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PHASE PORTRAIT ANALYSIS FOR NAVIER–STOKES EQUATIONS IN A STRIP WITH OMITTED PRESSURE TERM

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ABSTRACT
In this paper, we study Navier – Stokes Equations in an infinite strip $\Omega = \mathbb{R} \times [-1,1]$. We used the Uniformly Local Spaces as with focus on model situation where pressure has been omitted in the momentum equation. We obtain a priori estimates for the solutions and use the phase – portrait analysis to obtain bounds for them. We caution that the method used in this paper cannot be applied in general situation but is peculiar to only circumstances where the pressure term has been artificially removed.

Keywords: Phase-Portrait, Uniformly local Spaces, Weighted Energy Estimates

INTRODUCTION

In this paper we seek to find the weighted energy estimate, in uniformly local spaces, of the Navier – Stokes Equations in an infinite strip by omitting the pressure term. The phase portrait method has been used to study the solutions and obtain bounds for them.

The Navier Stokes Systems:

$$\begin{cases} \partial_t u + (u,\nabla) - \Delta u + \nabla p = f \\ divu = 0,\ \ u|_{\partial\Omega} = 0,\ \ u|_{t=0} = u_0 \end{cases}$$
(1.1)

is considered in $\Omega = \mathbb{R} \times [-1,1]$.

The global in time estimate for $2D$ Navier – Stokes equations was first obtained for bounded domains in the works of Ladyzhenkiya(1972). Later on, the unbounded domain case was treated by Abergel(1979) and Babin(1992), and the forcing term was required to lie in some weighted space. However the dimension estimate of the attractor for more general forces.

We know that, based on energy estimate, we can obtain energy solutions for (1.1) by multiplying through by u and integrating over the domain Ω and use the fact that the nonlinear term disappears:

$$\big((u,\nabla),u\big) = \int_{x\in\Omega}(u(x),\nabla)u(x).u(x)dx \equiv 0$$
(1.2)

For every divergence free vector field with Dirichlet boundary conditions.

The situation is completely different when the domain Ω is unbounded because the space of square integrable (divergence free) vector field is not a convenient phase space to work with as we are unable to multiply u because doing so the integral will not make sense. In an unbounded domain, the quest for estimates is intended to have the assumption that $u \in L^2(\Omega) \Rightarrow u(x) \to 0\ as\ |x| \to \infty$ which is too restrictive a decay condition. So under this choice of the phase space many hydrodynamical objects like Poiseuille flows (infinite energy), Kolmogorov flows etc. cannot be considered in the circumstances; because of the above restrictions hold on our model equation (1.1) we are unable to consider constant solutions space periodic solutions etc. which will hinder us from capturing physically relevant solutions.

Overcoming the above obstacles is a work in Zelik(2007) where the weighted energy theory was fully developed for $2D$ Navier – Stokes problem in a strip $\Omega = \mathbb{R} \times [-1,1]$. In this paper, we want to neglect the pressure term of Navier – Stokes system by not adopting any specific method available to excluding pressure. We work in uniformly local spaces and use the phase portrait method to determine bounds for the amended Navier – Stokes system.

PRELIMINARIES: UNIFORM AND WEIGHTED ENERGY SPACES

In this section, we introduce and briefly discuss the weighted and uniformly local spaces which are the main technical tools to deal with infinite-energy solutions, see Zelik (2007) for more detailed exposition. These tools will help us to obtain estimates for our equations (1.1-1.6) in unbounded domain $\Omega = \mathbb{R}[-1,1]$.We explain the space as follows: Let us define $B^1_{x0} = 1$- a unit rectangle centered at $(x_0,0)$ represented as:

$$B^1_{x0} = \left(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right) \times (-1,1), x_0 \in \Re \qquad (2.1)$$

Let us briefly state the definition and basic properties of weight functions and weighted functional spaces as presented by Zelik (2003), Anthony and Zelik (2014), Zelik (2013) and the references therein. Which will be systematically used throughout this project (see also Efendiev and Zelik (2002) for more details). We start with the class of admissible weight functions.

Definition 2.1. *A function* $\phi \in C_{loc}(\mathbb{R})$ *is weight function of exponential growth rate* $\mu > 0$ *if the following inequalities hold:*

$$\phi(x+y) \le C_\phi \phi(x) e^{\mu|y|}, \phi(x) > 0, \qquad (2.2)$$

For a $x, y \in \Omega = \mathbb{R}$

We now introduce a class of weighted Sobolev spaces in a regular unbounded domains associated with weights introduced above. We need only the case where $\Omega = \mathbb{R} \times [-1,1]$ is a strip which obviously have regular boundary. One would like to ask why we need weighted Sobolev Spaces; recall that the uniformly local spaces encountered some deficiencies in that they are not differentiable when the supremum is involved but the weighted energy spaces resolve this problem.

Definition 2.2.

$$L^p_\phi(\Omega) = \{u \in L^p_{loc}(\Omega), \|u\|^p_{L^p_\phi} = \int \phi^p(x)|u(x)|^p dx < \infty\}$$

And

$$L^p_{b,\phi}(\Omega) = \{u \in L^p_{loc}(\Omega), \|u\|^p_{L^p_{b,\phi}} = \sup_{x_0 \in \Re}(\phi(x)\|u\|_{L^p(B^1_{x_0})})\}$$

$x, y \in \Omega = \mathbb{R}$

The uniformly local space $L^p_{b,\phi}(\Omega)$ consists of all functions $u \in L^p_{loc}(\Omega)$ for which the following norm is finite

$$\|u\|^p_{L^p_b} = \sup_{x_0 \in \Re}\|u\|_{L^p(B^1_{x_0})} < \infty$$

If $u \in L^\infty \Rightarrow u \in L^2_b$, and $\|u\|_{L^2_b} \le C\|u\|_{L^\infty}$. This is because all functions that are bounded in L^∞ are also bounded in L^2_b but the reverse is not true.

$$\|u\|_{L^p(B^1_{x_0})} \le |B^1_{x_0}|\|u\|_{L^\infty(B^1_{x_0})} \le C\|u\|_{L^\infty(\Omega)}$$

Similarly, the uniformly local Sobolev spaces $H^s_b(\Omega)$ consist of all functions $u \in H^s_{loc}(\Omega)$ for which the following norm is finite:

$$\|u\|_{H^s_b(\Omega)} = \sup_{x_0 \in \Re}\|u\|_{H^s(B^1_{x_0})} < \infty$$

Where H^s is the space of all distributions whose derivative up to order s is in L^2. The following Lemma establishes the relationship between the spaces L^2_ϕ and L^2_b.

Lemma 2.3. *Let* ϕ *be a weight function of exponential growth rate, where* $\phi_{x_0}(x) = \phi_{x_0}(x - x_0)$, *satisfying* $\int \phi^2 dx < \infty$ *then the following inequalities hold*

$$\|u\|_{L^2_\phi} \le C_1\|u\|^2_{L^2_b} \bullet \int \phi^2 dx \qquad (2.3)$$

$$\|u\|^2_{L^2_\phi} \le C_2 \sup_{x_0 \in \Re}\|u\|^2_{L^2_\phi x_0} \qquad (2.4)$$

Where C_1 *and* C_2 *depend only on* C_ϕ *and* τ.

Proof. *If* $u \in L^2_b$ *then* $\|u\|_{L^p(B^1_{x_0})} < C, x_0 \in \Re$ *is bounded uniformly and*

$$\le \sum \|\phi\|^2{}_{L^\infty[N,N+1]}\|u\|^2{}_{L^2[N,N+1]}$$

$$\le \sum_N \sup_N \|u\|^2{}_{L^2[N,N+1]}\|\phi\|^2{}_{L^\infty[N,N+1]}$$

$$\le \|u\|^2_{L^2_\phi}\sum_{N=-\infty}^{\infty}\|\phi\|^2{}_{L^\infty[N,N+1]}$$

$$\le \|u\|^2_{L^2_\phi}\sum \sup_{x \in [N,N+1]}\|\phi^2(x)\|$$

$$= C\|u\|^2_{L^2_b}\sum \phi^2(N)$$

This follows from the integral criterion for convergence i.e. $\sum_{N=-\infty}^{\infty}\phi^2(N)$ converges if and only if $\int_{-\infty}^{\infty}\phi^2(x)dx < \infty$.

The opposite side of the proof takes $u \in \sup_{x_0 \in \Re}\|u\|^2_{L^2_\phi}$ so we need to estimate the norm $\|u\|_{L^2_{B^1_{x_0}}}$ hence $\min_{x \in B^1_{x_0}} \phi_{x_0}(x) \ge c_0$ and using a specific weight function of the form

$$(x + y) \le Ce^{\alpha|x|}\phi(y) \qquad (2.5)$$

so that

$$\phi(y) \ge C^{-1}e^{-\alpha|x|}\phi(x+y)$$

Take x=-y to get

$$\phi(y) \ge C^{-1}e^{-\alpha|x|}\phi(0) \ge C^{-1}\phi(0)$$

And hence,

$$\int_\Re \phi^2_{x_0}(x)|u(x)|^2 dx \ge c_0^2\int_{B^1_{x_0}}|u(x)|^2 dx = c_0^2\|u\|^2_{L^2_{B^1_{x_0}}}$$

$$C_2 \sup_{x_0 \in \Re}\|u\|^2_{L^2_\phi x_0} \le \|u\|^2_{L^2_\phi} \le C_1 \sup_{c_0 \in \Re}\|u\|^2_{L^2_\phi x_0} \qquad (2.6)$$

Before we conclude this section, we introduce a special weight function which would be essentially used in the sequel.

Let

$$\int_\Omega \phi^2(x)|u(x)|^2 dx = \sum_{N=-\infty}^{\infty} \int_N^{N+1} \phi^2(x)|u(x)|^2 dx$$

$$\phi_\varepsilon(x) = (1+\varepsilon^2|x|^2)^{-\frac{1}{2}} \qquad (2.7)$$

We see that

$$\phi_\varepsilon'(x) = -\frac{\varepsilon^2|x|}{(1+\varepsilon^2|x|^2)^{-\frac{3}{2}}}$$

$$\phi_\varepsilon'(x) \le -C\frac{\varepsilon^2|x|}{(1+\varepsilon^2|x|^2)^2}$$

$$\le C\varepsilon \frac{\varepsilon\ |x|}{\sqrt{1+\varepsilon^2 x^2}} \bullet \frac{1}{(1+\varepsilon^2 x^2)}$$

$$\phi_\varepsilon'(x) \le C\varepsilon\phi_\varepsilon^2(x)$$

This weight, in addition, to (2.5) (which holds for every positive μ), satisfies the following property:

$$\left|\phi_\varepsilon'(x)\right| \le C\varepsilon\phi_\varepsilon(x)^2 < C\varepsilon\phi_\varepsilon(x)$$

A bit more general are the weights $\phi_\varepsilon(x)^N, N \in \Re, N \neq 0,$ which are also the weights of exponential growth rate μ for any $\mu > 0$ and satisfy the analog of (2.9) where the exponent 2 is replaced by $\frac{N+1}{N}$.

In the sequel, we will need the Sobolev embedding and interpolation inequalities for the case of weighted spaces with the embedding constants independent of $\varepsilon \to 0$. Following Babin and Vishik (1990); and Robinson (2001), such inequalities can be derived from the analogous non-weighted situation utilizing the isomorphism $T_{\phi_\varepsilon} u = \phi_\varepsilon u$ between weighted and non-weighted spaces.

Lemma 2.4. *Let* ϕ_ε *be a weight function defined by (2.7). Then, for all l and of exponential growth rate, the map* T_{ϕ_ε} is an isomorphism between $W^{l,p}(\Omega)$ *and* $W_\phi^{l,p}(\Omega)$ *and the following inequalities hold:*

$$C_1\|\phi u\|_{W^{l,p}}^2 \le \|u\|_{W^{l,p}}^2 \le C_2\|\phi u\|_{W^{l,p}}^2 \qquad (2.10)$$

Where C_1 and C_2 are independent of ε but may depend on l and p

Proof let us take $V = \phi u,$ *then for* $u \in L_\phi^2$ we have:

$$\|u\|_{L_\phi^2}^2 = \int \phi^2(x)|u|^2 dx = \int |V|^2 dx = \|V\|_{L^2}^2 \qquad (2.11)$$

(2.11 shows isometry between spaces L_ϕ^2 and L_ϕ^2. The same procedure give isomorphism between L_ϕ^p and L^p. We show that $T_{\phi\varepsilon}$ is an isomorphism between $W^{1,2}$ and $W_{\phi\varepsilon}^{1,2}$. Let $u \in W_\phi^{1,2}(\Omega)$.

Then,

$$\|V\|_{W^{1,2}}^2 = \int (|\nabla V|^2 + |V|^2)dx$$

$$\|\phi u\|_{W^{1,2}}^2 = \int (|\nabla(\phi u)|^2 dx + \int |\phi u|^2 dx$$

$$= \int (|\phi\nabla u| + |\phi' u|)^2 dx + \int |\phi u|^2 dx$$

By Young's inequalities and the relation that $\phi' \le \frac{1}{2}\phi$ (since only small values of ε are important for us, for convenience, we assume that $\varepsilon \le \frac{1}{2}$), we have

$$\|\phi_\varepsilon u\|_{W^{1,2}}^2 < 2\int (\phi^2|\nabla u|^2 + \phi^2|u|^2)dx = C_1\|u\|_{W_\phi^{1,2}}^2$$

$$(2.12)$$

This gives the left inequality of (2.10) (in the particular case $l = 1, p = 2$). Let us prove now the right one:

$$\|u\|_{W_\phi^{1,2}}^2 = \int \phi^2|\nabla u|^2 dx + \int \phi^2|u|^2 dx$$

$$= \int |\nabla(\emptyset^2 u)|^2\, dx - \int \emptyset'|u|^2\, dx - 2\int \emptyset'\emptyset|u|^2\, \nabla u dx + \int \emptyset^2\, |u|^2 dx.$$

By Cauchy Schwartz, Young's inequalities and using the fact that $\phi' \le \frac{1}{2}\phi$ we obtain

$$||u||_{W_\emptyset^{1,2}}^2 \le 2(\int|\nabla(\emptyset u)|^2\, dx + \int \emptyset^2|u|^2\, dx) = 2(\int|\nabla V|^2\, dx + \int |V|^2\, dx) = 2(\int |\nabla V|^2 dx + \int |V|^2 dx) = C_2||V||_{W^{1,2}}^2 \qquad (2.13)$$

This gives the right inequality of (2.10).Thus, for the particular case $l = 1, p = 2$, (1.17) is proved. The proof in a general case is analogous and we leave it to the reader; we have the following relation

$$\|\phi_\varepsilon u\|_{W^{1,2}}^2 < 2\int (\phi^2|\nabla u|^2 + \phi^2|u|^2)dx = C_1\|u\|_{W_\phi^{1,2}}^2$$

$$(2.14)$$

As required

Next corollary gives the weighted analogue of one interpolation inequality useful for what follows, see Triebel (1978).

Corollary 2.5. *Let ϕ_ε be defined (2.7) and let $u \in W_{\phi\varepsilon}^{1,2}$ then the following interpolation inequalities hold*

$$\|u\|_{L_\phi^3}^3 \le C\|u\|_{L_\phi^2}^2 \|u\|_{H_\phi^1} \qquad (2.15)$$

Where the constant C is independent of $\varepsilon \ge 0$.

Proof

We know that for unweighted case, the embedding of H^α into L^3 is performed by the following Sobolev embedding theorem:

$$\frac{0}{2} - \frac{1}{3} \ge \frac{l}{6} - \frac{1}{2} \qquad (2.16)$$

Therefore $H^{\frac{1}{3}} \to L^3$; interpolating $H^{\frac{1}{3}}$ between L^3 and H^1 we have the following equation with exponent θ:

$$\|V\|_{L_\phi^3}^3 \le \|V\|_{L_\phi^3}^3 \le C\|V\|_{L_\phi^2}^0 \|V\|_{H^1}^{1-0} \qquad (2.17)$$

For any $V \in W^{1,2}$ where:

$$\frac{1}{3} = 0 \times \theta + 1 \times (1-0) \Rightarrow \theta = \frac{1}{3} \qquad (2.18)$$

And also

$$\|V\|_{L_\phi^3}^3 \le \|V\|_{H^{\frac{1}{3}}}^3 \le C(\|V\|_{L_\phi^2}^0 \|V\|_{H^1}^{1-0})^3 \qquad (2.19)$$

Taking $V \in W^{1,2}$ in (1.26) and using Lemma 2.4, we obtain (2.14). This ends the proof.

Now, in order to obtain the proper estimates for the solutions in the uniformly local spaces, one can use the so-called weighted energy estimates as an intermediate step and utilize the relation

$$\|u\|_{L_\phi^3}^3 \sim \sup_{s \in \Re} \|u\|_{L_{\phi(-s)}^2}^2$$

Where ϕ is a property chosen (square integral) weight function, 2.7.

ESTIMATES FOR NAVIER – STOKES EQUATIONS WITHOUT PRESSURE

We want to prove that any sufficiently regular solution of the Navier – Stokes proble; (1.1) in a cylinder satisfies the uniformly local estimate:

$$\|u(t)\|_{L_b^2(\Omega)} \le Q\left(\|u(0)\|_{L_b^2(\Omega)}\right) + Q\left(\|f\|_{L_b^2(\Omega)}\right) \qquad (3.1)$$

for some monotone function Q. The first difficulty here is that, in contrast to the case of usual energy solutions, the function $t \to \|u(t)\|_{L_b^2(\Omega)}^2$ is not differentiable due to the presence of supremum in the definition of L_b^2 - norm. This does not allow us to obtain estimate (3.1) directly by multiplying the equation by the appropriate factor and use Gronwall's inequality. Instead, following the general strategy in, we deduce the weighted energy estimates as an intermediate step; multiplying the equation by $\phi^2 u$ where ϕ is a proper weight function described in equation 2.7. If we succeed to verify the analogue of (3.1) in all weighted spaces $L_{\phi x_0}^2(\mathbb{R})$ uniformly with respect to all shifts $x_0 \in \mathbb{R}$, the desired uniformly local estimate will obtained by taking the supremum with respect to $x_0 \in \mathbb{R}$ and using Lemma 2.2. Thus, we need to multiply equation (1.1) by $\phi^2 u$ where $\phi = \phi(x)$ is an appropriate weight function in x_1 direction. But the nonlinear term will still remain unresolved since it will not disappear as in the bounded case. In fact it will produce a cubic nonlinearity $\phi' u^3$. Note that the cubic term is not clear how to control the cubic term in order to produce an a priori estimate. Another setback is the fact that $\phi^2 u$ is not divergence free so the pressure p does not disappear in the weighted energy equality and $(\phi\phi' p, u)$ will pose a problem closely related with finding a reasonable extension of the Helmholtz projector (to divergence free vector fields) to uniformly local spaces. In summary, we have at least two hurdles to overcome in order to close our estimates: to estimate the cubic term $\phi' u^3$ produced by the nonlinear term and $(\phi' p, u)$ when we multiply the momentum equation (1.1) by $\phi^2 u$ and integrate over the domain Ω. Let us put this in perspective to have a clearer view of the terms when we multiply (1.1) by $\phi_\in^2(x_1)u$ $(where\ \phi_\in(x_1)$ is defined as in (2.7) and ϵ is a small positive constant to be determined later) and integrate over Ω to obtain:

$$(\partial_t u, \phi^2 u) + ((u,\nabla)u, \phi^2 u) - (\nu\Delta \mathrm{u}, \phi^2 u) + (\Delta \mathrm{p}, \phi\phi^2 u) = (f, \phi^2 u) \qquad (3.2)$$

and hence,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_\phi^2}^2 + ((u,\nabla)u, \phi^2 u) + \nu\|u\|_{L_\phi^2}^2 + (\Delta \mathrm{p}, \phi^2 u) = (f, \phi^2 u) - \nu(u\phi\phi'.\nabla u) \qquad (3.3)$$

Next, we try and resolve the nasty terms in (3.3) i.e. the second and fourth terms on the LHS; this is to help simplify them as much as possible. Using 2D coordinates; we now seek to estimate the non – linear term as follows:

$$-\sum_{i,j}^{2} \int_\Omega u_j \partial_j u_i . \phi^2 u_i dx$$

$$= -\sum_{i,j}^{2} (u_i u_j \phi^2 \partial_j u_i + u_i^2 \phi^2 \partial_j u_i + u_i^2 \partial_j u_i \phi^2) dx$$

applying the divergent – free condition and collecting like terms we obtain

$$2\sum_{i,j}^{2} \int_\Omega u_i u_j \phi^2 \partial_j u_i dx = -\sum_{i,j}^{2} \int_\Omega u_i^2 u_j \partial_j \phi^2 dx$$

Simplifying the above sum integral with careful consideration that ϕ is applied in the x_1 direction we obtain:

$$-\sum_{i,j}^{2} \int_\Omega u_i^2 u_j \partial_j \phi^2 dx = -\int_\Omega (u_1^2 u_1 \partial_{dx_1}\phi^2 + u_2^2 u_1 \partial_{dx_1}\phi^2) dx = -\frac{1}{2}\left(\partial_{dx_1}\phi^2, u_1(u_1^2 + u_2^2)\right) \le (\phi|\phi'|, |u|^3)$$

(3.4)

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Next, we have the pressure term to resolve but it will simply not disappear because the equation:

$$\int_\Omega \nabla p.\phi^2 u dx = -\int_\Omega p\nabla(\phi^2 u)dx$$
$$= -\int_\Omega p[\phi^2\nabla u + u\nabla\phi^2]dx$$
$$= 2(\phi\phi' p, u)$$

Butm the application of the divergence free condition did not help exclude the term with pressure $(\phi\phi' p, u)$ – a difficult piece to estimate.

Just for the moment, we shall proceed with other terms of Navier – stokes equation without the term containing the pressure and compute the estimate with the impression that we shall return later with good theory that will enable us overcome the difficulty posed by $(\phi\phi' p, u)$ and eventually close our estimate for Navier - Stokes equation in an unbounded domain $\Omega \subset R^2$.

Now, recall (3.3) using (3.4) and ignoring the term $(\phi\phi' p, u)$ we have:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} + (\phi|\phi'|, |u|^3) + \nu\|u\|^2_{L^2_\phi} + \leq (f, \phi u) - \nu(u\phi\phi'.\nabla u) \quad (3.5)$$

The non – linear term is then estimated using corollary 2.1 and Poincare inequality to obtain the following:

$$(\phi\phi', |u|^3) \leq (\epsilon\phi^3, |u|^3)_{L^1} \leq \epsilon\|u\|^3_{L^3_\phi} \leq C\epsilon\|u\|^2_{L^2_\phi} \quad (3.6)$$

Let us tidy up these bits of the Navier – Stokes equation and write (3.5) using (3.6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C\epsilon\|u\|^2_{L^2_\phi}\|\nabla u\|_{L^2_\phi} + \nu\|u\|^2_{L^2_\phi} + \leq (f, \phi u) - \nu(u\phi\phi'.\nabla u) \quad (3.7)$$

By Cauchy Schwartz inequality on the RHS of (3.7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C\epsilon\|u\|^2_{L^2_\phi}\|\nabla u\|_{L^2_\phi} + \nu\|\nabla u\|^2_{L^2_\phi} \leq \|f\|_{L^2_\phi}\|u\|_{L^2_\phi} - \nu\epsilon\|u\|_{L^2_\phi}\|\nabla u\|_{L^2_\phi} \quad (3.8)$$

By Young's inequality on (3.8) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C\epsilon^2\|u\|^4_{L^2_\phi} - \frac{\nu\|u\|^2_{L^2_\phi}}{4} + \nu\|u\|^2_{L^2_\phi} \leq \frac{\|f\|^2_{L^2_\phi}}{2\gamma} + \frac{\gamma\|u\|^2_{L^2_\phi}}{2} + \frac{\nu\epsilon^2\|u\|^2_{L^2_\phi}}{2} + \frac{\nu\|\nabla u\|^2_{L^2_\phi}}{2} \quad (3.9)$$

This simplifies to:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C\epsilon^2\|u\|^4_{L^2_\phi} - \frac{\nu\|\nabla u\|^2_{L^2_\phi}}{4} + \frac{\nu\|\nabla u\|^2_{L^2_\phi}}{2} \leq \frac{\|f\|^2_{L^2_\phi}}{2\gamma} + \frac{\gamma\|u\|^2_{L^2_\phi}}{2} + \frac{\nu\epsilon^2\|u\|^2_{L^2_\phi}}{2} \quad (3.10)$$

Where γ is a positive constant to be determined later. Applying Poincare inequality on the second term of the RHS of (3.10) and assuming $\epsilon \ll 1$ is small enough; the equation reduces to:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C\epsilon^2\|u\|^4_{L^2_\phi} + \frac{\nu\|\nabla u\|^2_{L^2_\phi}}{4} \leq \frac{\|f\|^2_{L^2_\phi}}{2\gamma} + \frac{C\gamma\|\nabla u\|^2_{L^2_\phi}}{2} \quad (3.11)$$

Take that $\frac{\nu}{4} \geq \frac{C\gamma}{2}$ for the purpose of achieving a positive linear term that will afford us not only global existence of solution but also dissipative.

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C\epsilon^2\|u\|^4_{L^2_\phi} + \frac{3}{4}C\gamma\|\nabla u\|^2_{L^2_\phi} \leq \frac{\|f\|^2_{L^2_\phi}}{2\gamma} \leq \epsilon^{-1}C_1\|f\|^2_{L^2_\phi} \quad (3.12)$$

Where estimate (2.1) from lemma 2.1 and $\int\phi^2_\epsilon(x)dx = \frac{C}{\gamma}$ are used in order to obtain the RHS of (3.12) and constant C and C_1 are independent of $\epsilon \geq 0$. We concisely get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2_\phi} - C_2\epsilon^2\|u\|^4_{L^2_\phi} + C_3\gamma\|\nabla u\|^2_{L^2_\phi} \leq \frac{\|f\|^2_{L^2_\phi}}{2\gamma} \leq \epsilon^{-1}C_1\|f\|^2_{L^2_\phi} \quad (3.13)$$

We take $\|u\|^2_{L^2_\phi} = y(t)$ to obtain the following differential inequality:

$y'(t) + C_3\gamma y(t) \leq \epsilon^{-1}$[illegible]$_1\|$[illegible]$\|^2_{[illegible]^2}$ + [illegible]$_2$[illegible]^2[illegible]^2([illegible])

(3.14)

By change of variable [illegible] = [illegible] and upon using the fact that $\|$[illegible]$(0)\|^2_{[illegible]^2} \leq \|$[illegible]$(0)\|^2_{[illegible]^2}\int\phi$([illegible]) [illegible] $= \|$[illegible]$(0)\|^2_{[illegible]^2}$ [illegible] $< \infty$; we have, first that

[illegible]$(0) = \|u(0)\|^2_{L^2_\phi} \leq \frac{C}{\epsilon}\|u(0)\|^2_{L^2_\phi}$

(3.15)

Now, recall (3.14) and write it as: $y'(t) + \gamma y(t) \leq \frac{C_f}{\epsilon} + \epsilon^2 y^2(t)$ so that for the initial data $y(0) \leq C\epsilon^{-1}\|u(0)\|^2_{L^2_\phi}$ and by the above scaling, with its initial condition$z(0) \leq C\|u(0)\|^2_{L^2_b}$.

We shall seek to solve (3.14) to prove that it has global bounds for solutions because of the positive linear term on the LHS of (3.14). we state a Lemma involving a two stage proof of the estimate for (3.14): The first part of the proof considers the case of a simple ODE, while the next considers a system of ODE:

Theorem 4.1 Let y(t) = $y_\epsilon(t)$ be absolutely continuous and satisfy for every small ϵ the following differential inequality:

$$y'(t) + \frac{\gamma}{2}y(t) \leq \epsilon^{-1}C_f + y(t)(\epsilon^2 y(t) - \frac{\gamma}{2}) \quad (4.1)$$

$$y(0) \leq \epsilon^{-1}C_0 \quad (4.2)$$

For some C_f, C_0 $independent\ of\ \epsilon$. Then $y(t)\ is\ globally\ bounded\ for\ all\ t \geq 0$:

$$y(t) \leq \epsilon^{-1}(C_0 + \frac{C_f}{\gamma/2}) \quad (4.3)$$

If $\epsilon\ is\ small\ enough$:

$$\epsilon \leq C(C_f + C_0) \quad (4.4)$$

And constant $C\ is\ independent\ of\ \epsilon \to 0$.

Proof:

This portion of the proof will use the phase-potrait technique. Given

$$z'(t) + \gamma z(t) \leq C_f + \epsilon z^2(t), \quad z(0) = \epsilon y(0) \leq C_0 \quad (4.5)$$

Let $z(t) \leq w(t)$

$$so\ thatthe\ inequality\ part\ of\ (4.5)\ is\ exactly\ w(t)$$
$$w' + \gamma w(t) = C_f + \epsilon w^2(t),$$
$$w(0) = z(0) \quad (4.6)$$

Bearing in mind that all solutions (4.5) lies below (4.6) with the same initial condition, we conclude that $z(t) \leq w(t)$ $forall\ t. So\ it\ is\ enough\ to\ estimate\ w(t)$.

To study the dynamics of ODE (4.6), we find the following equilibrium solutions:

$$w_1(\epsilon) = \frac{2C_f}{(\gamma + \sqrt{\gamma^2 - 4\epsilon C_f})} = \frac{C_f}{\gamma} - \frac{2C_f\epsilon}{\gamma^3} + \cdots - \approx \frac{C_f}{\gamma},$$

and

$w_2(\epsilon) = \frac{\gamma + \sqrt{\gamma^2 - 4\epsilon C_f}}{2\epsilon} = \frac{\gamma}{\epsilon} - \frac{C_f}{\gamma} \dots - \approx \frac{2\gamma}{\epsilon}$.

And from $w_1(\epsilon)$ we see that $\lim_{\epsilon \to 0} \frac{(\gamma + \sqrt{\gamma^2 - 4\epsilon C_f})}{2\epsilon} = \frac{C_f}{\gamma}$. We see that $w_1(\epsilon)$ remains bounded as $\epsilon \to 0$ and $w_1(0) = \lim_{\epsilon \to 0} w_1(\epsilon) = \frac{C_f}{\gamma}$, *however, is exactly the root of the limit linear e*

$$w_2(\epsilon) \to \infty \text{ as } \epsilon \to \infty.$$

Looking at the phase-portrait of the ODE, we get full description about the dynamics: There is a stable equilibrium near $w_1 = \frac{C_f}{\gamma}$, and all solutions around it which are less that w_2 are attracted to it. We know that w(t) will decay for all time tending to $w_1(\epsilon)$ *if* $w(0) \in [w_1(\epsilon), w_2(\epsilon)]$. Thus in that case we simply have $z(t) \leq w(0) \leq z(0)$. The above choice of the initial condition is possible if ϵ *is such that* $\gamma^2 - 4\epsilon C_f > 0$, otherwise w(t) will blow up in finite time, we also exclude the choice of ϵ *that considers* $z(0) > w_2(\epsilon)$.

But in the case of $z(0) < w_1(\epsilon)$ the solution will grow tending to $w_1(\epsilon)$ *as* $t \to \infty$. So in that case, $z(t) < w_1(\epsilon)$. Finally, in both cases, $z(t) \leq z(0) + w_1(\epsilon)$ *and since* $w_1(\epsilon) \leq \frac{2C_f}{\gamma}$ we obtain

$$y(t) \leq \epsilon^{-1} z(t) \leq \epsilon^{-1}(C_0 + C_f)$$

We have shown how to derive the desired a priori estimate for the NSE in uniformly local spaces in the model situation where the pressure term is not taken into account. The phase portrait analysis has been used to get bounds for the solutions. Existence, uniqueness and further regularity of solutions may be obtained in a standard way by the Galerkin approximation. We must emphasize here, however, that this method applies only to model situation where pressure is formally omitted.

REFERENCES

Abergel F. (1979) Attractors for a Navier-Stokes ow in an unbounded domain, Math.Mod.Num.Anal., 23:3,359-370.

Anthony P. & Zelik S. (2014). *Infinite energy solutions for Navier-Stokes equations in a strip revisited* Comm.on Pure and Applied Analysis13(4): 1361-1393.

Babin A. (1992), *The attractor of a Navier-Stokes system in an unbounded channel-like domain*, J. Dynam. Differential Equations, 4:4, 555-584

Efendiev M. & Zelik S. (2002), *Upper and lower bounds for the Kolmogorov entropy of the attractor for an RDE in an unbounded domain*, J. Dyn. Di_. Eqns 14:369-403.

Ladyzhenskaya O. (1972). *On the dynamical system generated by the Navier-stokes equation*, J. of Soviet Maths 27:91-114.

Robinson J. (2001), *Infinite-dimensional dynamical systems*. Cambridge University Press.

Triebel H. (1978), *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland.

Zelik, S.(2007), *Spatially nondecaying solutions of the 2D Navier-Stokes equation in a strip*. Glasg. Math. J., 49: 3, 525-588.

Zelik S. (2003), *Attractors of reaction-diffusion systems in unbounded domains and their spatial complexity,*Comm. Pure Appl. Math., 56: 5; 584-637.