariant functor $F \in \mathfrak{J}_0 Top$, Weiss calculus constructs the $n-$polynomial approximations $T_n F$ and the n homogeneous approximations $D_n F$. These can be clearly shown in a tower of brations. For all *n* ≥ 0 there exist a sequence of fibration $D_n F \to T_n F \to T_{n-1} F$ which can be arranged as below.



Tower of fibration

For this tower of fibration to be useful we must understand the functor F, the polynomial approximation of the functor F and also the homogeneous functors as well (Barnes and Oman, 2013).

1.3. Derivative of a functor

We will denote $\mathbb{R}^{\infty}$ with *μ* (as infinite-dimensional vector space with a positive definite inner product) with the standard inner product, and regard all finite dimensional vector spaces $\mathbb{R}^{\infty}$ as subspaces of *μ*, inheriting its inner product. Througout our work we will denote our nite dimensional vector spaces with object *U, V, W* and denote the one point compactification of V with V^C. We write $R^n \otimes V$ to mean $n.V$. Let' think of $\mathbb{R}^n$ to be a suitable subspace of $\mathbb{R}^{\infty}$, so that $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \ldots \subset \mathbb{R}^{\infty}$ then $0.V \subset 1.V \subset 2.V \subset 3.V \subset \ldots \subset n.V$ this will also denote the one point compactification.
Let's consider *Mor* (*V, W*) to be a linear isometries from *V* to *W*. which preserves the inner product.

Also lets consider the category $\mathfrak{J}$ of vector spaces preserving inner product with objects been *U, V,* and *W* such that *Mor* (*V, W*) is the set of maps from *V* to *W*.

Also let $\mathfrak{J}_n$ be of the same object as $\mathfrak{J}$ with the category of objects *U, V, W. . . .* and with $mor_n(U, V)$ as the set of maps from U to V. $\mathfrak{J}_n$ is considered as a topological category which is pointed with class of objects that are discrete. The morphisms set are topological spaces that are pointed. Just as the non equivairiant case, we can form inclusiion $\mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \mathfrak{J}_2 \subset \mathfrak{J}_3 \subset \ldots$ and a notion of derivatives. More that $\mathfrak{J}_0$ differs slightly from $\mathfrak{J}$ such that $mor_0(V, W)$ is *mor* (*V, W*) with an added base point. We now concentrate on functors that are continuous; i.e. if E is a covariant functor $\mathfrak{J}_0 \to (Top_*)$ pointed spaces, then it has a derivative $E^{(1)} : \mathfrak{J}_1 \to Top_*$ which itself has a derivative $E^{(2)} : \mathfrak{J}_2 \to Top_*$ which will also have the derivative $E^{(3)} : \mathfrak{J}_3 \to Top_*$ and bso on as in the non equivariant case. The derivative is defined in terms of the adjoint to the restriction functor.

Thus restriction from ξ_n to ξ_m for m, n with *m* ≤ *n* gives us a natural transformation res_m^n. we can think of m, n as positive integers. more generally we can obtain a restriction map $res_m^n : \xi_n \to \xi_m$ for *m* ≤ *n* by successive composition. There exist a map which helps to transform one functor to the other or preserves the structure of functor which is known as the natural transformation. Hence the natural transformation above will be continuous if it is invertible and a homomorphism. E.g. $res_m^n : \mathrm{nat}_n(E, F) \to nat_m(res_m^n E, res_m^n F)$ is continuous.

Proposition 1.1. A fuctor res_m^n has $ind_m^n : \xi_m \to \xi_n$ as its right adjoint, and its defined as $(ind_m^n X)(V) = Nat_{\xi m}(Mor_n(V, -), X)$ With the right hand side denoting the topological space of the morphism between two objects of ξ_m.

Proposition 1.2. For all V and $W \in \mathfrak{J}_0$ and all *n* ≥ 0 there is a natural homotopy cofiber sequence

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$$Mor_n(\mathbb{R}\oplus V,W)\wedge S^n \to$$
$$Mor_n(V,W)\to Mor_{n+1}(V,W)$$

Proof. Identifying S^n as the closure of the subspace $(i,x)\in\gamma_n(V,\mathbb{R}\oplus V)\$,$ where i is the standard inclusion, the composition map

$$Mor_n(\mathbb{R}\oplus V,W)\wedge Mor_n(V,\mathbb{R}\oplus V)\to$$
$$Mor_n(V,W)$$

Restricts to a morphism

$Mor_n(\mathbb{R}\oplus V,W)\wedge S^n \to Mor_n(V,W).$ The homotopy cofiber of the restriction is then the quotient of $[0,\infty]\times\gamma_n(\mathbb{R}\oplus V,W)\times\mathbb{R}^n$. The desired homeomorphism, away from the base point, is indeed by the association below.

Consider a quadruple

$$\begin{pmatrix} t\in[0,\infty],\ f\in Mor(\mathbb{R}\oplus V,W,) \\ y\in\mathbb{R}^n\otimes(W-f(\mathbb{R}\oplus V)),z\in\mathbb{R}^n \end{pmatrix}$$

We send this to the element $(f\mid v,x)\in Mor_{n+1}(V,W)$ where $x=y+(f\mid_{\mathbb{R}^*})(z)+t\omega(f\mid_{\mathbb{R}^*}(1))\$,$ and $\omega:W\to\mathbb{R}^{n+1}\otimes W$ identifies

$$W\cong(\mathbb{R}^n\otimes W)^{\perp}\subset\mathbb{R}^{n+1}\otimes W$$

From this cofiber sequence we can make a fiber sequence by applying the functor

$$Nat_{\varepsilon_n}(-,F)\$for\$F\in\varepsilon_n\$.$$

Lemma 1.0.1 For all $V\in\mathfrak{I}_n$ and $F\in\varepsilon_n$ there is a natural homotopy fiber sequence.

$$res_n^{n+1}ind_n^{n+1}F(V)\to F(V)\to\Omega^n F(\mathbb{R}\oplus V)$$

2. Structure to Polynomial Functors

Which functor E in $\mathfrak{I}_0Top$ deserves to be called polynomial functors of degree $\leq n$? This question has to be certainly answered at some point in time if we want do calculus. One easy to see requirement of the n-polynomial functor is that it's $(n+1)-Th$ derivative of the functor E should vanish. However this does not hold for all cases especially the case *n* = 0 shows that this definition is not enough.

A functor E deserves to be called polynomial of degree 0 iff E(f) is a homotopy equivalence for all nonzero morphisms f in $\mathfrak{I}_0$.

2.1. Polynomial Functor

Polynomial functors has appeared to be very important in physics, also in mathematics with special areas like topology (Bisson and Joyal, 1995), (Pirashvili, 2000), and in algebra (Macdonald, 1998) and also nd it route in mathematical logic (Girard, 1988),(Moerdijk and Palmgren, 2000) and computer science (theoretical),(Jay and Cockett, 1994),(Abbott *et al.,* 2003), (Setzer and Hancock, 2005) and useful in the representation theory of symmetric groups (Macdonald, 1998). Hence we want to study a well behaved collection of such functors in ξ_0 those whose derivatives are eventually trivial. By analogy with functions on the real numbers, we call these functors polynomial. In this section we introduce this class of functors and examine their structures.

Definition 2.1. For $E\in\mathfrak{I}_0Top$ or for $E\in\xi_0$ define $\tau_nE(V)=\underset{0\neq U\subset R^{n+1}}{ho\lim}E(U\oplus V)$ We can think of the covariant functor E to be n-polynomial if the canonical map $\rho_E^n(V):E(V)\to\tau_nE(V)$ Is homotopy equivalence for every genre vector space V of $\mathfrak{I}$

Remark. The non-zero linear subspace $U\subset\mathbb{R}^{n+1}$ form a poset P where $T\leq U$ means $T\subset U$

With the above theorem we sometimes think of such functor E to be n-polynomial. The value of the functor E which is n-polynomial at the vector space V is determined up to homotopy by the values $E(U\oplus V)$ and the arrows between them for the nonlinear subspace $U\subset\mathbb{R}^{n+1}$

This definition captures the idea of the value of the functor E at some vector space V being recoverable from the value of E at vector spaces of higher dimension.

I.e. we can think of the n-polynomial functor as one where it is possible to extrapolate the information of $E(V)$ from the spaces $E(U\oplus V)$.

The homotopy fiber of

$\rho_E^n(V):E(V)\to\tau_nE(V)$ measures how far E is from being *n*-polynomial, its always helpful for us identifying what the fibers are. Also let's recall that a sphere bundle $S\gamma_{n+1}(V,W)\xrightarrow{p}mor(V,W)$ if we fix V and vary W, we will get a natural transformation $S\gamma_{n+1}(V,-)\to mor(V,-)$ We then have a map

$$\rho^*:nat(mor(V,-),E)\to$$
$$nat(S\gamma_{n+1}(V,-),E)$$

Hence by the yonneda lemma we get $\rho^*:E(V)\to nat(S\gamma_{n+1}(V,-),E)$ And its polynomial of degree $\leq n$ iff

$\rho^*:E(V)\to nat(S\gamma_{n+1}(V,-),E)$ is homotopy equivalence for all V

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Definition 2.2 For $E \in \xi_0$, we define $\tau_n E \in \xi_0$ such that $(\tau_n E)(V) = Nat_{\xi_0}\left(S\gamma_{n+1}(V,-)_+, E\right)$ We also have natural transformation of self functors on $\rho_n : Id \to \tau_n$ This natural transformation comes from the map $S\gamma_{n+1}(V,W)_+ \to Mor_0(V,W)$ And by yonneda lemma.

By Michael Weiss there is another description of $S\gamma_{n+1}(-,-)$ It is the homotopy colimit :

$$S\gamma_{n+1}(V,A)_+ \cong \underset{0\neq U\subset\mathbb{R}^{n+1}}{hocolim}\, E(U \oplus V)$$

Where the right hand side is the Bousfield-Kan formula for the homotopy colimit of the functor $U \to Mor_0(U \oplus V)$ as U varies over the topological category of nonzero subspace of $\mathbb{R}^{n+1}$ and inclusions. Thus we see that $\tau_n E(V) = \underset{0\neq U\subset\mathbb{R}^{n+1}}{holim}\, E(U \oplus V)$.

We choose to define τ_n in terms $S\gamma_{n+1}(-,-)$ and we then define polynomial functors in terms of τ_n (Barnes and Oman 2013).

Proposition 2.1. For any $E \in \xi_0$, and any $n \in \mathbb{N}$, the sequence,

$$res_0^{n+1} ind_0^{n+1} E(V) \xrightarrow{\ \mu\ } E(V) \xrightarrow{\ \rho\ } \tau_n E(V)$$

Is a fibration sequence up to homotopy and hence $res_0^{n+1} ind_0^{n+1} E(V)$ vanishes if E is a polynomial of degree $\leq n$

Proof: Let's define

$$res_0^{n+1} ind_0^{n+1} E(V) = Nat_{\xi_0}\left(mor_{n+1}(V,-), E\right)$$

then for the natural co-fiber sequence

$$S\gamma_{n+1}(V,A)_+ \to Mor_0(V,A)_+ \to Mor_{n+1}(V,A)$$

Which is natural in A with respect to $\Im_0$. This converge to give a cofiber sequence of : $\Im_0 - spaces$

$$S\gamma_{n+1}(V,-)_+ \to Mor_0(V,-) \to Mor_{n+1}(V,-)$$

Considering the induced maps of spaces

$$Nat_{\xi_0}\left(Mor_{n+1}(V,-), E\right) \to$$
$$Nat_{\xi_0}\left(Mor_0(V,-), E\right) \to$$
$$Nat_{\xi_0}\left(S\gamma_{n+1}(V,-)_+, E\right).$$

Hence the above can be identified with

$$\left(res_0^n ind_0^{n+1} E\right)(V) \to E(V) \to (\tau_n E)(V)$$

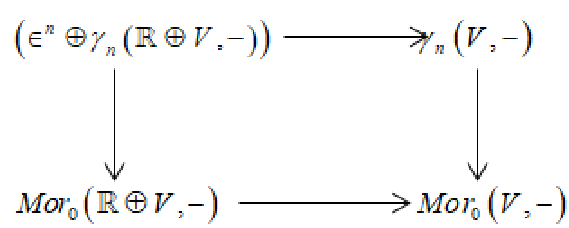
Which is a fibration sequence up to homotopy for all vector space V. (Barnes and Oman, 2013)

Proposition 2.2 If E in ξ is polynomial of degree $\leq n-1$, then it is polynomial of $\leq n$ degree.

Proof. We will actually show that any $S_n - equivalence$ is an $S_{n-1} - equivalence$ Thus we are to prove that $S_n = \left\{S\gamma_{n+1}(V,-)_+ \to Mor_0(V,-) \middle| V \in \Im_0\right\}$ is an $S_{n-1} - equivalence$ for any V. We can reduce this to proving that the map $\alpha : S\gamma_n(V,-)_+ \to S\gamma_{n+1}(V,-)_+$ is an S_{n-1} - equivalence. The standard inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induces a map of vector bundles $\gamma_n(V,W) \to \gamma_{n+1}(V,W)$ and hence a map of their respective unit spheres bundles: $\alpha : S\gamma_n(V,-)_+ \to S\gamma_{n+1}(V,-)_+$ We can write $S\gamma_{n+1}(V,-)_+$ as the fiberwise product over $Mor_0(V,-)$ (denoted $\otimes$) of $S\gamma_n(V,-)_+ \$and\$ S\gamma_1(V,-)_+$ Thus we can write $S\gamma_{n+1}(V,-)_+$ as the homotopy pushout of the following diagram

$$\begin{array}{c} S\gamma_n(V,-) \xleftarrow{\ \rho_1\ } S\gamma_n(V,-) \otimes \\ S\gamma_1(V,-) \xrightarrow{\ \rho_2\ } S\gamma_1(V,-) \end{array}$$

Where ρ_1 and ρ_2 are the projection maps. Now we can identify the codomain ρ_2 with the stiefel manifold $Mor(\mathbb{R} \oplus V,-)\$$ and in fact ρ_2 itself is just the bundle. Writing $\in^n$ for the n-dimensional trivial bundle, it clear that there is a pullback square:



The projection map ρ_2 can be identified with $S\left(\in^n \oplus \gamma_n(\mathbb{R}\oplus V,-)\right)_+ \to Mor_0(\mathbb{R}\oplus V,-)$.

Hence the vector bundle $S\gamma_{n+1}(V,-)_+$

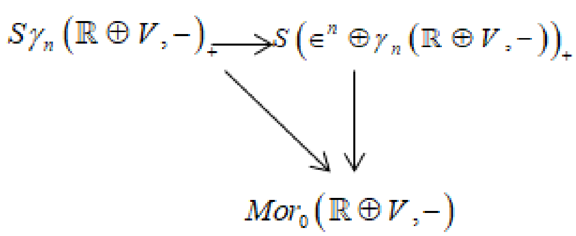
is the homotopy pushout of $S\gamma_n(V,-)_+ \leftarrow S\left(\in^n \oplus \gamma_n(\mathbb{R}\oplus V,-)\right)_+ \xrightarrow{\rho_2} Mor_0(\mathbb{R}\oplus V,-)$.

If ρ_2 is an S_{n-1}-equivalence, then so is its homotopy pushout, which is α.

The unit sphere of the Whitney sum of vector bundles is equal to the fiberwise join of the unit sphere bundles. Hence we can write domain of ρ_2 as the homotopy pushout

$$S\gamma_n(\mathbb{R}\oplus V,-)_+ \leftarrow S_+^{n-1} \wedge S\gamma_n(\mathbb{R}\oplus V,-)_+ \xrightarrow{\delta} S_+^{n-1} \wedge Mor_0(\mathbb{R}\oplus V,-).$$

The map δ is an S_{n-1} -equivalence, hence the top map in the commutative diagram below is an S_{n-1} -equivalence:



Since the diagonal map is an element of S_{n-1} it follows that ρ_2 is an S_{n-1}-equivalence, as desired (Barnes and Oman, 2013).

Proposition 2.3. Let $g: F \to E$ be a map in ξ_0 such that $ind_0^{n+1}E$ is object wise contractible and F in n-polynomial. Then the covariant functor $V \mapsto hofiber\left[F(V) \xrightarrow{g} E(V)\right]$ is also polynomial of degree $\leq n$.

Remark. In particular, it proves that the homotopy fiber of a map between n-polynomial objects is n-polynomial.

Proposition 2.4. We say that a functor $E \in \xi_0$ is connected at infinity if the space $hoco\lim_k E(\mathbb{R}^k)$ is connected.

Remark. Polynomial functors can be determined by their behaviour at very high dimension. i.e. by considering the behaviour of the vector space V at a very high dimension and which is always the best possible approximation to the functor in question.

If a functor E is polynomial functor of degree $\leq n$, then all morphisms in the diagram $E \xrightarrow{\rho} \tau_n E \xrightarrow{\rho} \tau_n\tau_n E \xrightarrow{\rho} \ldots$ are equivalences.

For arbitrary E in ξ_0 the space $E(\mathbb{R}^\infty) := hoco\lim_i E(\mathbb{R}^i)$, and the spectra $\Theta E^{(1)}, \Theta E^{(2)}, \Theta E^{(3)}, \ldots$ are determined up to homotopy equivalence by the behaviour of E at infinity.

Proposition 2.5. For a morphism $g: E \to F$ in ξ_0 such that $hofiber\left[E(V) \xrightarrow{g} F(V)\right]$ is contractible for all V. Lets think of F to be connected at infinity, and that the covariant functors E and F are polynomial of degree $\leq n$.

Then g is a homotopy equivalence.

Proof. The problem lies in the fact that at each stage of V, the homotopy fiber is defined via a fixed choice of base point in F(V), but we need an isomorphism of homotopy groups between E(V) and F (V) for all choices of base points.

Let $F_b(V)$ be the Subspace of F(V) consisting of only the basepoint component of F(V). $F_b \to F$ We prove that $F_b \to F$ is an equivalence after applying the functor $\tau_n = hoco\lim_k \tau_n^k$.

Note that since E and F are n–polynomial, the maps $E \to \tau_n E$ and $F \to \tau_n F$ are objectwise weak equivalences.

Consider the map $hoco\lim_k \tau_n F_b \to hoco\lim_k \tau_n^k F\$.$

For each choice of basepoint, the homotopy fiber of $\tau_n^k F_b \to \tau_n^k F$ is empty or contractible.

If C is some component in $F(V) \simeq \tau_n^k F(V)\$,$ then because f is connected at infinity, there is some l such that the image of C in $\tau_n^l F(V)$ is in the basepoint component.

This holds since $\tau_n^l F(V)$ is defined using only the terms $F(V \oplus U)$ for U of dimension greater than or equal to l.

Hence C is contained in $\tau_n^l F_b(V)$ and there can be no empty fibers. We thus have objectwise weak equivalences $T_n F_b \to T_n F\$.$

Consider the map $T_n E(V) \to T_n F(V)$ and choose some basepoint x in $T_n F(V)$, then we see that $x \in \tau_n^k F(V)$ for some k.

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As k increases, eventually X is in the same component as the canonical basepoint of $\tau_n^k F(V)$\$. Hence by our assumptions, the homotopy fibre for this choice x is contractible. So $T_n E \to T_n F$ is an objectwise weak equivalence and it follows that $E \to F$ is a objectwise weak equivalence.

Now we show from Weiss that τ_m preserves n–polynomial functors. The proof is simply that homotopy limits commute, $(\tau_n \tau_m = \tau_m \tau_n)$ and that homotopy limits preserve weak equivalences.

Lemma 2.0.1. If E is an n-polynomial object of ξ_0, then so is $\tau_m E$ for any $m \geq 0$ (Weiss, 1995).

Proof. We Start by showing that the canonical map $\tau_m E(V) = \underset{0 \neq U \subset \mathbb{R}^{m+1}}{ho\lim} E(U \oplus V) \to$

$\underset{0 \neq W \subset \mathbb{R}^{n+1}}{ho\lim} \underset{0 \neq U \subset \mathbb{R}^{m+1}}{ho\lim} E(W \oplus U \oplus V)$

Is a homotopy equivalence, for all generic object V in $\Im$. Target can be written as $\underset{0 \neq W \subset \mathbb{R}^{m+1}}{ho\lim} \underset{0 \neq U \subset \mathbb{R}^{n+1}}{ho\lim} E(W \oplus U \oplus V)$.

3. Homogeneous Functors.

When working with actual smooth functions, the n-th Taylor approximation (around 0) to $f : \mathbb{R} \to \mathbb{R}$ is giving by $T_n(x) = \sum_{i=0}^{n} f^{(n)}(0) \frac{x^n}{n!}$. In particular, the difference between two consecutive Taylor approximations is giving by $T_n(x) - T_{n-1}(x) = f^{(n)}(0) \frac{x^n}{n!}$.

The analogue of taking the “difference”, when working with (stable) ∞-categories, is to find the fiber of the map $T_n F \to T_{n-1} F$.

The classification of homogeneous functors takes a similar form. It is the space of sequence of a fibration whose fibers are the derivatives $F^{(k)}(\phi)$ with orthogonal group actions.

Let’s consider some examples of homogeneous functors and also define what it means for a functor to be homogeneous and consider some examples and also define what makes a functor homogeneous.

Definition 3.1. Let $F : \Im_0 \to Top_*$ be a functor. Define $D_n F$ to be the fiber of the natural transformation $T_n F \to T_{n-1} F$, then $D_n F$ is a homogeneous functor of degree n.

If it is a polynomial functor of degree $\leq n$ and $T_{n-1}F(V)$ is contractible for every $V \in \Im_0$

i.e. $T_{n-1} D_n F(V) \simeq *$ for all $V \in \Im_0$

Remark. For contravariant functor F, choose a basepoint in $\Im_0$. This bases F(V) for all $V \in \Im_0$. This is then a homogeneous functor of degree n. That is the polynomial of degree $\leq n$.

To see that $T_{n-1} D_n F(V) \simeq *$ for every V, first observe that T_{n-1} commutes with homotopy fibers and next observe that $T_{n-1} T_n F \simeq T_{n-1} F$.

Theorem 3.1 The full subcategory of n-homogenous functors inside $Ho(\Im_0 Top)$ is equivalent to the homotopy category of spectra with the orthogonal group action on n.

For a given spectrum Ψ_E with orthogonal group action on n the functor below is an n-homogeneous functor of $\Im_0 Top.$

$$V \mapsto \Omega^\infty \left[\left(S^{\mathbb{R}^n \otimes V} \wedge \Psi_E \right) / ho(n) \right]$$

We can think of, $S^{\mathbb{R}^n \otimes V}$ from the theorem as the one-point compactification of $\mathbb{R}^n \otimes V$.

This has orthogonal group action (O(n)-action) induced from the regular representation of the smash product is equipped with the diagonal action of $O(n)$, Ψ_E indicates a spectrum with the orthogonal group action $O(n)$. \$O(n)\$ denotes homotopy orbits alias the Borel construction.

We now look at how to obtain the spectra Ψ_E. We begin by recalling that $\Im$ denotes the category of finite dimensional inner product space with maps the linear maps that preserves the internal structures. Let define a vector bundle over $\Im(U,V)$ for $U,V \in \Im$

$$\gamma_n(U,V) = \left\{ (f,x) \mid f : U \to V, x \in^n \otimes (V - f(U)) \right\}$$

The total space of the vector bundle has a natural action of O(n) due to the $\mathbb{R}^n$ factor. We assume $\Im_n(U,V) := T\gamma_n(U,V)$, the associated Thom space. Hence this is the cofiber in the sequence:

$$S\gamma_n(U,V) \to D\gamma_n(U,V) \to T\gamma_n(U,V)$$
$$\{(f,x) \mid \|x\| = 1\} \quad \{(f,x) \mid \|x\| \leq 1\}$$

Recall that $T(\mathbb{R}^n \to *) = S^n$ and $T(X = X) = X_+$ as defined already. In particular if we choose $n = 0$, then $\Im_0(U,V) = \Im(U,V)_+$

When looking at the vector bundles there exist a natural composition

$$\gamma_n(V,W) \times \gamma_n(U,V) \to \gamma_n(U,W)$$

$$(g,y) \qquad (f,x) \mapsto \left(g \circ f, y + \left(\mathbb{R}^n \otimes g\right) x\right)$$

where $\left(\mathbb{R}^n \otimes g\right) : \mathbb{R}^n \otimes \left(V - f(U)\right) \to \mathbb{R}^n \otimes W$

This composition induces associative and unital maps $\mathfrak{I}_n(V,W) \wedge \mathfrak{I}_n(U,V) \to \mathfrak{I}_n(U,W)$ Which are O(n) equivariant and functorial in the inputs (Barnes and Oman, 2013).

4. Conclusion

The study explained that calculus is not only about derivatives or fluxions but is also about approximation by polynomials.

This was shown by splitting our functor $F(V)$ into tower of fibrations where T_nF is the *n* polynomial approximation and D_nF is the *n* homogeneous functors where $D_nF \to T_nF \to T_{n-1}F$, is the sequence of fibration for all $n..2$.

The study also reviewed continuous functors and derivatives of orthogonal calculus of functors by concentrating on categories of vector spaces to pointed spaces.

Finally our research work has analyzed some structures of polynomial and homogeneous functors in the orthogonal calculus

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