Science World Journal Vol. 15(No 2) 2020
www.scienceworldjournal.org
ISSN 1597-6343
Published by Faculty of Science, Kaduna State University

Full Length Research Article

A NEW MODIFIED PRECONDITIONED ACCELERATED OVERRELAXATION (AOR) ITERATIVE METHOD FOR L-MATRIX LINEAR ALGEBRAIC SYSTEMS

[1]Isyaku Abdullahi and [2]Abdulrahman Ndanusa

[1,2]Department of Mathematics, Federal University of Technology, Minna, Nigeria

Authors Email Addresses: [1]isyakuabdullahi010@gmail.com and [2]as.ndanusa@futminna.edu.ng

ABSTRACT
A new preconditioner of the type $P = I + \bar{S} + S'$ which generalizes the preconditioners of Evans *et al.* (2001) and Ndanusa and Adeboye (2012) is proposed. Theoretical investigation of the new preconditioned AOR method is undertaken by advancement of some convergence theorems with well-known procedures. In order to validate the results of theoretical convergence analysis, numerical investigation with sample problems is done. Numerical results of comparison of the proposed preconditioner with some available preconditioners in literature are presented. The results show that convergence of the proposed preconditioned AOR method is faster than that of the unpreconditioned AOR as well as the preconditioned methods in current use.

Keywords: Accelerated Overrelaxation Method, Preconditioner, Convergence, $L-\mathrm{matrix}$, spectral radius

INTRODUCTION

Recent advancements in Physics, numerical weather forecasting, fluid dynamics, oil and gas resource development, image processing, optimization, simulated nuclear explosion and so on are modeled as partial differential equations, which are eventually transformed into large sparse linear systems of equations through finite element or finite difference discretization. Typically, the large sparse linear system of equations can be expressed as

$$Ax = b \qquad (1)$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular matrix with nonvanishing diagonal elements, and where $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are respectively vectors of unknown and preassigned variables. For some time, a great many research has been devoted to iterative solution methods for approximating the solution of (1). Recall the usual splitting of the coefficient matrix $A \in \mathbb{R}^{n\times n}$,

$$A = D_A - L_A - U_A \qquad (2)$$

where $D_A = diag(a_{11}, a_{22}, \cdots, a_{nn})$ is the diagonal part of A, $-L_A$ and $-U_A$ its strictly lower and strictly upper parts, respectively. If $a_{ii} \neq 0$ for all $i \in \mathbb{N}$ $(i = 1,2,\cdots,n)$, then we can multiply the linear system (1) by ${D_A}^{-1}$, arising therefrom the splitting of the matrix

$$A = I - L - U \qquad (3)$$

where $I = {D_A}^{-1}D_A$, $L = {D_A}^{-1}L_A$ and $U = {D_A}^{-1}U_A$.

Suppose $A = M - N$ is a regular splitting of the coefficient matrix $A = (a_{ij})$, then the basic iterative method for the solution of system (1) can be expressed in the form

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \qquad k = 0,1,2,\cdots \qquad (4)$$

where $M^{-1}N$ is known as the iteration matrix of the method. The iteration (4) is known to converge to the exact solution $x = A^{-1}b$ for any initial vector value $x^{(0)} \in \mathbb{R}^n$ if and only if the spectral radius $\rho(M^{-1}N) < 1$. The smaller the spectral radius the faster the convergence speed of the iterative method. The classical AOR iterative method given in Hadjidimos (1978) for solving (1) is defined as

$$x^{(k+1)} = \mathcal{L}_{r,\omega}x^{(k)} + (I - rL)^{-1}\omega b \qquad k = 0,1,2,\cdots \qquad (5)$$

with the iteration matrix $\mathcal{L}_{r,\omega}$ given as

$$\mathcal{L}_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U] \qquad (6)$$

where ω and r are real parameters with $\omega \neq 0$.

Besides the iteration matrix and parameters involved in the iterative methods, convergence is also dependent on the nature of the linear systems of equations themselves. Therefore, in order to improve efficiency of the iterative method (4), the linear system (1) is transformed into the equivalent preconditioned system

$$PAx = Pb \qquad (7)$$

where P is a nonsingular matrix called a preconditioner. Different preconditioners have been advanced by several researchers for the preconditioned system (7). Evans *et al.* (2001), Li *et al.* (2007), Wu *et al.* (2007), Wu and Huang (2007), Yun and Kim (2008), Wang and Song (2009), Darvishi *et al.* (2011), Li (2011), Ndanusa and Adeboye (2012), Huang *et al.* (2016), Behzadi (2019) and Wang (2019) are some instances of application of the preconditioned system (7) to improve the convergence of the AOR method. The preconditioner P of Evans *et al.* (2001) is described by $P = I + S'$, where

$$S' = (s_{ij}) = \begin{cases} -a_{1n} & \\ 0 & \text{otherwise} \end{cases} \qquad (8)$$

$P = I + \bar{S}$ is another preconditioner proposed by Ndanusa and Adedboye (2012), with

$$\bar{S} = (s_{ij}) = \begin{cases} -a_{i1}, & i = 2,\cdots,n \\ -a_{i,i+1}, & i = 1,\cdots,n-1 \\ 0, & \text{otherwise } 00 \end{cases} \qquad (9)$$

Using the idea of previous works, we consider the preconditioner $P = I + \hat{S}$, where

$$\hat{S} = \bar{S} + S' = \begin{cases} -a_{1n} & \\ -a_{i1}, & i = 2,\cdots,n \\ -a_{i,i+1}, & i = 1,\cdots,n-1 \\ 0, & \text{otherwise} \end{cases} \qquad (10)$$

The present work is aimed at investigating the modified preconditioner $P = I + \hat{S}$ applied to AOR iteration, through theoretical proofs, corroborated by numerical verifications, in order to improve upon the rate of convergence of the method.

MATERIALS AND METHODS

Preliminaries

For convenience, some notations, definitions and lemmas that will be used in the succeeding sections are briefly explained. Suppose $A = (a_{ij}) \in R^{n\times n}$ is a real matrix. The notation $diag(A)$ implies the $n \times n$ diagonal matrix coinciding in its diagonal with a_{ii}. For a square matrix A, $\rho(A)$ denotes the spectral radius of A.

Definition 1 (Young, (1971)). A matrix $A = (a_{ij})_{i,j=1,2,\cdots,n}$ is an L −matrix if $a_{ii} > 0$ and $a_{ij} \leq 0\ (i \neq j)$.

Definition 2 (Saad (2000)). An L −matrix $A = (a_{ij})$ where A is nonsingular and $A^{-1} \geq 0$ is called an M −matrix.

Definition 3 (Saad (2000)). A matrix $A = (a_{ij})_{i,j=1,2,\cdots,n}$ is called nonnegative, nonpositive and positive if $a_{ij} \geq 0$, $a_{ij} \leq 0$ and $a_{ij} > 0$, respectively.

Definition 4 (Dehghan and Hajarian (2009). The decomposition of a real matrix $A \in R^{n\times n}$ into the form $A = M - N$, where M is a nonsingular matrix is called a splitting of A. Such splitting is called

- Regular if $M^{-1} \geq 0$ and $N \geq 0$
- Nonnegative if $M^{-1}N \geq 0$
- Convergent if $\rho(M^{-1}N) < 1$
- M −splitting if M is a nonsingular M −matrix and $N \geq 0$.

Definition 5 (Saad (2000)). $A = (a_{ij})$ is called an irreducible matrix if the directed graph associated to A is strongly connected.

Lemma 1 (Varga (1962)). Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

i. A has a positive real eigenvalue equal to its spectral radius.
ii. To $\rho(A)$ there corresponds an eigenvector $x > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Gunawardena *et al.* (1991)). Let A be a nonnegative matrix. Then

i. If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
ii. If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3 (Varga (1962)). Suppose $A = M - N$ is an $M -$ splitting of A. Then the splitting is convergent iff A is a nonsingular $M -$ matrix.

Lemma 4 (Varga (1962)). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq O$. If $N_2 \geq N_1 \geq O$, then

$$1 > \rho\left({M_2}^{-1}N_2\right) \geq \rho\left({M_1}^{-1}N_1\right) \geq O.$$

If moreover, $A^{-1} > O$ and if $N_2 \geq N_1 \geq O$, equality excluded (meaning that neither N_1 nor $N_2 - N_1$ is the null matrix), then

$$1 > \rho\left({M_2}^{-1}N_2\right) > \rho\left({M_1}^{-1}N_1\right) > O.$$

The Preconditioned AOR iterative method

Application of the preconditioner P to system (1) results in the preconditioned linear system

$$\hat{A}x = \hat{b} \qquad (11)$$

where $\hat{A} = (I + \hat{S})A$ and $\hat{b} = (I + \hat{S})b$ with

$$\hat{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & 0 \\ -a_{31} & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & -a_{n-1,n} \\ -a_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

A usual splitting of the preconditioned coefficient matrix $\hat{A}$ of (11) into its diagonal $(\widehat{D})$, strictly lower $(-\hat{L})$ and strictly upper $(-\widehat{U})$ components is obtained thus

$$\hat{A} = \widehat{D} - \hat{L} - \widehat{U}$$

with the following resultant representations

$$\widehat{D} = \begin{bmatrix} 1 - a_{12}a_{21} - a_{1n}a_{n1} & 0 & \cdots & 0 & 0 \\ 0 & 1 - a_{12}a_{21} - a_{23}a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 - a_{1,n-1}a_{n-1,1} - a_{n-1,n}a_{n,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 - a_{1n}a_{n1} \end{bmatrix}$$

$$-\hat{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_{23}a_{31} & 0 & \vdots & 0 & 0 \\ \vdots & -a_{31}a_{12} - -a_{34}a_{42} + a_{32} & \ddots & \vdots & \vdots \\ -a_{n-1,n}a_{n1} & \vdots & \ddots & 0 & 0 \\ 0 & -a_{n1}a_{12} + a_{n2} & \cdots & -a_{n1}a_{1,n-1} + a_{n,n-1} & 0 \end{bmatrix}$$

and

$$-\widehat{U} = \begin{bmatrix} 0 & -a_{1n}a_{n2} & -a_{12}a_{23} - a_{1n}a_{n3} + a_{13} & \cdots & -a_{12}a_{2n} \\ 0 & 0 & -a_{21}a_{13} & \cdots & -a_{21}a_{1n} - a_{23}a_{3n} + a_{2n} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & -a_{n-1,1}a_{1n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

It is observed that $\widehat{D} = I + D_1$, $\hat{L} = L + L_{\hat{S}} + L_1$, $\widehat{U} = U + U_{\hat{S}} + U_1$, $\hat{S} = -L_{\hat{S}} - U_{\hat{S}}$, and $-\hat{S}L - \hat{S}U = D_1 - L_1 - U_1$; where $D_1,\ -L_1,\ \text{and}\ -U_1$ are the diagonal, strictly lower and strictly upper parts of $-\hat{S}L - \hat{S}U$ respectively; and $-L_{\hat{S}}$ and $-U_{\hat{S}}$ are the strictly lower and strictly upper parts of $\hat{S}$ respectively. Application of the AOR method to the preconditioned linear system (11) results in the corresponding preconditioned AOR method whose iterative matrix is defined by

$$\hat{\mathcal{L}}_{r,\omega} = (\widehat{D} - r\hat{L})^{-1}\left[(1-\omega)\widehat{D} + (\omega - r)\hat{L} + \omega\widehat{U}\right] \qquad (12)$$

Theorem 1 Let $\mathcal{L}_{r,\omega}$ and $\hat{\mathcal{L}}_{r,\omega}$ be the AOR iterative matrices corresponding to the linear systems (1) and (11) respectively. Suppose $0 \leq r \leq \omega \leq 1\ (\omega \neq 0, r \neq 1)$, A is an irreducible $L -$ matrix with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1\ (i = 2(1)n-1)$ and $0 < a_{12}a_{21} + a_{in}a_{n1} < 1$. Then $\mathcal{L}_{r,\omega}$ and $\hat{\mathcal{L}}_{r,\omega}$ are nonnegative and irreducible matrices.

PROOF: Since A is an $L -$ matrix, $L \geq 0$ and $U \geq 0$. Thus $(I - rL)^{-1} = I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1} \geq 0$. And, from (6), we have

$$\mathcal{L}_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]$$

$$= [I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}][(1-\omega)I + (\omega - r)L + \omega U]$$
$$= (1-\omega)I + (\omega - r)L + \omega U + rL(1-\omega)I + rL[(\omega - r)L + \omega U]$$
$$+(r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-\omega)I + (\omega - r)L + \omega U]$$
$$= (1-\omega)I + [(\omega - r)L + rL(1-\omega)I] + \omega U + rL[(\omega - r)L + \omega U]$$
$$+(r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-\omega)I + (\omega - r)L + \omega U]$$
$$= (1-\omega)I + \omega(1-r)L + \omega U + T$$

where

$$T = rL[(\omega - r)L + \omega U] + (r^2L^2 + \cdots + r^{n-1}L^{n-1}) \times [(1-\omega)I + (\omega - r)L + \omega U] \geq 0.$$

It is clear that $(1-\omega)I + \omega(1-r)L + \omega U \geq 0$. Consequently, $\mathcal{L}_{r,\omega} = (1-\omega)I + \omega(1-r)L + \omega U + T \geq 0$. Hence, $\mathcal{L}_{r,\omega}$ is a nonnegative matrix. Since $A = I - L - U$ is irreducible, so also is $(1-\omega)I + \omega(1-r)L + \omega U$ since the coefficients of $I, L, \text{and } U$ are different from zero and less than 1 in absolute value. Hence, $\mathcal{L}_{r,\omega}$ is an irreducible matrix.

Now, consider the preconditioned AOR iterative matrix

$$\hat{\mathcal{L}}_{r,\omega} = (\hat{D} - r\hat{L})^{-1}\left[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}\right]$$

Equation (7) ensures that the $L-\text{matrix}$ structure of A is preserved in $\hat{A}$. Since $\hat{A}$ is an $L-\text{matrix}$, it is evident that $\hat{L} \geq 0$ and $\hat{U} \geq 0$. Also, by the conditions of Theorem 1, it is easy to get that $\hat{D} \geq 0$. Thus,

$$\hat{\mathcal{L}}_{r,\omega} = \left[\hat{D}(I - r\hat{D}^{-1}\hat{L})\right]^{-1}\left[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}\right]$$
$$= (I - r\hat{D}^{-1}\hat{L})^{-1}\hat{D}^{-1}\left[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}\right]$$
$$= (I - r\hat{D}^{-1}\hat{L})^{-1}\left[(1-\omega)I + (\omega - r)\hat{D}^{-1}\hat{L} + \omega\hat{D}^{-1}\hat{U}\right]$$
$$= \left[I + r\hat{D}^{-1}\hat{L} + r^2\left(\hat{D}^{-1}\hat{L}\right)^2 + \cdots + r^{n-1}\left(\hat{D}^{-1}\hat{L}\right)^{n-1}\right] \times \left[(1-\omega)I + (\omega - r)\hat{D}^{-1}\hat{L} + \omega\hat{D}^{-1}\hat{U}\right]$$
$$= (1-\omega)I + \omega(1-r)\hat{D}^{-1}\hat{L} + \omega\hat{D}^{-1}\hat{U} + \hat{T}$$

where

$$\hat{T} = r\hat{D}^{-1}\hat{L}\left[(\omega - r)\hat{D}^{-1}\hat{L} + \omega\hat{D}^{-1}\hat{U}\right] + \left[r^2\left(\hat{D}^{-1}\hat{L}\right)^2 + \cdots + r^{n-1}\left(\hat{D}^{-1}\hat{L}\right)^{n-1}\right] \times \left[(1-\omega)I + (\omega - r)\hat{D}^{-1}\hat{L} + \omega\hat{D}^{-1}\hat{U}\right] \geq 0$$

Using similar arguments, it is conclusive that $\hat{\mathcal{L}}_{r,\omega}$ is also nonnegative and irreducible. ☐

The following equalities are essential to prove Theorem 2.

A1: $\hat{U} = U + U_{\hat{S}} + U_1$

A2: $\hat{D} = \bar{D} - S'\bar{S} = I + D_1$

A3: $\hat{L} = \bar{L} = L + L_{\hat{S}} + L_1$

A4: $\hat{D} - \hat{L} = \bar{D} - S'\bar{S} - \bar{L}$

Theorem 2 Let $\mathcal{L}_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]$ and $\hat{\mathcal{L}}_{r,\omega} = (\hat{D} - r\hat{L})^{-1}[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}]$ be the AOR and preconditioned AOR iterative matrices respectively. Suppose $0 \leq r \leq \omega \leq 1$ $(\omega \neq 0, r \neq 1)$, A is an irreducible $L-\text{matrix}$ with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$ $(i = 2(1)n-1)$ and $0 < a_{12}a_{21} + a_{in}a_{n1} < 1$. Then

(i) $\rho\left(\hat{\mathcal{L}}_{r,\omega}\right) < \rho(\mathcal{L}_{r,\omega})$, if $\rho\left(\mathcal{L}_{r,\omega}\right) < 1$;

(ii) $\rho\left(\hat{\mathcal{L}}_{r,\omega}\right) = \rho(\mathcal{L}_{r,\omega})$, if $\rho\left(\mathcal{L}_{r,\omega}\right) = 1$;

(iii) $\rho\left(\hat{\mathcal{L}}_{r,\omega}\right) > \rho(\mathcal{L}_{r,\omega})$, if $\rho\left(\mathcal{L}_{r,\omega}\right) > 1$.

PROOF: It is established, from Theorem 1, that $\mathcal{L}_{r,\omega}$ and $\hat{\mathcal{L}}_{r,\omega}$ are nonnegative and irreducible matrices. Therefore, suppose $\eta = \rho(\mathcal{L}_{r,\omega})$, then by Lemma 1 there exists a positive vector y, such that

$$\mathcal{L}_{r,\omega}y = \eta y$$

Equivalently,

$$(I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]y = \eta y$$
$$[(1-\omega)I + (\omega - r)L + \omega U] = \eta(I - rL)$$
$$\omega U = (\eta + \omega - 1)I + (r - \omega - \eta r)L \qquad (13)$$

Therefore, for this $y > 0$,

$$\hat{\mathcal{L}}_{r,\omega}y - \eta y = \left(\hat{D} - r\hat{L}\right)^{-1}\left[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}\right]y - \eta y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}\left[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}\right]y - \eta\left(\hat{D} - r\hat{L}\right)^{-1}\left(\hat{D} - r\hat{L}\right)y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U} - \eta\left(\hat{D} - r\hat{L}\right)]y$$

From the identity,

$$\eta\left(\hat{D} - r\hat{L}\right) = \eta(1-r)\hat{D} + \eta r\left(\hat{D} - \hat{L}\right),$$

it implies

$$\hat{\mathcal{L}}_{r,\omega}y - \eta y = \left(\hat{D} - r\hat{L}\right)^{-1}[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U} - \eta(1-r)\hat{D} - \eta r(\hat{D} - \hat{L})]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[\hat{D} - \omega\hat{D} + \omega\hat{L} - r\hat{L} + \omega\hat{U} - \eta\hat{D} + \eta r\hat{D} - \eta r\hat{D} + \eta r\hat{L}]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[\hat{D} - r\hat{D} - \eta\hat{D} + \eta r\hat{D} - \omega\hat{D} + r\hat{D} - \eta r\hat{D} + \omega\hat{L} - r\hat{L} + \eta r\hat{L} + \omega\hat{U}]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(1-\eta)(1-r)\hat{D} - (\omega - r + \eta r)\left(\hat{D} - \hat{L}\right) + \omega\hat{U}]y$$

By employing A1 to A4, we obtain

$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(1-\eta)(1-r)(I + D_1) - (\omega - r + \eta r)(I + D_1) + (\omega - r + \eta r)(L + L_{\hat{S}} + L_1) + \omega(U + U_{\hat{S}} + U_1)]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(1 - \omega - \eta)(I + D_1) + (\omega - r + \eta r)(L + L_{\hat{S}} + L_1) + \omega(U + U_{\hat{S}} + U_1)]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(1 - \omega - \eta)I + \omega U - (r - \omega - \eta r)L + (1 - \omega - \eta)D_1 + (\omega - r + \eta r)(L_{\hat{S}} + L_1) + \omega(U_{\hat{S}} + U_1)]y$$

From (13),

$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(1-\eta)D_1 - \omega(D_1 - L_1 - U_1) + \omega(L_{\hat{S}} + U_{\hat{S}}) - r(1-\eta)(L_{\hat{S}} + L_1)]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta - 1)(-D_1) + (\eta - 1)(rL_{\hat{S}} + rL_1) - \omega\left(-\hat{S}L - \hat{S}U\right) + \omega(-\hat{S})]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta - 1)(-D_1 + rL_{\hat{S}} + rL_1) + \omega\hat{S}L + \omega\hat{S}U - \omega\hat{S}]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta - 1)(-D_1 + rL_{\hat{S}} + rL_1) + (1-\omega)\hat{S} + \omega\hat{S}U - \hat{S}(I - \omega L)]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta - 1)(-D_1 + rL_{\hat{S}} + rL_1) + (1-\omega)\hat{S} + \omega\hat{S}L - r\hat{S}L + \omega\hat{S}U - \hat{S}I + r\hat{S}L)]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta - 1)(-D_1 + rL_{\hat{S}} + rL_1) + (1-\omega)\hat{S} + (\omega - r)\hat{S}L + \omega\hat{S}U - \hat{S}(I - rL)]y$$
$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta - 1)(-D_1 + rL_{\hat{S}} + rL_1) + \hat{S}\{(1-\omega) + (\omega - r)L + \omega U\} - \hat{S}(I - rL)]y$$

And from (13),

$$(1-\omega)I + (\omega - r)L + \omega U = \eta(I-r)L \quad (14)$$

$$\hat{\mathcal{L}}_{r,\omega}y - \eta y = \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta-1)(-D_1 + rL_{\hat{S}} + rL_1) + \eta\hat{S}(I-rL) - \hat{S}(I-rL)]y$$

$$= \left(\hat{D} - r\hat{L}\right)^{-1}[(\eta-1)(-D_1 + rL_{\hat{S}} + rL_1) + (\eta - 1)\hat{S}(I-rL)]y$$

By employing (14),

$$= (\eta-1)\left(\hat{D} - r\hat{L}\right)^{-1}[-D_1 + rL_{\hat{S}} + rL_1 + [(1-\omega)\hat{S} + (\omega-r)\hat{S}L + \omega\hat{S}U]/\eta]y$$

$$= [(\eta-1)/\eta]\left(\hat{D} - r\hat{L}\right)^{-1}[-\eta D_1 + r\eta L_{\hat{S}} + r\eta L_1 + (1-\omega)\hat{S} + (\omega-r)\hat{S}L + \omega\hat{S}U]y$$

It is obvious that $-\eta D_1 + r\eta L_{\hat{S}} + r\eta L_1 \geq 0$, provided $a_{q,q+1}a_{q+1,1} + a_{q1} \geq 0\ (q = 2, \cdots, n-1)$ and $\eta r a_{n1} - (1-\omega)a_{n1} \geq 0$, $(1-\omega)\hat{S} \geq 0$, $(\omega - r)\hat{S}L \geq 0$ and $\omega\hat{S}U \geq 0$. Suppose $\hat{D} - r\hat{L}$ is a splitting of some matrix X. From observation, $\hat{D}$ is an $M-\text{matrix}$ and $r\hat{L} \geq 0$. Consequently, $\hat{D} - r\hat{L}$ is an $M-\text{splitting of } X$. Also, $r\hat{D}^{-1}\hat{L}$, being a strictly lower triangular matrix, has its eigenvalues lying on its main diagonal, and they are all zeros. Therefore, $\rho(r\hat{D}^{-1}\hat{L}) = 0 < 1$. And by Lemma 3, X is a nonsingular $M-\text{matrix}$. consequently, $X^{-1} = \left(\hat{D} - r\hat{L}\right)^{-1} \geq 0$. We are now ready to deduce $(i)-(iii)$, by employing Lemma 2 thus.

(1) If $\eta < 1$, then $\hat{\mathcal{L}}_{r,\omega}y - \eta y \leq 0$ but not equal to 0. Therefore, $\hat{\mathcal{L}}_{r,\omega}y \leq \eta y$. By Lemma 2, we obtain $\rho(\hat{\mathcal{L}}_{r,\omega}) < \eta = \rho(\mathcal{L}_{r,\omega})$.

(2) If $\eta = 1$, then $\hat{\mathcal{L}}_{r,\omega}y - \eta y = 0$. Therefore, $\hat{\mathcal{L}}_{r,\omega}y = \eta y$. By Lemma 2, we obtain $\rho(\hat{\mathcal{L}}_{r,\omega}) = \eta = \rho(\mathcal{L}_{r,\omega})$.

(3) If $\eta > 1$, then $\hat{\mathcal{L}}_{r,\omega}y - \eta y \geq 0$ but not equal to 0. Therefore, $\hat{\mathcal{L}}_{r,\omega}y \geq \eta y$. By Lemma 2, we obtain $\rho(\hat{\mathcal{L}}_{r,\omega}) > \eta = \rho(\mathcal{L}_{r,\omega})$.

□

Following Kohno *et al.* (1997), a more general case of the preconditioner introduced in this work is advanced by introducing the preconditioner $P = I + \hat{s}(\alpha)$, where

$$\hat{s}(\alpha) = \begin{bmatrix} 0 & -\alpha_1 a_{12} & 0 & \cdots & -a_{1n} \\ -a_{21} & 0 & -\alpha_2 a_{23} & \cdots & 0 \\ -a_{31} & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & -\alpha_{n-1}a_{n-1,n} \\ -\alpha_n a_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

for the preconditioned linear system

$$\hat{A}(\alpha)x = \hat{b}(\alpha) \quad (15)$$

Correspondingly, the iteration matrix of the preconditioned AOR method takes the form

$$\hat{\mathcal{L}}_{r,\omega}(\alpha) = (\hat{D}(\alpha) - r\hat{L}(\alpha))^{-1}\left[(1-\omega)\hat{D}(\alpha) + (\omega - r)\hat{L}(\alpha) + \omega\hat{U}(\alpha)\right]$$

Theorem 3 Let $\mathcal{L}_{r,\omega}$ and $\hat{\mathcal{L}}_{r,\omega}(\alpha)$ be the AOR iterative matrices corresponding to the linear systems (1) and (15) respectively. Suppose $0 \leq r \leq \omega \leq 1\ (\omega \neq 0, r \neq 1)$, A is an irreducible $L-\text{matrix}$ with $0 < \alpha_n a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + \alpha_i a_{i,i+1}a_{i+1,i} < 1\ (i = 2(1)n-1)$ and $0 < \alpha_1 a_{12}a_{21} + a_{in}a_{n1} < 1$. Then $\mathcal{L}_{r,\omega}$ and $\hat{\mathcal{L}}_{r,\omega}$ are nonnegative and irreducible matrices.

Theorem 4 Let $\mathcal{L}_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]$ and $\hat{\mathcal{L}}_{r,\omega}(\alpha) = (\hat{D}(\alpha) - r\hat{L}(\alpha))^{-1}\left[(1-\omega)\hat{D}(\alpha) + (\omega - r)\hat{L}(\alpha) + \omega\hat{U}(\alpha)\right]$ be the AOR and preconditioned AOR iterative matrices respectively. Suppose $0 \leq r \leq \omega \leq 1\ (\omega \neq 0, r \neq 1)$, A is an irreducible $L-\text{matrix}$ with $0 < \alpha_n a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + \alpha_i a_{i,i+1}a_{i+1,i} < 1\ (i = 2(1)n-1)$ and $0 < \alpha_1 a_{12}a_{21} + a_{in}a_{n1} < 1$. Then

(i) $\rho(\hat{\mathcal{L}}_{r,\omega}(\alpha)) < \rho(\mathcal{L}_{r,\omega})$, if $\rho(\mathcal{L}_{r,\omega}) < 1$;

(ii) $\rho(\hat{\mathcal{L}}_{r,\omega}(\alpha)) = \rho(\mathcal{L}_{r,\omega})$, if $\rho(\mathcal{L}_{r,\omega}) = 1$;

(iii) $\rho(\hat{\mathcal{L}}_{r,\omega}(\alpha)) > \rho(\mathcal{L}_{r,\omega})$, if $\rho(\mathcal{L}_{r,\omega}) > 1$.

Proofs of Theorems 3 and 4 follow the same pattern as in the proofs of Theorems 1 and 2. Thus omitted.

Theorem 5 Let $0 < r_1 < r_2 \leq \omega \leq 1$ and $A^{-1} \geq 0$. Under the hypothesis of Theorem 2, then $1 > \rho(\hat{\mathcal{L}}_{r_1,\omega}) > \rho(\hat{\mathcal{L}}_{r_2,\omega}) > 0$, if $0 < \eta < 1$.

PROOF: Let

$$\hat{A} = \hat{M}_{r,\omega} - \hat{N}_{r,\omega}$$

where $\hat{M}_{r,\omega} = (1/\omega)(\hat{D} - r\hat{L})$ and $\hat{N}_{r,\omega} = (1/\omega)[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}]$. Suppose also that $\hat{A} = \hat{M}_{r_1,\omega} - \hat{N}_{r_1,\omega}$ and $\hat{A} = \hat{M}_{r_2,\omega} - \hat{N}_{r_2,\omega}$ are two regular splittings of $\hat{A}$, where $\hat{M}_{r_1,\omega} = (1/\omega)(\hat{D} - r_1\hat{L})$, $\hat{N}_{r_1,\omega} = (1/\omega)[(1-\omega)\hat{D} + (\omega - r_1)\hat{L} + \omega\hat{U}]$, $\hat{M}_{r_2,\omega} = (1/\omega)(\hat{D} - r_2\hat{L})$ and $\hat{N}_{r_2,\omega} = (1/\omega)[(1-\omega)\hat{D} + (\omega - r_2)\hat{L} + \omega\hat{U}]$. Since $0 < r_1 < r_2 \leq \omega \leq 1$, then $\hat{N}_{r_1,\omega} \geq \hat{N}_{r_2,\omega} \geq 0$, equality excluded, then in the light of Lemma 4, we have that

$$1 > \rho(\hat{\mathcal{L}}_{r_1,\omega}) > \rho(\hat{\mathcal{L}}_{r_2,\omega}) > 0$$

□

Consequently, the following theorem applies.

Theorem 6 Let $0 < r_1 < r_2 \leq \omega \leq 1$ and $A^{-1} \geq 0$. Under the hypothesis of Theorem 4, then $1 > \rho(\hat{\mathcal{L}}_{r_1,\omega}(\alpha)) > \rho(\hat{\mathcal{L}}_{r_2,\omega}(\alpha)) > 0$, if $0 < \eta < 1$.

When $\omega = r$ in (6), the AOR method reduces to the SOR method. As a consequence, the following corollaries are easily obtained.

Corollary 1 Let $\mathcal{L}_\omega = (I - \omega L)^{-1}[(1-\omega)I + \omega U]$ and $\hat{\mathcal{L}}_\omega = (\hat{D} - \omega\hat{L})^{-1}[(1-\omega)\hat{D} + \omega\hat{U}]$ be the SOR and preconditioned SOR iterative matrices respectively. Suppose $0 < \omega < 1$, A is an irreducible $L-\text{matrix}$ with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1\ (i = 2(1)n-1)$ and $0 < a_{12}a_{21} + a_{in}a_{n1} < 1$. Then

(i) $\rho(\hat{\mathcal{L}}_\omega) < \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) < 1$;

(ii) $\rho(\hat{\mathcal{L}}_\omega) = \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) = 1$;

(iii) $\rho(\hat{\mathcal{L}}_\omega) > \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) > 1$.

Corollary 2 Let $\mathcal{L}_\omega = (I - \omega L)^{-1}[(1-\omega)I + \omega U]$ and $\hat{\mathcal{L}}_\omega(\alpha) = (\hat{D}(\alpha) - \omega\hat{L}(\alpha))^{-1}\left[(1-\omega)\hat{D}(\alpha) + \omega\hat{U}(\alpha)\right]$ be the AOR and preconditioned AOR iterative matrices respectively. Suppose $0 < \omega < 1$, A is an irreducible $L-\text{matrix}$ with $0 < \alpha_n a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + \alpha_i a_{i,i+1}a_{i+1,i} < 1\ (i = 2(1)n-1)$ and $0 < \alpha_1 a_{12}a_{21} + a_{in}a_{n1} < 1$. Then

(i) $\rho(\hat{\mathcal{L}}_\omega(\alpha)) < \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) < 1$;

(ii) $\rho(\hat{\mathcal{L}}_\omega(\alpha)) = \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) = 1$;

(iii) $\rho(\hat{\mathcal{L}}_\omega(\alpha)) > \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) > 1$.

Corollary 3 Let $0 < \omega_1 < \omega_2 \leq 1$ and $A^{-1} \geq 0$. Under the hypothesis of Corollary 1, then $1 > \rho(\hat{\mathcal{L}}_{\omega_1}) > \rho(\hat{\mathcal{L}}_{\omega_2}) > 0$, if $0 < \eta < 1$.

Corollary 4 Let $0 < \omega_1 < \omega_2 \leq 1$ and $A^{-1} \geq 0$. Under the hypothesis of Corollary 2, then $1 > \rho(\hat{\mathcal{L}}_{\omega_1}(\alpha)) > \rho(\hat{\mathcal{L}}_{\omega_2}(\alpha)) > 0$, if $0 < \eta < 1$.

RESULTS AND DISCUSSION

Numerical Examples

Numerical examples are presented to verify the theorems. These calculations were performed using Maple 2019 software package. Let the coefficient matrix A of the linear system (1) be given by

$$A = \begin{pmatrix} 1 & -1/6 & -1/7 & -1/8 & -1/6 & -1/7 \\ -1/8 & 1 & -1/6 & -1/7 & -1/8 & -1/6 \\ -1/6 & -1/8 & 1 & -1/6 & -1/7 & -1/8 \\ -1/7 & -1/6 & -1/8 & 1 & -1/6 & -1/7 \\ -1/8 & -1/7 & -1/6 & -1/8 & 1 & -1/6 \\ -1/6 & -1/8 & -1/7 & -1/6 & -1/8 & 1 \end{pmatrix}$$

The corresponding iterative matrices of the matrix A for various iteration processes discussed here are computed alongside their spectral radii. The results are presented in Tables 1 – 4. In the following tables, $\mathcal{L}_{r,\omega}(1)$ denote the iterative matrix in Ndanusa and Adeboye (2012) under the conditions of Theorem 2 and $\mathcal{L}_{r,\omega}(2)$ denote the iterative matrix in Wang (2019) under the conditions of Theorem 2.

Table 1: Numerical validation of Theorem 2

ω	r	$\rho(\hat{\mathcal{L}}_{r,\omega})$	$\rho(\mathcal{L}_{r,\omega}(1))$	$\rho(\mathcal{L}_{r,\omega}(2))$	$\rho(\mathcal{L}_{\omega})$
0.95	0.85	0.4820339009	0.4827830342	0.5172376634	0.6205255277
0.90	0.80	0.5268521180	0.5276840729	0.5606287531	0.6518574112
0.80	0.70	0.6059612644	0.6068707293	0.6364965183	0.7083014149
0.70	0.65	0.6652871216	0.6661423662	0.6921471055	0.7516743194
0.60	0.50	0.7352605039	0.7361121449	0.7584515241	0.8026767336
0.50	0.40	0.7897395742	0.7904979556	0.8090612095	0.8429614522
0.40	0.30	0.8391415217	0.8397793239	0.8545524533	0.8796743773
0.30	0.20	0.8843014020	0.8847984990	0.8958008930	0.9133536720
0.20	0.10	0.9258516890	0.9261931550	0.9334669460	0.9444202400
0.10	0.05	0.9636211849	0.9637940570	0.9674142960	0.9727210830

Table 1 displays the results of comparing the spectral radii of the proposed preconditioned method, for varied values of ω and r, with those of other preconditioned methods in literature. It reveals that the rate of convergence of the proposed method is faster than those of Ndanusa and Adeboye (2012) and Wang (2019), even as all the preconditioned methods proved faster than the unpreconditioned AOR.

Table 2: Numerical validation of Theorem 4

α	ω	r	$\rho(\mathcal{L}_{r,\omega}(\alpha))$	$\rho(\hat{\mathcal{L}}_{r,\omega})$	$\rho(\mathcal{L}_{\omega})$
(1,4,1,3,5,2)	**0.95**	0.85	0.2933986209	0.4820339009	0.6205255277
(2,1,3,1,1,2)	**0.90**	0.80	0.4654973618	0.5268521180	0.6518574112
(1,1,2,1,1,1)	**0.80**	0.70	0.5911674768	0.6059612644	0.7083014149
(2,1,2,3,6,8)	**0.70**	0.65	0.6177208139	0.6652871216	0.7516743194

In Table 2, the effect of parameterization of the proposed preconditioned method is evident in the convergence rate of the parameterized method being faster than that of the unparameterized preconditioned method.

Table 3: Numerical validation of Theorem 5

ω	r_1	r_2	$\rho(\hat{\mathcal{L}}_{r_2,\omega})$	$\rho(\hat{\mathcal{L}}_{r_1,\omega})$
0.95	0.80	0.85	0.4820339009	0.5005661270
0.95	0.70	0.75	0.5171135993	0.5320790053
0.95	0.60	0.65	0.5457468061	0.5583268321
0.95	0.50	0.55	0.5699791445	0.5808291307

Results of Table 3 affirms that for two regular splitting of the preconditioned matrix $\hat{A} = \hat{M}_{r_1,\omega} - \hat{N}_{r_1,\omega} = \hat{M}_{r_2,\omega} - \hat{N}_{r_2,\omega}$ with $\hat{N}_{r_1,\omega} \geq \hat{N}_{r_2,\omega} \geq 0$ (since $0 < r_1 < r_2 \leq \omega \leq 1$), then $1 > \rho(\hat{\mathcal{L}}_{r_1,\omega}) > \rho(\hat{\mathcal{L}}_{r_2,\omega}) > 0$.

Table 4: Numerical validation of Corollary 3

ω_1	ω_2	$\rho(\hat{\mathcal{L}}_{\omega_2})$	$\rho(\hat{\mathcal{L}}_{\omega_1})$
0.10	0.15	0.9432965130	0.9629258430
0.40	0.50	0.7793837568	0.8317916620
0.60	0.65	0.6891951876	0.7210485253
0.80	0.90	0.4892856621	0.5794241055

Similar to Table 3, Table 4 seeks to validate Corollary 3, where there exists two regular splitting of the preconditioned matrix $\hat{A} = \hat{M}_{\omega_1} - \hat{N}_{\omega_1} = \hat{M}_{\omega_2} - \hat{N}_{\omega_2}$ such that $\hat{N}_{\omega_1} \geq \hat{N}_{\omega_2} \geq 0$, then $1 > \rho(\hat{\mathcal{L}}_{\omega_1}) > \rho(\hat{\mathcal{L}}_{\omega_2}) > 0$.

Conclusion

In this study, we investigated the modified preconditioned AOR (SOR) iterative method in order to discover the most effective method for accelerating its convergence speed towards solution of linear systems. Results of numerical experiments undertaken to validate the proposed convergence theorems, demonstrated the effectiveness of the new method by not only improving the convergence rate of the AOR method, but also in its outperformance among three preconditioned methods considered.

REFERENCES

Behzadi, R. (2019). A new class AOR preconditioner for *L*-matrices. *Journal of Mathematical Research with Applications,* 39(1): 101-110.

Darvishi, M. T., Hessari, P. and Shin, B. C. (2011). Preconditioned modified AOR method for systems of linear equations. *International Journal for numerical methods in biomedical engineering,* 27: 758–769.

Dehghan, M. and Hajarian, M. (2009). Improving preconditioned SOR-type iterative methods for *L*-matrices. *International Journal for Numerical Methods in Biomedical Engineering,* 27: 774-784.

Evans, D. J., Martins, M. M., Trigo, M. E. (2001). The AOR iterative method for new preconditioned linear systems. *Journal of Computational and Applied Mathematics*, 132: 461–466.

Gunawardena, A. D., Jain, S. K. and Snyder, L. (1991). Modified iterative methods for consistent linear systems. *Linear Algebra and its Applications,* 154-156, 123-143.

Hadjidimos, A. (1978). Accelerated overrelaxation method. *Mathematics of Computation*, (32) 141: 149-157.

Huang, Z., Wang, L., Xu, Z. and Cui, J. (2016). Convergence analysis of some new preconditioned AOR iterative methods for *L*-matrices. *IAENG International Journal of Applied Mathematics,* 46(2): 202-209.

Kohno, T., Kotakemori, H., Niki, H. and Usui, M. (1997). Improving modified Gauss–Seidel method for Z-matrices. *Linear Algebra Appl.*, 267: 113–123.

Li, A. (2011). Improving AOR iterative methods for irreducible *L* – matrices. *Engineering Letters,* 19(1): 46 – 49.

Li, Y., Li, C., Wu, S. (2007). Improvements of preconditioned AOR iterative method for *L*-matrices. *Journal of Computational and Applied Mathematics*, 206: 656 – 665.

Li, W. and Sun, W. (2000). Modified Gauss – Seidel type methods and Jacobi type methods for *Z* - matrices. *Linear Algebra and its Applications,* 317, 227-240.

Ndanusa, A. and Adeboye, K. R. (2012). Preconditioned SOR iterative methods for L-matrices. *American Journal of Computational and Applied Mathematics,* 2(6): 300-305.

Saad, Y. (2000). *Iterative methods for sparse linear* systems. 2nd ed. Philadelphia: SIAM.

Varga, R. S. (1962). *Matrix Iterative Analysis.* Englewood Cliffs, New York: Prentice -Hall.

Wang, H. (2019). A preconditioned AOR iterative scheme for systems of linear equations with *L*-matrices. *Open Mathematics*, 17: 1764–1773.

Wang, L. and Song, Y. (2009). Preconditioned AOR iterative methods for *M* – matrices. *Journal of Computational and Applied Mathematics,* 226: 114 – 124.

Wu, S. L. and Huang, T. (2007). A modified AOR-type iterative method for *L*-matrix linear systems. *Anziam Journal,* 49: 281 – 292.

Wu, M., Wang, L., Song, Y. (2007). Preconditioned AOR iterative method for linear systems. *Applied Numerical Mathematics*, 57: 672–685.

Young, D. M. (1971). *Iterative solution of large linear systems.* New York-London: Academic Press.

Yun, J. H. and Kim, S. W. (2008). Convergence of the preconditioned AOR method for irreducible *L*-matrices. *Applied Mathematics and Computation*, 201: 56–64