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Full Length Research Article

AN ANALYSIS OF ALGEBRAIC PATTERN OF A FIRST ORDER AND AN EXTENDED SECOND ORDER RUNGE-KUTTA TYPE METHOD

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ABSTRACT
The algebraic pattern of a 6-Stage Block Hybrid Runge –Kutta Type Methods (BHRKTM) for the solution of Ordinary Differential Equations (ODEs) is carefully analyzed. The analysis of the methods expressed in the Butcher Tableau led to the evolvement of two equations that satisfy the Runge – Kutta consistency conditions. The reason behind the uniform order and error constant for the developed first order and extended second order methods is explained using the theory of linear transformation and monomorphism. The pattern was retained during the transformation.

Keywords: Linear Transformation, Implicit, Runge-Kutta type, Algebraic pattern

INTRODUCTION
Ordinary Differential Equations arise frequently in the study of the physical problems. Unfortunately, many cannot be solved exactly (Akinfenwa et *al,* 2011). This is why the ability to solve these equations numerically is important. Onumanyi (2004) describes initial value problems as one of the most frequently occurring mathematical problems in numerical analysis.
Traditionally, mathematicians have used one of two classes of methods for solving numerically ordinary differential equations. These are Runge –Kutta methods and Linear Multistep Methods (LMM) (Rattenbury,2005).
Runge- Kutta (RK) methods are very popular because of their symmetrical forms, they have simple coefficients, are very efficient and numerically stable (Badmus, 2013). The methods are fairly simple to program, easy to implement and their truncation error can be controlled in a straighter manner than multistep methods (Muhammad R, *et al* 2015a).
The application of Runge-Kutta methods have provided many satisfactory solutions to many problems that have been regarded as insolvable (Mackenzie, 2000). The popularity and the explosive growth of these methods, coupled with the amount of research effort being undertaken are further evidence that the applications are still the leading source of inspiration for mathematical creativity (Muhammad R, *et al* 2015b).
A linear transformation (Homomorphism) can be defined as when a function T between two vector spaces $T: V \rightarrow W$ preserves the operations of addition if v_1 and $v_2 \in V$ then

$$T(v_1 + v_2) = T(v_1) + T(v_2) \tag{1}$$

And scalar multiplication if $v \epsilon\, V$ and $r \epsilon R$, then

$$T(r.v) = rT(v) \tag{2}$$

(Agam, 2013).

A homomorphism that is one to one or a mono is called a monomorphism.
The monomorphism Transformation preserves its algebraic structure and the order of the Domain into its Range.

METHODOLOGY
A first and second order 6-Stages Block Hybrid Runge-Kutta Type Methods (BHRKTMs) of uniform order $(5,5,5,5,5)^T$ are given in equations (3a|&3b) and (4a&4b) respectively.

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + h(0k_1 + \tfrac{242}{225}k_2 - \tfrac{4807}{5760}k_3 + \tfrac{2197}{5760}k_4 - \tfrac{1427}{9600}k_5 + \tfrac{151}{5760}k_6) \\
y_{n+1} &= y_n + h\left(0k_1 + \tfrac{2008}{1575}k_2 - \tfrac{179}{360}k_3 + \tfrac{119}{360}k_4 - \tfrac{79}{600}k_5 + \tfrac{59}{2520}k_6\right) \\
y_{n+2} &= y_n + h\left(0k_1 + \tfrac{1856}{1575}k_2 + \tfrac{4}{45}k_3 + \tfrac{41}{45}k_4 - \tfrac{16}{75}k_5 + \tfrac{11}{315}k_6\right) \\
y_{n+3} &= y_n + h\left(0k_1 + \tfrac{216}{175}k_2 - \tfrac{3}{40}k_3 + \tfrac{63}{40}k_4 + \tfrac{51}{200}k_5 + \tfrac{3}{280}k_6\right) \\
y_{n+4} &= y_n + h\left(0k_1 + \tfrac{256}{225}k_2 + \tfrac{8}{45}k_3 + \tfrac{52}{45}k_4 + \tfrac{88}{75}k_5 + \tfrac{16}{45}k_6\right)
\end{aligned}\right\} \quad (3a)$$

Where

$$\left.\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \tfrac{1}{2}h, y_n + h\left(0k_1 + \tfrac{242}{225}k_2 - \tfrac{4807}{5760}k_3 + \tfrac{2197}{5760}k_4 - \tfrac{1427}{9600}k_5 + \tfrac{151}{5760}k_6\right)) \\
k_3 &= f(x_n + h, y_n + h\left(0k_1 + \tfrac{2008}{1575}k_2 - \tfrac{179}{360}k_3 + \tfrac{119}{360}k_4 - \tfrac{79}{600}k_5 + \tfrac{59}{2520}k_6\right)) \\
k_4 &= f(x_n + 2h, y_n + h\left(0k_1 + \tfrac{1856}{1575}k_2 + \tfrac{4}{45}k_3 + \tfrac{41}{45}k_4 - \tfrac{16}{75}k_5 + \tfrac{11}{315}k_6\right)) \\
k_5 &= f(x_n + 3h, y_n + h\left(0k_1 + \tfrac{216}{175}k_2 - \tfrac{3}{40}k_3 + \tfrac{63}{40}k_4 + \tfrac{51}{200}k_5 + \tfrac{3}{280}k_6\right)) \\
k_6 &= f(x_n + 4h, y_n + h\left(0k_1 + \tfrac{256}{225}k_2 + \tfrac{8}{45}k_3 + \tfrac{52}{45}k_4 + \tfrac{88}{75}k_5 + \tfrac{16}{45}k_6\right))
\end{aligned}\right\} \quad (3b)$$

And

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \tfrac{1}{2}hy'_n + h^2\left(0k_1 + \tfrac{583}{1500}k_2 - \tfrac{24937}{57600}k_3 + \tfrac{5339}{19200}k_4 - \tfrac{13297}{96000}k_5 + \tfrac{1711}{57600}k_6\right), \\
y'_{n+\frac{1}{2}} &= y'_n + h\left(0k_1 + \tfrac{242}{225}k_2 - \tfrac{4807}{5760}k_3 + \tfrac{2197}{5760}k_4 - \tfrac{1427}{9600}k_5 + \tfrac{151}{5760}k_6\right) \\
y_{n+1} &= y_n + hy'_n + h^2\left(0k_1 + \tfrac{7804}{7875}k_2 - \tfrac{58}{75}k_3 + \tfrac{797}{1800}k_4 - \tfrac{301}{1500}k_5 + \tfrac{169}{4200}k_6\right), \\
y'_{n+1} &= y'_n + h\left(0k_1 + \tfrac{2008}{1575}k_2 - \tfrac{179}{360}k_3 + \tfrac{119}{360}k_4 - \tfrac{79}{600}k_5 + \tfrac{59}{2520}k_6\right) \\
y_{n+2} &= y_n + 2hy'_n + h^2\left(0k_1 + \tfrac{1952}{875}k_2 - \tfrac{208}{225}k_3 + \tfrac{76}{75}k_4 - \tfrac{148}{375}k_5 + \tfrac{118}{1575}k_6\right), \\
y'_{n+2} &= y'_n + h\left(0k_1 + \tfrac{1856}{1575}k_2 + \tfrac{4}{45}k_3 + \tfrac{41}{45}k_4 - \tfrac{16}{75}k_5 + \tfrac{11}{315}k_6\right) \\
y_{n+3} &= y_n + 3hy'_n + h^2\left(0k_1 + \tfrac{2988}{875}k_2 - \tfrac{87}{100}k_3 + \tfrac{459}{200}k_4 - \tfrac{54}{125}k_5 + \tfrac{129}{1400}k_6\right), \\
y'_{n+3} &= y'_n + h\left(0k_1 + \tfrac{216}{175}k_2 - \tfrac{3}{40}k_3 + \tfrac{63}{40}k_4 + \tfrac{51}{200}k_5 + \tfrac{3}{280}k_6\right) \\
y_{n+4} &= y_n + 4hy'_n + h^2\left(0k_1 + \tfrac{5248}{1125}k_2 - \tfrac{24}{25}k_3 + \tfrac{856}{225}k_4 + \tfrac{104}{375}k_5 + \tfrac{16}{75}k_6\right), \\
y'_{n+4} &= y'_n + h\left(0k_1 + \tfrac{256}{225}k_2 + \tfrac{8}{45}k_3 + \tfrac{52}{45}k_4 + \tfrac{88}{75}k_5 + \tfrac{16}{45}k_6\right)
\end{aligned}\right\} \quad (4a)$$

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$$
\left.\begin{aligned}
k_1 &= f(x_n, y_n, y_n')\\
k_2 &= f\Big(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hy_n' + h^2\Big(0k_1 + \tfrac{583}{1500}k_2 - \tfrac{24937}{57600}k_3 + \tfrac{5339}{19200}k_4 - \tfrac{13297}{96000}k_5 + \tfrac{1711}{57600}k_6\Big),\\
&\qquad y_n' + h\Big(0k_1 + \tfrac{242}{225}k_2 - \tfrac{4807}{5760}k_3 + \tfrac{2197}{5760}k_4 - \tfrac{1427}{9600}k_5 + \tfrac{151}{5760}k_6\Big)\Big)\\
k_3 &= f\Big(x_n + h, y_n + hy_n' + h^2\Big(0k_1 + \tfrac{7804}{7875}k_2 - \tfrac{58}{75}k_3 + \tfrac{797}{1800}k_4 - \tfrac{301}{1500}k_5 + \tfrac{169}{4200}k_6\Big),\\
&\qquad y_n' + h\Big(0k_1 + \tfrac{2008}{1575}k_2 - \tfrac{179}{360}k_3 + \tfrac{119}{360}k_4 - \tfrac{79}{600}k_5 + \tfrac{59}{2520}k_6\Big)\Big)\\
k_4 &= f\Big(x_n + 2h, y_n + 2hy_n' + h^2\Big(0k_1 + \tfrac{1952}{875}k_2 - \tfrac{208}{225}k_3 + \tfrac{76}{75}k_4 - \tfrac{148}{375}k_5 + \tfrac{118}{1575}k_6\Big),\\
&\qquad y_n' + h\Big(0k_1 + \tfrac{1856}{1575}k_2 + \tfrac{4}{45}k_3 + \tfrac{41}{45}k_4 - \tfrac{16}{75}k_5 + \tfrac{11}{315}k_6\Big)\Big)\\
k_5 &= f\Big(x_n + 3h, y_n + 3hy_n' + h^2\Big(0k_1 + \tfrac{2988}{875}k_2 - \tfrac{87}{100}k_3 + \tfrac{459}{200}k_4 - \tfrac{54}{125}k_5 + \tfrac{129}{1400}k_6\Big),\\
&\qquad y_n' + h\Big(0k_1 + \tfrac{216}{175}k_2 - \tfrac{3}{40}k_3 + \tfrac{63}{40}k_4 + \tfrac{51}{200}k_5 + \tfrac{3}{280}k_6\Big)\Big)\\
k_6 &= f\Big(x_n + 4h, y_n + 4hy_n' + h^2\Big(0k_1 + \tfrac{5248}{1125}k_2 - \tfrac{24}{25}k_3 + \tfrac{856}{225}k_4 + \tfrac{104}{375}k_5 + \tfrac{16}{75}k_6\Big),\\
&\qquad y_n' + h\Big(0k_1 + \tfrac{256}{225}k_2 + \tfrac{8}{45}k_3 + \tfrac{52}{45}k_4 + \tfrac{88}{75}k_5 + \tfrac{16}{45}k_6\Big)\Big)
\end{aligned}\right\} \quad (4b)
$$

Numerical Experiment

Consider the problem

$$y'' = -y \qquad y(0) = 1, \quad y'(0) = 1 \qquad h = 0.1, \qquad 0 \leq x \leq 1$$

Exact Solution

$y(x) = \cos x + \sin x$

Applying the Runge-Kutta Type Method (RKTM) to this problem yield the following results.

Table 1: Absolute Error for Problem Using the methods

x	Exact Solution	Computed Solution	Error
0.1	1.094837582	1.094837561	2.1E-08
0.2	1.178735909	1.178735881	2.8E-08
0.3	1.250856696	1.250856674	2.2E-08
0.4	1.310479336	1.310479296	4.0E-08
0.5	1.357008101	1.357008037	6.3E-08
0.6	1.389978088	1.389978017	7.1E-08
0.7	1.409059874	1.409059810	6.5E-08
0.8	1.4140628	1.494062714	8.6E-08
0.9	1.404936878	1.404936770	1.1E-07
1.0	1.381773291	1.381773179	1.1E-07

RESULTS AND DISCUSSION

The equations $(3a)$ and $(4a)$ are respectively expressed in the Table 2 and Table 3

Table 2

0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{242}{225}$	$-\frac{4807}{5760}$	$\frac{2197}{5760}$	$-\frac{1427}{9600}$	$\frac{151}{5760}$
4	0	$\frac{256}{225}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$
3	0	$\frac{216}{175}$	$-\frac{3}{40}$	$\frac{63}{40}$	$\frac{51}{200}$	$\frac{3}{280}$
2	0	$\frac{1856}{1575}$	$\frac{4}{45}$	$\frac{41}{45}$	$-\frac{16}{75}$	$\frac{11}{315}$
1	0	$\frac{2008}{1575}$	$\frac{-179}{360}$	$\frac{119}{360}$	$\frac{-79}{600}$	$\frac{59}{2520}$
	0	$\frac{2008}{1575}$	$\frac{-179}{360}$	$\frac{119}{360}$	$\frac{-79}{600}$	$\frac{59}{2520}$

Table 3

0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{242}{225}$	$-\frac{4807}{5760}$	$\frac{2197}{5760}$	$-\frac{1427}{9600}$	$\frac{151}{5760}$	0	$\frac{583}{1500}$	$-\frac{24937}{57600}$	$\frac{5339}{19200}$	$-\frac{13297}{96000}$	$\frac{1711}{57600}$
4	0	$\frac{256}{225}$	$\frac{8}{45}$	$\frac{52}{45}$	$\frac{88}{75}$	$\frac{16}{45}$	0	$\frac{5248}{1125}$	$-\frac{24}{25}$	$\frac{856}{225}$	$\frac{104}{375}$	$\frac{16}{75}$
3	0	$\frac{216}{175}$	$-\frac{3}{40}$	$\frac{63}{40}$	$\frac{51}{200}$	$\frac{3}{280}$	0	$\frac{2988}{875}$	$-\frac{87}{100}$	$\frac{459}{200}$	$-\frac{54}{125}$	$\frac{129}{1400}$
2	0	$\frac{1856}{1575}$	$\frac{4}{45}$	$\frac{41}{45}$	$-\frac{16}{75}$	$\frac{11}{315}$	0	$\frac{1952}{875}$	$-\frac{208}{225}$	$\frac{76}{75}$	$-\frac{148}{375}$	$\frac{118}{1575}$
1	0	$\frac{2008}{1575}$	$\frac{-179}{360}$	$\frac{119}{360}$	$\frac{-79}{600}$	$\frac{59}{2520}$	0	$\frac{7804}{7875}$	$-\frac{58}{75}$	$\frac{797}{1800}$	$-\frac{301}{1500}$	$\frac{169}{4200}$
	0	$\frac{2008}{1575}$	$\frac{-179}{360}$	$\frac{119}{360}$	$\frac{-79}{600}$	$\frac{59}{2520}$	0	$\frac{7804}{7875}$	$-\frac{58}{75}$	$\frac{797}{1800}$	$-\frac{301}{1500}$	$\frac{169}{4200}$

The Table 2 satisfies the Runge –Kutta conditions for solution of first order since

$(i) \quad \sum_{j=1}^{s} a_{ij} = c_i \qquad (5)$

$(ii) \quad \sum_{j=1}^{s} b_j = 1 \qquad (6)$

Also the Table 3 satisfies the Runge –Kutta conditions for solution of second order since

$(iii) \quad \sum_{j=1}^{s} b_j = \frac{1}{2} \qquad (7)$

We consider the general second order differential equation in the form

$$y'' = f(x, y, y'), y(x_0) = y_0 \quad y'(x_0) = y_0' \qquad (8)$$

$$y'' = f(v), \qquad v = (x, y, y') \qquad (9)$$

$$T(V_i) = T\big(x + c_i h, y + \textstyle\sum_{j=1}^{s} a_{ij} T(V_j), y' + \sum_{j=1}^{s} a_{ij} T'(V_j)\big) \qquad (10)$$

$$= h\,(y' + \textstyle\sum_{j=1}^{6} a_{ij}\, T'(V_j)) = h\,(y' + \sum_{j=1}^{6} a_{ij}\, hm_j) \qquad (11)$$

$$T'(V_j) = hm_j \qquad (12)$$

$$T(V_1) = hy' \qquad (13)$$

$$T(V_2) = h(y' + \tfrac{242}{225}hm_2 - \tfrac{4807}{5760}hm_3 + \tfrac{2197}{5760}hm_4 - \tfrac{1427}{9600}hm_5 + \tfrac{151}{5760}hm_6) \qquad (14)$$

$$T(V_3) = h\begin{pmatrix} y' + \frac{2008}{1575}hm_2 - \frac{179}{360}hm_3 + \\ \frac{119}{360}hm_4 - \frac{79}{600}hm_5 + \frac{59}{2520}hm_6 \end{pmatrix} \quad (15)$$

$$T(V_4) = h\begin{pmatrix} y' + \frac{1856}{1575}hm_2 + \frac{4}{45}hm_3 + \frac{41}{45}hm_4 - \\ \frac{16}{75}hm_5 + \frac{11}{315}hm_6 \end{pmatrix} \quad (16)$$

$$T(V_5) = h(y' + \frac{216}{175}hm_2 - \frac{3}{40}hm_3 + \frac{63}{40}hm_4 + \frac{51}{200}hm_5 + \frac{3}{280}hm_6) \quad (17)$$

$$T(V_6) = h(y' + \frac{256}{225}hm_2 + \frac{8}{45}hm_3 + \frac{52}{45}hm_4 + \frac{88}{75}hm_5 + \frac{16}{45}hm_6) \quad (18)$$

$$m_1 = 0 \quad (19)$$

$$m_2 = f(x + \frac{1}{2}h; y + \frac{1}{2}hy' + \frac{583}{1500}h^2m_2 - \frac{24937}{57600}h^2m_3 + \frac{5339}{19200}h^2m_4 - \frac{13297}{96000}h^2m_5 + \frac{1711}{57600}h^2m_6; y' + \frac{242}{225}hm_2 - \frac{4807}{5760}hm_3 + \frac{2197}{5760}hm_4 - \frac{1427}{9600}hm_5 + \frac{151}{5760}hm_6) \quad (20)$$

$$m_3 = f(x + h; y + hy' + \frac{7804}{7875}h^2m_2 - \frac{58}{75}h^2m_3 + \frac{797}{1800}h^2m_4 - \frac{301}{1500}h^2m_5 + \frac{169}{4200}h^2m_6; y' + \frac{2008}{1575}hm_2 - \frac{179}{360}hm_3 + \frac{119}{360}hm_4 - \frac{79}{600}hm_5 + \frac{59}{2520}hm_6) \quad (21)$$

$$m_4 = f(x + 2h; y + 2hy' + \frac{1952}{875}h^2m_2 - \frac{208}{225}h^2m_3 + \frac{76}{75}h^2m_4 - \frac{148}{375}h^2m_5 + \frac{118}{1575}h^2m_6; y' + \frac{1856}{1575}hm_2 + \frac{4}{45}hm_3 + \frac{41}{45}hm_4 - \frac{16}{75}hm_5 + \frac{11}{315}hm_6) \quad (22)$$

$$m_5 = f(x + 3h; y + 3hy' + \frac{2988}{875}h^2m_2 - \frac{87}{100}h^2m_3 + \frac{459}{200}h^2m_4 - \frac{54}{125}h^2m_5 + \frac{129}{1400}h^2m_6; y' + \frac{216}{175}hm_2 - \frac{3}{40}hm_3 + \frac{63}{40}hm_4 + \frac{51}{200}hm_5 + \frac{3}{280}hm_6) \quad (23)$$

$$m_6 = f(x + 4h; y + 4hy' + \frac{5248}{1125}h^2m_2 - \frac{24}{25}h^2m_3 + \frac{856}{225}h^2m_4 + \frac{104}{375}h^2m_5 + \frac{16}{75}h^2m_6; y' + \frac{256}{225}hm_2 + \frac{8}{45}hm_3 + \frac{52}{45}hm_4 + \frac{88}{75}hm_5 + \frac{16}{45}hm_6) \quad (24)$$

The direct method for solving $y'' = f(x, y, y')$ is

$$y_{n+1} = y_n + b_1T(V_1) + b_2T(V_2) + b_3T(V_3) + b_4T(V_4) + b_5T(V_5) + b_6T(V_6) \quad (25)$$

$$y_{n+1} = y_n + 0T(V_1) + \frac{2008}{1575}T(V_2) - \frac{179}{360}T(V_3) + \frac{119}{360}T(V_4) - \frac{79}{600}T(V_5) + \frac{59}{2520} \quad (26)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{63000}(62432m_2 - 48720m_3 + 27895m_4 - 12642m_5 + 2535m_6) \quad (27)$$

$$y'_{n+1} = y'_n + \frac{h}{12600}(16064m_2 - 6265m_3 + 4165m_4 - 1659m_5 + 295m_6) \quad (28)$$

We made use of the coefficients of the butcher table of the first order RKTM to prove to the second order RKTM. Equation (27) and (28) satisfy the Runge-Kutta consistency conditions of second and first order respectively. This further shows that it is a monomorphism.

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