Science World Journal Vol 11 (No 1) 2016
www.scienceworldjournal.org
ISSN 1597-6343

Full Length Research Article

ON COUPLED SYSTEM OF NAVIER-STOKES EQUATIONS AND TEMPERATURE IN INFINITE CHANNEL

Peter Anthony

Department of Mathematics, Kaduna State University – Nigeria.

Email: p.anthony@kasu.edu.ng

ABSTRACT
This paper deals with the coupled system of Navier-Stokes equations and temperature (Thermohydraulics) in a strip in the class of spatially non-decaying (infinite-energy) solutions belonging to the properly chosen uniformly local Sobolev spaces. The global well-posedness and dissipativity of the Navier-Stokes equations in a strip in such spaces has been established in Zelik (2007). Similar protocol is observed using maximum principle to obtain bounds for the temperature solutions in Boussinesq approximation.

Keywords: Naviernavier-Stokes Equations, Unbounded Domains, Infinite-Energy Solutions, Uniformly Local Spaces, Boussinesq Approximation, Maximum Principle

INTRODUCTION
In this paper, we consider the equation of coupled system of fluid and temperature in Boussinesq approximation.
We consider Boussinesq equation in nondimensional form in the strip $\Omega = \mathbb{R} \times [0,1]$.

$$\delta_t u + u \cdot \nabla u + \nabla p = \Delta u + +e_n T \ldots \quad (1.1)$$

$$u\big|_{x_2=1} = 0, u\big|_{x_2=0} = 0 \quad (1.2)$$

$$\nabla \cdot u = 0 \quad (1.3)$$

$$\delta_t T + \left(u\nabla\right) T - \mu \Delta T = 0 \quad (1.4)$$

$$T\big|_{x_2=1} = 0, T\big|_{x_2=0} = 1 \quad (1.5)$$

$$T\big|_{t=0} = T_0 \quad (1.6)$$

Which satisfies the zero flux condition; where $u = (u_0, u_1)$ is the velocity vector, $T_0 = 1$ is the temperature at the bottom $T_1 = 0$ is the temperature at the top, e_n is the standard coordinate basis in $\mathbb{R}^2$and the kinematic viscosity $\mu > 0$:

We consider the problem of thermal convection (heat transfer) by an incompressible Newtonian fluid. We give a brief overview as presented in Doering and Gibbon (2004) :

In the first approximation, the local temperature of the fluid may be considered a passive scalar, i.e., a quantity characteristic of each particular fluid element whose space-time evolution is thus controlled by the fluid's motion. Thermal conduction between neighboring fluid elements is then taken into account by including a diffusive term and introducing another material parameter, the thermal diffusion coefficient. The influence of the temperature field on the incompressible fluid's motion is taken into account by introducing a buoyancy force into the velocity evolution equation. The origin of the buoyancy force is the observation that temperature variations typically lead to density variations which, in the presence of a gravitational field, lead to pressure gradients. The inclusion of density variations in the buoyancy force – while neglecting them in the continuity equation - and the neglect of the local heat source due to viscous dissipation constitute the approximate formulation known as the Boussinesq equations.

Preliminaries: Uniform and Weighted Energy Spaces
In this section, we introduce and briefly discuss the weighted and uniformly local spaces which are the main technical tools to deal with infinite-energy solutions, see Zelik (2007) for more detailed exposition. These tools will help us to obtain estimates for our equations (1.1-1.6) in unbounded domain $\Omega = \mathbb{R}[-1,1]$.We explain the space as follows: Let us define $B^1_{x0} = 1$- a unit rectangle centered at $(x_0, 0)$ represented as:

$$B^1_{x0} = \left(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right) \times (-1,1), x_0 \in \Re \quad (2.2)$$

Let us briefly state the definition and basic properties of weight functions and weighted functional spaces as presented by Zelik (2003. Anthony and Zelik (2014). Zelik (2013) and the references therein. Which will be systematically used throughout this project (see also Efendiev and Zelik (2002) for more details). We start with the class of admissible weight functions.

Definition 2.1. A function $\phi \in C_{loc}(\mathbb{R})$ is weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:

$$\phi(x+y) \leq C_\phi \phi(x) e^{\mu|y|}, \phi(x) > 0, \quad (2.2)$$

For a $x, y \in \Omega = \mathbb{R}$
We now introduce a class of weighted Sobolev spaces in a

regular unbounded domains associated with weights introduced above. We need only the case where $\Omega = \mathbb{R} \times [-1,1]$ is a strip which obviously have regular boundary. One would like to ask why we need weighted Sobolev Spaces; recall that the uniformly local spaces encountered some deficiencies in that they are not differentiable when the supremum is involved but the weighted energy spaces resolve this problem.

Definition 2.2.

$$L_\phi^p(\Omega) = \{u \in L_{loc}^p(\Omega), \|u\|_{L_\phi^p}^p = \int \phi^p(x)|u(x)|^p dx < \infty\}$$

And

$$L_{b,\phi}^p(\Omega) = \{u \in L_{loc}^p(\Omega), \|u\|_{L_{b,\phi}^p}^p = \sup_{x_0 \in \Re}(\phi(x)\|u\|_{L^p(B_{x_0}^1)})\}$$

$x, y \in \Omega = \mathbb{R}$

The uniformly local space $L_{b,\phi}^p(\Omega)$ consists of all functions $u \in L_{loc}^p(\Omega)$ for which the following norm is finite

$$\|u\|_{L_b^p}^p = \sup_{x_0 \in \Re}\|u\|_{L^p(B_{x_0}^1)} < \infty$$

If $u \in L^\infty \Rightarrow u \in L_b^2$, and $\|u\|_{L_b^2} \le C\|u\|_{L^\infty}$. This is because all functions that are bounded in L^∞ are also bounded in L_b^2 but the reverse is not true.

$$\|u\|_{L^p(B_{x_0}^1)} \le |B_{x_0}^1| \|u\|_{L^\infty(B_{x_0}^1)} \le C\|u\|_{L^\infty(\Omega)}$$

Similarly, the uniformly local Sobolev spaces $H_b^s(\Omega)$ consist of all functions $u \in H_{loc}^s(\Omega)$ for which the following norm is finite:

$$\|u\|_{H_b^s(\Omega)} = \sup_{x_0 \in \Re}\|u\|_{H^s(B_{x_0}^1)} < \infty$$

.

Where H^s is the space of all distributions whose derivative up to order s is in L^2. The following Lemma establishes the relationship between the spaces L_ϕ^2 and L_b^2.

Lemma 2.3. *Let ϕ be a weight function of exponential growth rate, where $\phi_{x_0}(x) = \phi_{x_0}(x - x_0)$, satisfying $\int \phi^2 dx < \infty$ then the following inequalities hold*

$$\|u\|_{L_\phi^2} \le C_1\|u\|_{L_b^2}^2 \bullet \int \phi^2 dx \qquad (2.3)$$

$$\|u\|_{L_\phi^2}^2 \le C_2 \sup_{x_0 \in \Re}\|u\|_{L_{\phi x_0}^2}^2 \qquad (2.4)$$

(2.4)

Where C_1 and C_2 depend only on C_ϕ and τ.

Proof. *If $u \in L_b^2$ u 2 L2 b then $\|u\|_{L^p(B_{x_0}^1)} < C, x_0 \in \Re$ is bounded uniformly and*

$$\int_\Omega \phi^2(x)|u(x)|^2 dx = \sum_{N=-\infty}^{\infty} \int_N^{N+1} \phi^2(x)|u(x)|^2 dx$$

$$\le \sum \|\phi\|^2{}_{L^\infty[N,N+1]}\|u\|^2{}_{L^2[N,N+1]}$$

$$\le \sum_N \sup_N \|u\|^2{}_{L^2[N,N+1]}\|\phi\|^2{}_{L^\infty[N,N+1]}$$

$$\le \|u\|_{L_\phi^2}^2 \sum_{N=-\infty}^{\infty} \|\phi\|^2{}_{L^\infty[N,N+1]}$$

$$\le \|u\|_{L_\phi^2}^2 \sum \sup_{x \in [N,N+1]} \|\phi^2(x)\|$$

$$= C\|u\|_{L_b^2}^2 \sum \phi^2(N)$$

This follows from the integral criterion for convergence i.e. $\sum_{N=-\infty}^{\infty}\phi^2(N)$ converges if and only if $\int_{-\infty}^{\infty}\phi^2(x)dx < \infty$.

The opposite side of the proof takes $u \in \sup_{x_0 \in \Re}\|u\|_{L_\phi^2}^2$ so we need to estimate the norm $\|u\|_{L_{B_{x_0}^1}^2}$ hence $\min_{x \in B_{x_0}^1} \phi_{x_0}(x) \ge c_0$ and using a specific weight function of the form

$$(x + y) \le Ce^{\alpha|x|}\phi(y) \qquad (2.5)$$ so that

$$\phi(y) \ge C^{-1}e^{-\alpha|x|}\phi(x+y)$$

Take x=-y to get

$$\phi(y) \ge C^{-1}e^{-\alpha|x|}\phi(0) \ge C^{-1}\phi(0)$$

And hence,

$$\int_\Re \phi_{x_0}^2(x)|u(x)|^2 dx \ge c_0^2 \int_{B_{x_0}^1}|u(x)|^2 dx = c_0^2\|u\|_{L_{B_{x_0}^1}^2}^2$$

$$C_2 \sup_{x_0\in\Re}\|u\|^2_{L^2_\phi x_0} \le \|u\|^2_{L^2_\phi} \le C_1 \sup_{c_0\in\Re}\|u\|^2_{L^2_\phi x_0} \quad (2.6)$$

Before we conclude this section, we introduce a special weight function which would be essentially used in the sequel.
Let

$$\phi_\varepsilon(x) = (1+\varepsilon^2|x|^2)^{-\frac{1}{2}} \quad (2.7)$$

We see that

$$\phi_\varepsilon{}'(x) = -\frac{\varepsilon^2|x|}{(1+\varepsilon^2|x|^2)^{-\frac{3}{2}}}$$

$$\phi_\varepsilon{}'(x) \le -C\frac{\varepsilon^2|x|}{(1+\varepsilon^2|x|^2)^2}$$

$$\le C\varepsilon\frac{\varepsilon\ |x|}{\sqrt{1+\varepsilon^2x^2}} \bullet \frac{1}{(1+\varepsilon^2x^2)}$$

$$\phi_\varepsilon{}'(x) \le C\varepsilon\phi_\varepsilon^2(x) \quad (2.8)$$

This weight, in addition, to (2.5) (which holds for every positive μ), satisfies the following property:

$$\left|\phi_\varepsilon{}'(x)\right| \le C\varepsilon\phi_\varepsilon(x)^2 < C\varepsilon\phi_\varepsilon(x) \quad (2.9)$$

A bit more general are the weights $\phi_\varepsilon(x)^N, N\in\Re, N\ne 0,$ which are also the weights of exponential growth rate μ for any $\mu > 0$ and satisfy the analog of (2.9) where the exponent 2 is replaced by $\frac{N+1}{N}$.

In the sequel, we will need the Sobolev embedding and interpolation inequalities for the case of weighted spaces with the embedding constants independent of $\varepsilon \to 0$. Following Babin and Vishik (1990) and Robinson (2001), such inequalities can be derived from the analogous non-weighted situation utilizing the isomorphism $T_{\phi_\varepsilon} u = \phi_\varepsilon u$ between weighted and non-weighted spaces.

Lemma 2.4. *Let ϕ_ε be a weight function defined by (2.7). Then, for all l and of exponential growth rate, the map T_{ϕ_ε} is an isomorphism between $W^{l,p}(\Omega)$ and $W_\phi^{l,p}(\Omega)$ and the following inequalities hold:*

$$C_1\|\phi u\|^2_{W^{l,p}} \le \|u\|^2_{W^{l,p}} \le C_2\|\phi u\|^2_{W^{l,p}} \quad (2.10)$$

Where C_1 and C_2 are independent of ε but may depend on l and p

Proof let us take $V = \phi u,$ *then for* $u \in L^2_\phi$ we have:

$$\|u\|^2_{L^2_\phi} = \int \phi^2(x)|u|^2 dx = \int |V|^2 dx = \|V\|^2_{L^2} \quad (2.11)$$

(2.11) shows isometry between spaces L^2_ϕ and L^2_ϕ. The same procedure give isomorphism between L^p_ϕ and L^p. We show that $T_{\phi\varepsilon}$ is an isomorphism between $W^{1,2}$ and $W^{1,2}_{\phi\varepsilon}$. Let $u \in W^{1,2}_\phi(\Omega)$.

Then,

$$\|V\|^2_{W^{1,2}} = \int (|\nabla V|^2 + |V|^2)dx$$

$$\|\phi u\|^2_{W^{1,2}} = \int (|\nabla(\phi u)|^2 dx + \int |\phi u|^2 dx$$

$$= \int (|\phi\nabla u| + |\phi' u|)^2 dx + \int |\phi u|^2 dx$$

By Youngs's inequalities and the relation that $\phi' \le \frac{1}{2}\phi$ (since only small values of ε are important for us, for convenience, we assume that $\varepsilon \le \frac{1}{2}$), we have

$$\|\phi_\varepsilon u\|^2_{W^{1,2}} < 2\int(\phi^2|\nabla u|^2 + \phi^2|u|^2)dx = C_1\|u\|^2_{W^{1,2}_\phi} \quad (2.12)$$

This gives the left inequality of (2.10) (in the particular case $l = 1, p = 2$). Let us prove now the right one:

$$\|u\|^2_{W^{1,2}_\phi} = \int \phi^2|\nabla u|^2 dx + \int \phi^2|u|^2 dx$$

$$= \int |\nabla(\phi^2 u|)^2 dx - \int \phi'|u|^2 dx - 2\int \phi'\phi|u|^2 \nabla u dx + \int \phi^2|u|^2 dx$$

By Cauchy Schwartz, Young's inequalities and using the fact that $\phi' \le \frac{1}{2}\phi$ we obtain (2.13)

$$\|u\|^2_{W^{1,2}_\phi} \le 2\left(\int|\nabla(\phi u)|^2 dx + \int\phi^2|u|^2 dx\right) = 2\left(\int|\nabla V|^2 dx + \int|V|^2 dx\right) = C_2\|V\|^2_{W^{1,2}}$$

This gives the right inequality of (2.10).Thus, for the particular case $l = 1, p = 2$, (1.17) is proved. The proof in a general case is analogous and we leave it to the reader; we have the following relation

$$\|\phi_\varepsilon u\|^2_{W^{1,2}} < 2\int(\phi^2|\nabla u|^2 + \phi^2|u|^2)dx = C_1\|u\|^2_{W^{1,2}_\phi} \quad (2.14)$$

As required

Next corollary gives the weighted analogue of one interpolation inequality useful for what follows, see Triebel (1978).

Corollary 2.5. *Let ϕ_ε be defined (2.7) and let $u \in W^{1,2}_{\phi\varepsilon}$ then the following interpolation inequalities hold*

$$\|u\|^3_{L^3_\phi} \le C\|u\|^2_{L^2_\phi}\|u\|_{H^1_\phi} \qquad (2.15)$$

Where the constant C is independent of $\varepsilon \ge 0$.

Proof

We know that for unweighted case, the embedding of H^α into L^3 is performed by the following Sobolev embedding theorem:

$$\frac{0}{2} - \frac{1}{3} \ge \frac{l}{6} - \frac{1}{2} \qquad (2.16)$$

Therefore $H^{\frac{1}{3}} \to L^3$; interpolating $H^{\frac{1}{3}}$ between L^3 and H^1 we have the following equation with exponent θ :

$$\|V\|^3_{L^3_\phi} \le \|V\|^3_{L^3_\phi} \le C\|V\|^0_{L^2_\phi}\|V\|^{1-0}_{H^1} \qquad (2.17)$$

or any $V \in W^{1,2}$ where:

$$\frac{1}{3} = 0 \times \theta + 1 \times (1-0) \Rightarrow \theta = \frac{1}{3} \qquad (2.18)$$

And also

$$\|V\|^3_{L^3_\phi} \le \|V\|^3_{H^{\frac{1}{3}}} \le C(\|V\|^0_{L^2_\phi}\|V\|^{1-0}_{H^1})^3 \qquad (2.19)$$

Taking $V \in W^{1,2}$ in (1.26) and using Lemma 2.4, we obtain (2.14). This ends the proof.

Now, in order to obtain the proper estimates for the solutions in the uniformly local spaces, one can use the so-called weighted energy estimates as an intermediate step and utilize the relation

$$\|u\|^3_{L^3_\phi} \sim \sup_{s\in\Re}\|u\|^2_{L^2_{\phi(.-s)}}$$

Where ϕ is a property chosen (square integral) weight function, 2.7.

We assume that (u,T) , is a sufficiently regular solution of the system (1.1) – (1.6) satisfying the following properties:

$$u \in L^\infty(\Re_+, L^2_b(\Omega)), \nabla u \in L^2_b(R_+ \times \Omega), divu = 0$$
(2.20)

And

$$T \in L^\infty(\Re_+, L^2_b(\Omega)), \nabla T \in L^2_b(R_+ \times \Omega). \qquad (2.21)$$

Where

$$0 \le T_0 \le 1 \qquad (2.22)$$

We want to obtain estimates for T and u in uniformly local spaces as the ones obtained for the solutions of the Navier Stokes system, see Zelik(2007) for more expositions. The key technical tool for that is the maximum/comparison principle for temperature which we will consider later.

Maximum Principle for Temperature

In this subsection, we consider equations (1.4)-(1.6) for temperature assuming that u is a given vector field satisfying (1.1) (u is not necessarily a solution of the Navier-Stokes equation. Our aim is to show that the inequality (2.22) at time moment $t = 0$ implies analogous inequality for all $t \ge 0$. To justify the maximum/comparison principle, we need the following properties of the truncation function.

Lemma 3.1. *Let $V \in H^1_b(\Omega)$. Then, the truncation functions*

$V_+(x) = \max\{V(x), 0\}$ *and*

$V_-(x) = \max\{-V(x), 0\}$ *belong to $H^1_b(\Omega)$ as well, and their distribution derivatives satisfy:*

$$\nabla V_+ = (\nabla V)_+ = \begin{cases} \nabla V & if\ V > 0 \\ 0 & if\ V < 0 \end{cases}$$

Similarly,

$$\nabla V_- = (\nabla V)_- = \begin{cases} \nabla V & if\ T < 0 \\ 0 & if\ T > 0 \end{cases}$$

In particular,

$$|T_-(x,t)| \cdot |T_+(x,t)| = 0 \qquad (3.1)$$

$$\nabla T_-(x,t) \cdot \nabla T_+(x,t) = 0 \qquad (3.2)$$

Almost everywhere

Now, we will state the prove of the main result of this subsection.

Theorem 3.2. *Let u and T satisfy equation (1.1)-(1.6), and $0 \le T(x,0) \le 1$ for almost all $x \in \Re$ then,*

$$0 \le \nabla T(x,t) \le 1, \forall (x,t) \in \Re_+ \times \Omega$$
(3.3)

Proof:

Let us first prove that $T > 0$ almost everywhere. To this end, we multiply (1.4) by $-T_-\phi^2$, where T_- is the truncation of T and $\phi = \phi(x)$ is the weight function $\phi_\varepsilon(x_1) = e^{-\varepsilon|x_1|}$. We integrate by parts and use $T = T_+ - T_-$ and $T_{-|\partial\Omega}$ to obtain T_- estimates as follows:

$$\frac{1}{2}\frac{d}{dt}\|T_-\|^2_{L^2_\phi} + \mu(\nabla T_+ - \nabla T_{-,} \nabla T_-) + (u(\nabla T_+ - \nabla T_-), T_-) = 0$$
(3.4)

then using (3.1) and (3.2) to obtain

$$\frac{1}{2}\frac{d}{dt}\|T_-\|_{L^2_\phi}^2+\mu\|\nabla T_-\|_{L^2_\phi}^2+(\nabla T_-,2T_-\varepsilon\phi^2)+((u\nabla)T_-,T_-\phi^2)=0 \quad (3.5)$$

By using that $divu=0$, we simplify the nonlinearity to get the following:

$$((u\nabla)T_-,T_-\phi^2)=(u_1\partial_{x_1}T_-,T_-\phi^2)+(u_2\partial_{x2}T_-,T_-\phi^2)$$

$$=\frac{1}{2}((u_1\partial_{x_1}T_-^2,\phi^2)+\frac{1}{2}(u_2\partial_{x2}(T_-^2),\phi^2)$$

$$=\frac{1}{2}\{((\partial_{x_1}u_1+\partial_{x2}u_2),T_-^2)-(u_1T_-^2,\phi\phi')\}=-(u_1T_-^2,\phi\phi')$$

Using that ϕ satisfies (2.9), we see that

$$\left|((u\nabla)T_-,T_-\phi^2)\right|\leq C\varepsilon(u_1T^2,\phi^2) \quad (3.6)$$

We estimate the RHS of (3.6) as follows:

$$C\varepsilon(u_1T^2,\phi^2)=C\varepsilon\sum_{N=-\infty}^{\infty}\int_{\Omega N,N+1}(u_1T^2,\phi^2)dx$$

$$\leq C\varepsilon\sum_{N=-\infty}^{\infty}\int_{\Omega N,N+1}|T_-^2u_1\|\phi|^2dx$$

$$\leq C\varepsilon\sum_{N=-\infty}^{\infty}\|T_-^2u_1\|_{L^1[\Omega N,N+1]}\|\phi\|_{L^\infty[N,N+1]}^2$$

$$\leq C\varepsilon\sum_{N}^{\infty}\sup_N\|u_1\|_{L^1[\Omega N,N+1]}\|T_-^2\|_{L^\infty[N,N+1]}\phi^2(N)$$

$$\leq C\varepsilon\|u_1\|_{L^2_B}\sum_{N}^{\infty}\|T_-\|_{L^4[\Omega N,N+1]}^2\phi^2(N)$$

Using Ladyzhenskaya inequality,

$$\|T_-\|_{L^4[\Omega N,N+1]}^2\leq\|T_-\|_{L^2[\Omega N,N+1]}\cdot\|\nabla T_-\|_{L^2(\Omega_N,N+1)} \quad (3.7)$$

And

$$\|T_-\|_{L^2_\phi}^2\sim\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|_{L^2(\Omega_N,N+1)}^2 \quad (3.8)$$

We finally obtain:

$$\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|_{L^4(\Omega_N,N+1)}^2$$

$$\leq C\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|_{L^2(\Omega_N,N+1)}\cdot\|\nabla T_-\|_{L^2(\Omega_N,N+1)} \quad (3.9)$$

$$\leq\partial\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|_{L^2(\Omega_N,N+1)}^2+C_\partial\sum_{N=-\infty}^{\infty}\phi^2(N)\|T_-\|_{L^2(\Omega_N,N+1)}^2$$

$$\leq\partial\|\nabla T_-\|_{L^2(\Omega_N,N+1)}^2+C_\partial\|T_-\|_{L^2_\phi}^2$$

Thus we have the estimate for the nonlinearity:

$$\left|((u,\nabla)T_-,T_-\phi^2)\right|\leq C\|u\|_{L^2_\phi}^2(\partial\|\nabla T_-\|_{L^2(\Omega_N,N+1)}^2+C_\partial\|T_-\|_{L^2_\phi}^2) \quad (3.10)$$

Substituting this estimate into (3.5) we obtain

$$\frac{d}{dt}\|T_-\|_{L^2_\phi}^2+\frac{v}{2}\|\nabla T_-\|_{L^2_\phi}^2\leq\partial\|u\|_{L^2_\phi}\|\nabla T_-\|_{L^2_\phi}^2+C_\partial\|u\|_{L^2_\phi}^2\|T_-\|_{L^2_\phi}^2+2\varepsilon^2v\|T_-\|_{L^2_\phi}^2 \quad (3.11)$$

Taking $\partial>0$ small enough reduce to:

$$\frac{d}{dt}\|T_-\|_{L^2_\phi}^2+c_1\|\nabla T_-\|_{L^2_\phi}^2\leq c_2\|T_-\|_{L^2_\phi}^2 \quad (3.12)$$

Dropping the second term on the LHS of the equation (3.12) we obtain:

$$\frac{d}{dt}\|T_-\|_{L^2_\phi}^2+c_2\|T_-\|_{L^2_\phi}^2\leq 0$$

And applying gronwall and taking supremum with respect to all shifts, we obtain the following estimate:

$$\|T_-\|_{L^2_\phi}^2\leq\|T_-(0)\|_{L^2_\phi}^2e^{c_2t} \quad (3.13)$$

Since we know $T_-(x,0)=0$ we may conclude from above that:

$$\|(T-1)_-(t)\|_{L^2_\phi}^2=0 \quad (3.14)$$

We proceed similarly bearing in mind that $(T_-(x,t)-1)_+\leq 0$ to prove that:

$$\|(T-1)_+(t)\|_{L^2_\phi}^2\leq\|(T-1)_+(0)\|_{L^2_\phi}^2e^{c_2t} \quad (3.15)$$

Where $\|(T-1)_+(0)\|_{L^2_\phi}^2=0$. By similar argument we conclude that:

$$\|(T-1)_+(t)\|_{L^2_\phi}^2=0$$

Thus, 3.3 follows immediately. This ends the proof. We state a corollary that will give us a bound in time for ∇T.

Corollary 3.3. *Let T be the solution of (1.4) then let,*

$$\|\nabla T\|_{L^2_\phi[0,T;\Omega]}^2\leq C'+C\|u\|_{L^2_\phi[0,T;\Omega]} \quad (3.17)$$

***Proof**: We first construct an auxiliary function* $\overline{T}(x) = \frac{1-x_2}{2}$ which satisfies (1.5) so that

$$T(t,x_2) = \quad \overline{T}(x_2) + \widetilde{T}(x_2,t) \qquad (3.18)$$

We recall (1.4) as repeated here:

$$\partial_t T + (u\nabla)T - \Delta T = 0$$

Then we substitute (3.18) into the equation to obtain the following:

$$\partial_t \widetilde{T} + (u,\nabla)\overline{T} + (u,\nabla)\widetilde{T} - \Delta\widetilde{T} = 0 \qquad (3.19)$$

Which simplifies to:

$$\partial_t \widetilde{T} + (u,\nabla)\widetilde{T} - \Delta\widetilde{T} = \frac{u_1}{2}$$

We perform our usual multiplication by $\widetilde{T}\phi^2$ and integrate over the domain to obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\widetilde{T}\right\|^2_{L^2_\phi} + \left\|\nabla\widetilde{T}\right\|^2_{L^2_\phi} + ((u,\nabla)\widetilde{T},\widetilde{T}\phi^2)$$

$$\le \frac{\left\|u_1\right\|^2_{L^2_\phi}}{8} + \frac{\left\|\widetilde{T}\right\|^2_{L^2_\phi}}{2} + C\varepsilon^2\frac{\left\|\widetilde{T}\right\|^2_{L^2_\phi}}{2} + \frac{\left\|\nabla\widetilde{T}\right\|^2_{L^2_\phi}}{2} \qquad (3.21)$$

Similarly, we find the estimate for nonlinearity in $\widetilde{T}$ and u analogous to the one derived earlier in (3.10). Thus,

$$\left|((u,\nabla)\overline{T},T\phi^2)\right| \le C\left\|u\right\|^2_{L^2_b}\left(\partial\left\|\nabla\widetilde{T}\right\|^2_{L^2_b} + C_\delta\left\|\widetilde{T}\right\|^2_{L^2_b}\right) \qquad (3.22)$$

Substituting (3.22) into (3.21) and absorbing similar terms we obtain: ()

$$\frac{d}{dt}\left\|\widetilde{T}\right\|^2_{L^2_b} + (1 - C\left\|u\right\|^2_{L^2_b}\partial)\left\|\nabla\widetilde{T}\right\|^2_{L^2_b} - (1 + C_\partial\left\|u\right\|^2_{L^2_b})\left\|\widetilde{T}\right\|^2_{L^2_b} \le \frac{\left\|u\right\|^2_{L^2_b}}{4} \qquad (3.23)$$

By simplifying the above we have:

$$\frac{d}{dt}\left\|\widetilde{T}\right\|^2_{L^2_b} + C\left\|\nabla\widetilde{T}\right\|^2_{L^2_\phi} \le \frac{\left\|u\right\|^2_{L^2_b}}{4} + C'\left\|\nabla\widetilde{T}\right\|^2_{L^2_\phi} \qquad (3.24)$$

Integrating (3.24) between t and t+1we obtain:

$$\left\|\widetilde{T}(t+1)\right\|^2_{L^2_b} + \int_t^{t+1}\left\|\nabla\widetilde{T}\right\|^2_{L^2_\phi}dt \le \left\|\widetilde{T}(t)\right\|^2_{L^2_\phi} + $$

$$+\frac{1}{4}\int_t^{t+1}\left\|u\right\|^2_{L^2_\phi}dt + C'\int_t^{t+1}\left\|\widetilde{T}\right\|^2_{L^2_\phi}dt \qquad (3.24)$$

We drop the first term on the LHS of (3.25) and use our already obtained bounds for $T(x,t)$ as given in (3.3):

$$\sup_{x_0}\sup_t\int_t^{t+1}\left\|\nabla\widetilde{T}\right\|^2_{L^2_\phi}dt \le C' + C\left\|u\right\|^2_{L^{2[0,T;\Omega]}_\phi} + \qquad (3.24)$$

This ends the proof.

Apriori Estimate for the Full Boussinesq System

We obtain estimate for the full Boussinesq system by using the results of previous Lemma and corollary.

Theorem 4.1. *If velocity u and temperature T satisfy equations (1.1)-(1.6); and for*

$$0 \le T(x,0) \le 1 \qquad (4.1)$$

the following hold true:

$$0 \le T(x,t) \le 1 \qquad (4.2)$$

Then, we can estimate (1.1)-(1.6) as follows:

$$\left\|\nabla T\right\|_{L^2_\phi} + \left\|u\right\|_{L^2_\phi} \le C(\left\|u_0\right\|_{L^2_\phi} + C)^2 + C \qquad (4.3)$$

Next, we apply the usual principle of treating (1.1) as we treated the nonlinear Navier-Stokes equation in the previous section for the case of zero flux condition with $e_n T$ serving as the forcing term. We sketch the steps of the proof without too many details. First, we take the scalar product of (1.1) with $\phi^2 u - v_\phi$:

$$(\partial_t u, \emptyset^2 u - v_\emptyset) + (u.\nabla u, \emptyset^2 u - v_\emptyset) -$$
$$-\mu(\Delta u, \emptyset^2 u - v_\emptyset) + (\nabla p, \emptyset^2 u - v_\emptyset) =$$
$$= (T, \emptyset^2 u - v_\emptyset) \qquad (4.4)$$

And like before, we obtain:

$$\frac{d}{dt}(\frac{1}{2}||u||^2_{L^2_\emptyset} - (u, v_\emptyset) +$$
$$+c||\nabla u||^2_{L^2_\emptyset})\left(1 - \varepsilon\sup_{t\epsilon[0,T]}||u(t)||^2_{L^2_\emptyset}\right) +$$
$$+\gamma||u||^2_{L^2_\emptyset} \le C||T||^2_{L^2_\emptyset} + \varepsilon^2\sup_{t\epsilon[0,T]}||u(t)||^2_{L^2_\emptyset} \qquad (4.5)$$

It is easy to see that $\varepsilon\sup_{t\in[0,T]}\left\|u(t)\right\|_{L^2_\phi} \le 1$, hence we obtain:

$$\left\|u(t)\right\|^2_{L^2_\phi} \le C\varepsilon^{-1}(\left\|u(0)\right\|_{L^2_b} + C)^2$$

By a particular choice of $\varepsilon = \frac{1}{4}\left((\left\|u(0)\right\|_{L^2_\phi} + C)^2\right)^{-1}$ we obtain the following estimate:

$$\|u\|_{L_b^2} \leq C(\|u(0)\|_{L_b^2}^2 + C)^2 \qquad (4.6)$$

Coupling estimates (3.17) and (4.6) we obtain full estimate for the Boussinesq system as in (4.3) repeated here:

$$\|\nabla T\|_{L_b^2} + \|u(t)\|_{L_b^2} \leq C(\|u_0\|_{L_b^2} + C)^2 + C, \quad t \geq 0 \qquad (4.7)$$

This completes the proof.
Uniqueness and further regularity results for the solutions may be obtained in a standard way like that of Navier-Stokes Equations; see Zelik(2007) for more on this.

REFERENCES

Anthony P. & Zelik S. (2014) *Infinite energy solutions for Navier-Stokes equations in a strip revisited* .Comm.on Pure and Applied Analysis13: 4, 1361-1393.

Babin A. & Vishik M. (1990), *Attractors of Partial Differential Evolution Equations in an Unbounded Domain*. Proc. Roy. Soc. Edinburgh Sect. A, vol. 116:3-4;221-243

Doering R. & Gibbon J.D (2004*) Applied Analysis of Navier Stokes Equation* .Cambridge University Press London.

Efendiev M. & Zelik S. (2002). *Upper and lower bounds for the Kolmogorov entropy of the attractor for an RDE in an unbounded domain*, J. Dyn. Diff. Eqns 14 :369-403.

Robinson J. (2001) *Infinite-dimensional dynamical systems*. Cambridge University Press.

Triebel H. (1978), *Interpolation Theory, Function Spaces, Differential Operators, North-Holland*.

Zelik S. (2007) *Spatially nondecaying solutions of the 2D Navier-Stokes equation in a strip*. Glasg. Math. J., 49: 3, 525-588.

Zelik S. (2003), *Attractors of reaction-diffusion systems in unbounded domains and their spatial complexity*, Comm. Pure Appl. Math., 56,: 5, 584-637.

Zelik S. (2013), *Infinite energy solutions for damped Navier-Stokes equations in R2*, Jour. Math. Fluid Mech.