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POISSON-GENERALIZED GAMMA EMPIRICAL BAYES MODEL FOR DISEASE MAPPING

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ABSTRACT

In spatial disease mapping, the use of Bayesian models of estimation technique is becoming popular for smoothing relative risks estimates for disease mapping. The most common Bayesian conjugate model for disease mapping is the Poisson-Gamma Model (PG). To explore further the activity of smoothing of relative risk estimates for Bayesian disease mapping, this study focused on the use of generalized gamma distribution as conjugate priors with respect to Poisson likelihood. Two new empirical Bayesian (EB) models are developed; these include Poisson-Generalized Gamma model (PGG) and modified Poisson-Generalized Gamma model (MPGG). The model simulation results indicated that PGG and MPGG models are more likely to handle dispersion in zero-deflated data, contaminated data and zero-inflated data for small and large sample data. Hence, the new EB models are highly competitive to improve the efficiency of relative risk estimation for disease mapping.

Key Words: Disease Mapping, Empirical Bayes, Generalized Gamma, Dispersion, Poisson

INTRODUCTION

The term “disease mapping" was first seen in Clayton & Kaldor (1987), and defined as the investigation, estimation and visual presentation of summary measures of health outcomes across constituent related regions. Disease mapping is mainly used for health risk assessment, management and policy making. A well-known empirical Bayes (EB) model to smooth

relative risks estimates was proposed by Clayton & Kaldor (1987), due to failure of the frequentist approaches to provide satisfactory results. Previously, EB techniques to estimate relative risk have been used in Manton *et al.* (1981), Tsutakawa *et al.* (1985).Subsequent works include Manton*et al.* (1987), Tsutakawa (1988), Manton *et al.* (1989),Clayton & Kaldor (1989). Various modifications with applications are found in Besag*et al.* (1991), Marshall (1991), Clayton & Bernardinelli (1992), Cressie (1992), Waller *et al.* (1997), Maiti (1998), Meza (2003), Gelman *et al.* (2004), MacNab*et al*. (2004), Venkatesan *et al.* (2012), Lawson (2013), Clement (2014), Srinivasan & Venkatesan (2014, 2015), Abbas *et al.* (2015), Coly *et al.* (2015). The EB concept was first proposed by Robbins (1955) in a non-parametric setting. Recent advances in Bayesian statistics have substituted the non-parametric methods for parametric methods. This development was described in Morris (1983), Casella (1985), Maritz & Lwin (1989),Carlin & Gelf and (1990), Raghunathan (1993), Altman & Casella (1995), Efron (1996), Carlin & Louis (2009), Gelman *et al*. (2004), which is the widely used two-level hierarchical models. Some previous scholarly works on EB include the work of Geisser (1965), Maritz (1970), Efron & Morris (1971), Duncan (1974), Rubin (1980), Deely& Lindley (1981). Recent applications includes false discovery rates(Chakraborty *et al*. 2013), microarray data analysis (Newton *et al*. 2011),regularizing maximum likelihood estimation in the matrix Gaussian Procrustes problem (Theobald &Wuttke, 2006),monthly students allowance (Mbata *et al.* 2010), vehicular traffic analysis (Okafor *et al.* 2010,Okafor & Mbata 2012), voting intentions and election forecasting(Okafor & Ogundeji 2012), investigation of disease patterns and analysis (Osei & Yibile 2015, Ngwira & Kazembe 2016).The hallmark of EB analysis is thatit has the capacity to remove the random variability which is present in data from small population counts (Böhning *et al*., 2000), as well as the capacity to combine independent but related studies; and spatial disease mapping provides such a framework. The WHO (1997) reported that the standard Poisson assumption on count data and spatial hierarchical models is appropriate and useful for EB disease mapping. However, the choice of prior for proper modeling may vary depending on the nature of study and, the conjugate prior seems more suitable and flexible. As discussed in Geedipally & Lord (2007), Poisson-Gamma (PG) model is usually preferred over Poisson-lognormal and Poisson-Normal models in point data estimation because it offers a simple way of accommodating over-dispersion which usually features in disease mapping. But one major problem of PG model, as highlighted in Lord *et al.* (2013), is its inability to handle under-dispersion. And its ability to handle over-dispersion for some certain data conditions such as contaminated data and zero-inflated data as well as small sample data is also in doubt. According to Famoye *et al*. (2011), when the data is over-dispersed or under-dispersed, it is pertinent to use a probability model that can handle the situation to maintain efficiency in parameter estimation. The current paper is proposing an empirical Bayes modeling using generalized gamma distribution as conjugate prior with respect to Poisson likelihood for disease mapping, and the investigation of model efficiency in handling dispersionin the estimation of relative risk of disease incidence underzero-deflated data, contaminated data and zero-inflated data for small and large sample size data.

MATERIALS AND METHODS

The EB method of estimation of relative risk parameter using generalized gamma distribution with respect to Poisson likelihood model is presented for EB disease mapping.

Notation

It is assumed that incidence of disease is being investigated in a population that is partitioned into K subpopulations or regions, where

$\boldsymbol{Y_i}$ = observed number of cases of disease in region $i\ (i = 1, \ldots, K)$

$\boldsymbol{E_i}$ = expected number of cases of disease in region $i\ (i = 1, \ldots, K)$

$\boldsymbol{N_i}$ = number of persons at risk for the disease in region $i\ (i = 1, \ldots, K)$

$\boldsymbol{\theta_i}$ = Maximum likelihood estimate of relative risk of disease in region $i\ (i = 1, \ldots, K)$

$\tilde{\boldsymbol{\theta}}_i$ = Posterior estimates of relative risk of disease in region $i\ (i - 1, \ldots, K)$

The $\boldsymbol{Y_i}$ are observed random variables in region$i$, while$\boldsymbol{E_i}$ are known functions of $\boldsymbol{N_i}$. Hence,$\boldsymbol{E_i}$ is obtained as;$\boldsymbol{E_i = N_i\bar{r} = N_i\left(\frac{\sum_{i=1}^{k} Y_i}{\sum_{i=1}^{k} N_i}\right)}$ while$\boldsymbol{\theta_i}$ is $\boldsymbol{\theta_i = \frac{Y_i}{E_i}(thus Y_i = E_i\theta_i).\bar{r}}$ is the overall disease risk in the entire study region(called internal standardization (IS).

1. **EB Modeling of Generalized Gamma Distributions with respect to Poisson Likelihood**

The generalized gamma distribution, introduced by Stacy (1962), is considered, on the fact that the parent gamma distribution is a natural conjugate prior to Poisson likelihood.

Derivation of Poisson-Generalized Gamma (PGG) Model: Based on the general Bayes' theorem, thus

Proposition: Let Y_i be the observed cases of disease in region $i\ (i = 1,2, \ldots, k)$, then

$Y_i|\theta_i \sim Poison\ (E_i\theta_i)$

$\theta_i|\alpha,\beta,\lambda \sim GG1\ (\alpha,\beta,\lambda)$

Therefore,

$$P(\tilde{\theta}_i|Y_i;\ \alpha,\beta,\lambda) = \frac{L(Y_i|\theta_i)\ P(\theta_i|\alpha,\beta,\lambda)}{\int_0^{\infty} L(Y_i|\theta_i)\ P(\theta_i|\alpha,\beta,\lambda)d\theta} \propto L(Y_i|\theta_i)\ P(\theta_i|\alpha,\beta,\lambda). \quad (1)$$

Data Likelihood Function: The Poisson likelihood,ignoring factors that are free of θ, is jointly given as

$$L(Y_i|\theta_i) = \prod_{i=1}^{n}\frac{1}{y_i!}(E_i\theta)^{y_i}e^{-\theta E_i} \propto \theta^{\sum y_i}e^{-\theta\sum E_i} = \theta^{Y}e^{-\theta E} \quad (2)$$

where,$Y = \sum y_i$ is the sufficient statistic, and $E = \sum E_i$ is the random effect.

Prior Distribution: The conjugate prior is the Generalized Gamma (GG) distributiongiven as

$$P(\theta_i|\alpha,\beta,\lambda) = \frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}\theta^{\alpha\lambda-1}e^{-(\beta\theta)^{\lambda}},\ \theta > 0 \quad (3)$$

Derivation of the Posterior Distribution: Therefore plugging the likelihood and the prior distributions; the joint posterior distribution is given as

$$P(\tilde{\theta}_i|Y_i;\ \alpha,\beta,\lambda) \propto \left[\theta^{Y}e^{-\theta E}\right]\left[\frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}\theta^{\alpha\lambda-1}e^{-(\beta\theta)^{\lambda}}\right] = \frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}\theta^{Y+\alpha\lambda-1}e^{-\left(Y+(\beta\theta)^{\lambda}\right)}\alpha,\beta,\lambda,\theta > 0$$

(4)

A proper posterior distribution is achieved by multiplying (4) with the constant of proportionality in (5). The constant of proportionality is the inverse of the marginal distribution (denominator) in the **Proposition.** So,

$\rho = \frac{1}{\int_n^\infty L(Y_i|\theta_i)\, P(\theta_i|\alpha,\beta,\lambda)\, d\theta}$, considering the marginal distribution;

$$\int_0^\infty L(Y_i|\theta_i)\, P(\theta_i|\alpha,\beta,\lambda)d\theta = \int_0^\infty \frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}\theta^{Y+\alpha\lambda-1}e^{-(Y+(\beta\theta)^\lambda)}d\theta$$

$$= \frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}\int_0^\infty \theta^{Y+\alpha\lambda-1}e^{-(Y+(\beta\theta)^\lambda)}d\theta$$

$= \frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}e^{-Y}\int_0^\infty \theta^{Y+\alpha\lambda-1}e^{-(\beta\theta)^\lambda}d\theta$. Integrating by substitution;

$$\rho = \frac{1}{\frac{\Gamma\left(\alpha-\frac{Y}{\lambda}\right)}{\beta^Y\Gamma(\alpha)}e^{-Y}} = \frac{\beta^Y\Gamma(\alpha)}{e^{-Y}\Gamma\left(\alpha+\frac{Y}{\lambda}\right)} \quad (5)$$

Thus, a proper posterior distribution (PGG)is described as in (6);

$$P\left(\tilde{\theta}_i|Y_i;\, \alpha,\beta,\lambda\right) = \left[\frac{\beta^Y\Gamma(\alpha)}{e^{-Y}\Gamma\left(\alpha+\frac{Y}{\lambda}\right)}\right]\left[\frac{\lambda\beta^{\alpha\lambda}}{\Gamma(\alpha)}\theta^{Y+\alpha\lambda-1}e^{-(Y+(\beta\theta)^\lambda)}\right] = \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)}\theta^{Y+\alpha\lambda-1}e^{-(\beta\theta)^\lambda},$$

$\alpha,\beta,\lambda,\theta > 0$ (6)

2. Proof of the probability density function (pdf) of PGG distribution integrates to unity

From the **Proposition,** given as

$$P\left(\tilde{\theta}_i|Y_i;\, \alpha,\beta,\lambda\right) = \frac{L(Y_i|\theta_i)\, P(\theta_i|\alpha,\beta,\lambda)}{\int_0^\infty L(Y_i|\theta_i)\, P(\theta_i|\alpha,\beta,\lambda)d\theta} = \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)}\theta^{Y+\alpha\lambda-1}e^{-(\beta\theta)^\lambda}.$$

Considering the probability axiom$\int_{-\infty}^{\infty}P(x)dx = 1$, therefore,

$$\int_0^\infty P\left(\tilde{\theta}_i|Y_i;\, \alpha,\beta,\lambda\right)d\theta = \int_0^\infty \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)}\theta^{Y+\alpha\lambda-1}e^{-(\beta\theta)^\lambda}d\theta = 1$$

$$\int_0^\infty P\left(\tilde{\theta}_i|Y_i;\, \alpha,\beta,\lambda\right)d\theta = \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)}\int_0^\infty \theta^{Y+\alpha\lambda-1}e^{-(\beta\theta)^\lambda}d\theta = 1$$

Integrating by substitution: Let $W = (\beta\theta)^\lambda \Longrightarrow \theta = \frac{W^{\frac{1}{\lambda}}}{\beta}$, $hence\, d\theta = \frac{W^{\frac{1}{\lambda}-1}}{\lambda\beta}dW$, therefore

$$\int_0^\infty P\left(\tilde{\theta}_i|Y_i;\, \alpha,\beta,\lambda\right)d\theta = \left[\frac{1}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)}\right]\left[\Gamma\left(\alpha+\frac{Y}{\lambda}\right)\right] = 1$$

Hence, the pdf of the PGG1 distribution is a proper distribution since it integrates to unity [1].

3. The rth Moment, Expectation and Variance of PGG Model

Using method of moment (MOM), therefore

$$E(\tilde{\theta}^r) = \int_0^\infty \theta^r \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)} \theta^{Y+\alpha\lambda-1} e^{-(\beta\theta)^\lambda} d\theta = \int_0^\infty \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)} \theta^{Y+\alpha\lambda+r-1} e^{-(\beta\theta)^\lambda} d\theta$$

$= \frac{\lambda\beta^{Y+\alpha\lambda}}{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)} \int_0^\infty \theta^{Y+\alpha\lambda+r-1} e^{-(\beta\theta)^\lambda} d\theta$.Integrating by substitution; the rth moment of PGG is obtained as in (7).

$$E(\tilde{\theta}^r) = \frac{\Gamma\left(\alpha+\frac{Y+r}{\lambda}\right)}{\beta^r \Gamma\left(\alpha+\frac{Y}{\lambda}\right)} \tag{7}$$

Therefore, the mean and the variance are obtained as

$$E(\tilde{\theta}) = \frac{\Gamma\left(\alpha+\frac{Y+1}{\lambda}\right)}{\beta\Gamma\left(\alpha+\frac{Y}{\lambda}\right)} \tag{8}$$

$$V(\tilde{\theta}) = \frac{\Gamma\left(\alpha+\frac{Y+2}{\lambda}\right)}{\beta^2\Gamma\left(\alpha+\frac{Y}{\lambda}\right)} - \frac{\Gamma^2\left(\alpha+\frac{Y+1}{\lambda}\right)}{\beta^2\Gamma^2\left(\alpha+\frac{Y}{\lambda}\right)} = \frac{\Gamma\left(\alpha+\frac{Y}{\lambda}\right)\Gamma\left(\alpha+\frac{Y+2}{\lambda}\right) - \Gamma^2\left(\alpha+\frac{Y+1}{\lambda}\right)}{\beta^2\Gamma^2\left(\alpha+\frac{Y}{\lambda}\right)} \tag{9}$$

4. Estimation of α and β for PGG Model

The method of moment estimation proposed by Huang and Hwang (2006) for estimation of three parameter GG family was employed to estimate the hyperparameters $(\alpha \text{ and } \beta)$ from the prior distribution (GG) to completely specify the posterior distribution (PGG).Therefore,

$$E(\theta^r) = \mu^r = \frac{\Gamma\left(\alpha+\frac{r}{\lambda}\right)}{\beta^r\Gamma(\alpha)} \tag{10}$$

$$\mu = \frac{\Gamma\left(\alpha+\frac{1}{\lambda}\right)}{\beta\Gamma(\alpha)} \tag{11}$$

$$\mu^2 = \frac{1}{K}\frac{\Gamma\left(\alpha+\frac{2}{\lambda}\right)}{\beta^2\Gamma(\alpha)} + \frac{(K-1)}{K}\frac{\Gamma^2\left(\alpha+\frac{1}{\lambda}\right)}{\beta^2\Gamma^2(\alpha)} = \frac{\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\lambda}\right)+(K-1)\Gamma^2\left(\alpha+\frac{1}{\lambda}\right)}{K\beta^2\Gamma^2(\alpha)} \tag{12}$$

$$\sigma^2 = \frac{\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\lambda}\right)-\Gamma^2\left(\alpha+\frac{1}{\lambda}\right)}{\beta^2\Gamma^2(\alpha)} \tag{13}$$

Therefore, square of the coefficient of variation is obtained as

$$\frac{\sigma^2}{\mu^2} = \frac{K\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\lambda}\right)-K\Gamma^2\left(\alpha+\frac{1}{\lambda}\right)}{\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\lambda}\right)+(K-1)\Gamma^2\left(\alpha+\frac{1}{\lambda}\right)} \tag{14}$$

For optimization, λ can assume value 0.5 or 1.5 since $\lambda = 1.0$ gives gamma distribution and $\lambda = 2.0$ is approximately a generalized normal distribution.

When$\lambda = 1/2$ in equation (11),$\hat{\beta} - \frac{\alpha(\alpha+1)}{\mu}$ (15)

When $\lambda - 1/2$ in equation (14), $\hat{\alpha} = \left[6\left(\frac{\mu^2}{\sigma^2} - \frac{1}{K}\right) + \left(\frac{1}{2} + \frac{2}{K} - \frac{2\mu^2}{\sigma^2}\right)^2\right]^{\frac{1}{2}} - \left(\frac{1}{2} + \frac{2}{K} - \frac{2\mu^2}{\sigma^2}\right)$ (16)

For further details about the prior distribution; Generalized Gamma Distribution and its moment's derivation, see Dadpay*et al*. (2007), Khodabin and Ahmadabadi (2010).

5. Estimation of Relative Risk (RR) and Variance from PGG Model

Having derived the conditional posterior distribution, the posterior mean and variance, the estimation of relative risk (RR) in each Sub-population is obtained as

$$\hat{\theta}_i^{PGG} = \frac{\Gamma\left(\alpha + \frac{Y_i+1}{\lambda}\right)}{\beta\,\Gamma\left(\alpha + \frac{Y_i}{\lambda}\right)} \tag{17}$$

and the estimation of variance of relative risk (RR) in each sub-population is obtained as

$$Var\left(\hat{\theta}_i^{PGG}\right) = \frac{\Gamma\left(\alpha + \frac{Y_i}{\lambda}\right)\Gamma\left(\alpha + \frac{Y_i+2}{\lambda}\right) - \Gamma^2\left(\alpha + \frac{Y_i+1}{\lambda}\right)}{\beta^2\Gamma^2\left(\alpha + \frac{Y_i}{\lambda}\right)} \tag{18}$$

6. Estimation of Relative Risk (RR) and Variance from MPGG Model

The modified model (MPGG) is derived by putting $\beta = 1$ in equation (6). The derived conditional posterior distribution of MPGGis described as

$$P\left(\theta_i | Y_i; \alpha, \lambda\right) = \frac{\lambda}{\Gamma\left(\alpha + \frac{Y_i}{\lambda}\right)}\theta^{Y+\alpha\lambda-1}e^{-\theta^\lambda},\ \alpha, \lambda, \theta > 0, \tag{19}$$

and the posterior mean and variance, the estimation of relative risk (RR) in each sub-population is obtained as

$$\hat{\theta}_i^{MPGG} = \frac{\Gamma\left(\alpha + \frac{Y_i+1}{\lambda}\right)}{\Gamma\left(\alpha + \frac{Y_i}{\lambda}\right)} \tag{20}$$

and the estimation of variance of relative risk (RR) in each Sub-population is obtained as

$$Var\left(\hat{\theta}_i^{MPGG}\right) = \frac{\Gamma\left(\alpha + \frac{Y_i}{\lambda}\right)\Gamma\left(\alpha + \frac{Y_i+2}{\lambda}\right) - \Gamma^2\left(\alpha + \frac{Y_i+1}{\lambda}\right)}{\Gamma^2\left(\alpha + \frac{Y_i}{\lambda}\right)} \tag{21}$$

The developed EB models with generalized gamma distribution as conjugate prior revealed a proper conjugate prior with respect to Poisson likelihood.

RESULTS AND DISCUSSIONS

This section presents the characteristics of the simulation study to investigate the effect of dispersion on the efficiency of the different EB models, including the existing Poisson-Gamma model, in the estimation of relative risk parameter under zero-deflated data, contaminated data and zero-inflated data. The investigation is carried out through a simulation study of 1000 samples using MCMC sampling technique. In order to consider variation, the simulated dispersion parameter is obtained for different sample sizes (Y) = 5, 10, 20, 50, 100, 1000; theoretical dispersion parameter (Ø) = 0.5, 1.0, 2.0; and fixed mean (θ) = 0.5, 1.0, 10. These are carried out for each scenario. Comparison is based on the precision of predicted dispersion parameter value to theoretical value, and the standard deviation obtained in different cases. The choice of theoretical values is based on the previous work of Lord (2006)

and Lord et al. (2013). The simulation results are presented in Tables 4.1 and 4.2 respectively. The corresponding diagnostic plots are in Figures 4.1 to 4.4 respectively, for further investigation.

Table 1: Simulation results for dispersion parameter values in zero-deflated data

	ϕ	0.5			1.0			2.0		
θ		PG	PGG	MPGG	PG	PGG	MPGG	PG	PGG	MPGG
n = 5										
0.5	$\tilde{\phi}$	0.2593	0.2649	0.5260	1.0058	0.9037	0.9037	4.0342	3.0228	1.5116
	SD	0.0097	0.0035	0.0014	0.0159	0.0062	0.0062	0.0764	0.0331	0.0084
1.0	$\tilde{\phi}$	0.2579	0.2640	0.5305	1.0083	0.9080	0.9080	4.0232	3.0116	1.5114
	SD	0.0095	0.0044	0.0032	0.0190	0.0130	0.0130	0.0675	0.0197	0.0164
10	$\tilde{\phi}$	0.2568	0.2696	0.5275	0.9996	0.9117	0.9117	4.0068	3.0412	1.5156
	SD	0.0113	0.0034	0.0033	0.0176	0.0167	0.0167	0.0553	0.0180	0.0126
n = 10										
0.5	$\tilde{\phi}$	0.2587	0.2644	0.5256	1.0038	0.9016	0.9016	4.0338	3.0146	1.5050
	SD	0.0096	0.0043	0.0021	0.0199	0.0137	0.0137	0.0672	0.0369	0.0162
1.0	$\tilde{\phi}$	0.2576	0.2638	0.5251	1.0051	0.9016	0.9016	3.9913	2.9935	1.5073
	SD	0.0088	0.0050	0.0035	0.0237	0.0083	0.0083	0.0488	0.0355	0.0183
10	$\tilde{\phi}$	0.2550	0.2686	0.5270	1.0031	0.9069	0.9069	3.9903	2.9923	1.5000
	SD	0.0109	0.0048	0.0036	0.0185	0.0070	0.0070	0.0499	0.0291	0.0159
n = 20										
0.5	$\tilde{\phi}$	0.2510	0.2636	0.5251	0.9974	0.9016	0.9016	3.9930	3.0077	1.5024
	SD	0.0086	0.0053	0.0022	0.0202	0.0111	0.0111	0.0712	0.0386	0.0178
1.0	$\tilde{\phi}$	0.2572	0.2626	0.5249	1.0012	0.9050	0.9050	3.9964	3.0045	1.5039
	SD	0.0087	0.0051	0.0043	0.0231	0.0153	0.0153	0.0677	0.0324	0.0191
10	$\tilde{\phi}$	0.2523	0.2630	0.5266	1.0027	0.9004	0.9004	4.0063	3.0031	1.5022
	SD	0.0098	0.0052	0.0040	0.0246	0.0143	0.0143	0.0582	0.0299	0.0160
n = 50										
0.5	$\tilde{\phi}$	0.2505	0.2623	0.5247	0.9944	0.9008	0.9008	4.0052	3.0126	1.5046
	SD	0.0085	0.0055	0.0028	0.0183	0.0126	0.0126	0.0618	0.0308	0.0150
1.0	$\tilde{\phi}$	0.2517	0.2623	0.5237	1.0002	0.9042	0.9042	4.0062	3.0136	1.5064
	SD	0.0085	0.0053	0.0045	0.0194	0.0133	0.0133	0.0633	0.0281	0.0154
10	$\tilde{\phi}$	0.2506	0.2622	0.5260	0.9961	0.9022	0.9022	3.9867	3.0057	1.5037
	SD	0.0089	0.0053	0.0045	0.0221	0.0149	0.0149	0.0633	0.0301	0.0163
n = 100										
0.5	$\tilde{\phi}$	0.2501	0.2619	0.5226	1.0012	0.9034	0.9034	4.0088	3.0115	1.5062
	SD	0.0082	0.0056	0.0052	0.0228	0.0139	0.0139	0.0667	0.0293	0.0157
1.0	$\tilde{\phi}$	0.2510	0.2613	0.5235	1.0011	0.9039	0.9039	4.0078	3.0098	1.5058
	SD	0.0084	0.0056	0.0046	0.0224	0.0150	0.0150	0.0598	0.0351	0.0146
10	$\tilde{\phi}$	0.2504	0.2621	0.5251	0.9980	0.9036	0.9036	4.0019	3.0115	1.5052
	SD	0.0085	0.0055	0.0049	0.0212	0.0148	0.0148	0.0587	0.0327	0.0163
n = 1000										
0.5	$\tilde{\phi}$	0.2500	0.2618	0.5224	1.0003	0.9027	0.9027	4.0017	3.0092	1.5046
	SD	0.0078	0.0056	0.0055	0.0221	0.0133	0.0133	0.0642	0.0324	0.0159
1.0	$\tilde{\phi}$	0.2498	0.2617	0.5234	1.0001	0.9031	0.9031	4.0040	3.0092	1.5046
	SD	0.0077	0.0058	0.0049	0.0220	0.0133	0.0133	0.0636	0.0319	0.0161
10	$\tilde{\phi}$	0.2501	0.2617	0.5239	0.9977	0.9031	0.9031	3.9995	3.0085	1.5048
	SD	0.0079	0.0056	0.0054	0.0227	0.0138	0.0138	0.0598	0.0320	0.0152

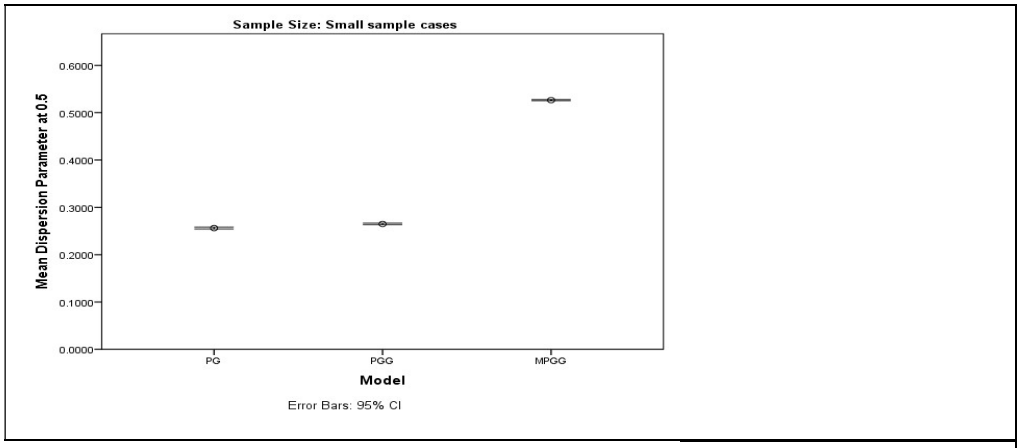
Note: θ = Fixed Mean. ϕ = Theoretical Dispersion parameter, $\tilde{\phi}$ = Posterior estimate, n = Sample size

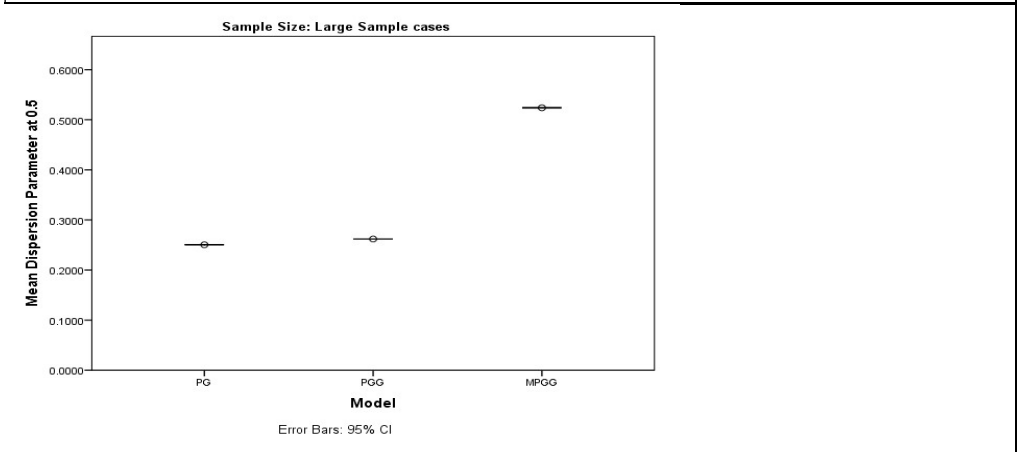
The simulation results of dispersion parameter for the different EB models have shown the following characteristics under zero-deflated data (Table 1):

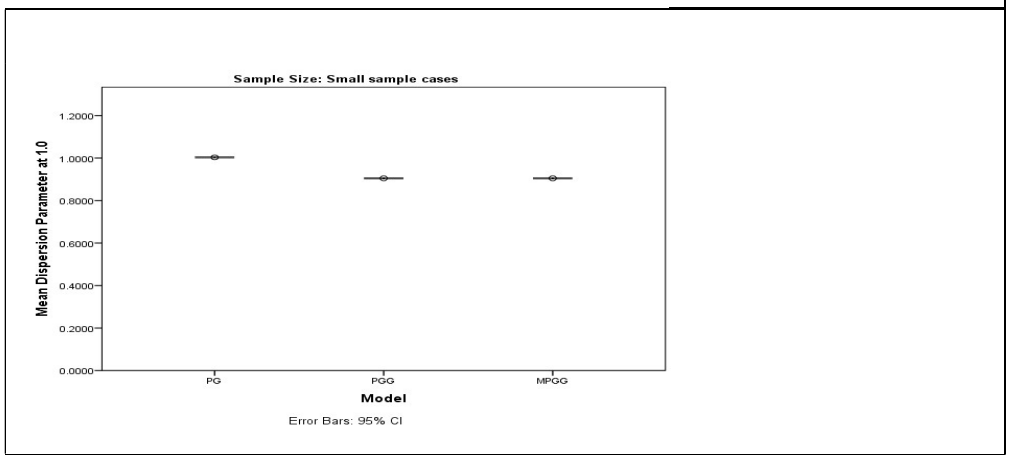
1. For sample size of 1000, 100, 50, 20, 10, 5 (both small and large sample sizes) with low or high mean at Ø= 0.5, MPGG EB model accurately predicted the theoretical value of dispersion parameter while PG and PGG models tend to underestimate the theoretical value (Ø= 0.5) by a factor of almost 2.
2. For sample size of 1000, 100, 50, 20, 10, 5 with low mean or high mean, all the EB models including the existing PG model predicted values very close to the theoretical value of the dispersion parameter for Ø= 1.0. But estimates from PGG models have smaller standard deviations.
3. For sample size of 1000, 100, 50, 20, 10, 5 with high mean, the dispersion parameter is likely to be unreliably estimated, no matter which EB model is used for Ø= 2.0. However, estimate of the predicted dispersion parameter value from MPGG model are much closer with lesser standard deviation to the theoretical value of the dispersion parameter.
4. As the sample size decreases, the distribution of the estimated values from PG becomes more skewed, which increases the mean of the estimated values of the dispersion parameter for Ø= 0.5 while as the sample size decreases, the distribution of the estimated values from PGG and MPGG becomes less skewed with increase in the mean of the estimated values of the dispersion parameter for Ø= 0.5. It implies that at small sample size, the PGG EB models are highly likely to perform competitively in the estimation of relative risk.
5. Small and large sample size data characterized with a low sample mean or a high sample mean are most likely to be highly dispersed for PG model while most likely less dispersed for PGG EB models. It implies that over-dispersion or under-dispersion is more likely to affect the efficiency of PG model in the estimation of relative risk parameter compared to PGG models. Since the standard deviations of the PGG models are generally less than the PG model, the PGG models are equally competitive in handling dispersion (over-dispersion or under-dispersion) in the estimation of relative risk parameter estimation for disease mapping, although the models may show the dispersion parameter is underestimated or overestimated in some cases. The simulation results of the posterior estimates of dispersion parameter are reliable, since there is convergence of MCMC sampling, as shown in the series plots (figure 4) for each model.

The summary analysis using diagnostic error bar plots presented in figure 1 depicted that under zero-deflated data (both small and large sample sizes) characterize with low or high sample mean bias MPGG model provided much more reliable estimates of theoretical values of dispersion parameter. The results implied that under zero-deflated data MPGG has a better model efficiency in handling dispersion in the estimation of relative risk parameter for disease mapping.

Figure 1: Diagnostic Error Bar Plots for Zero-deflated Data







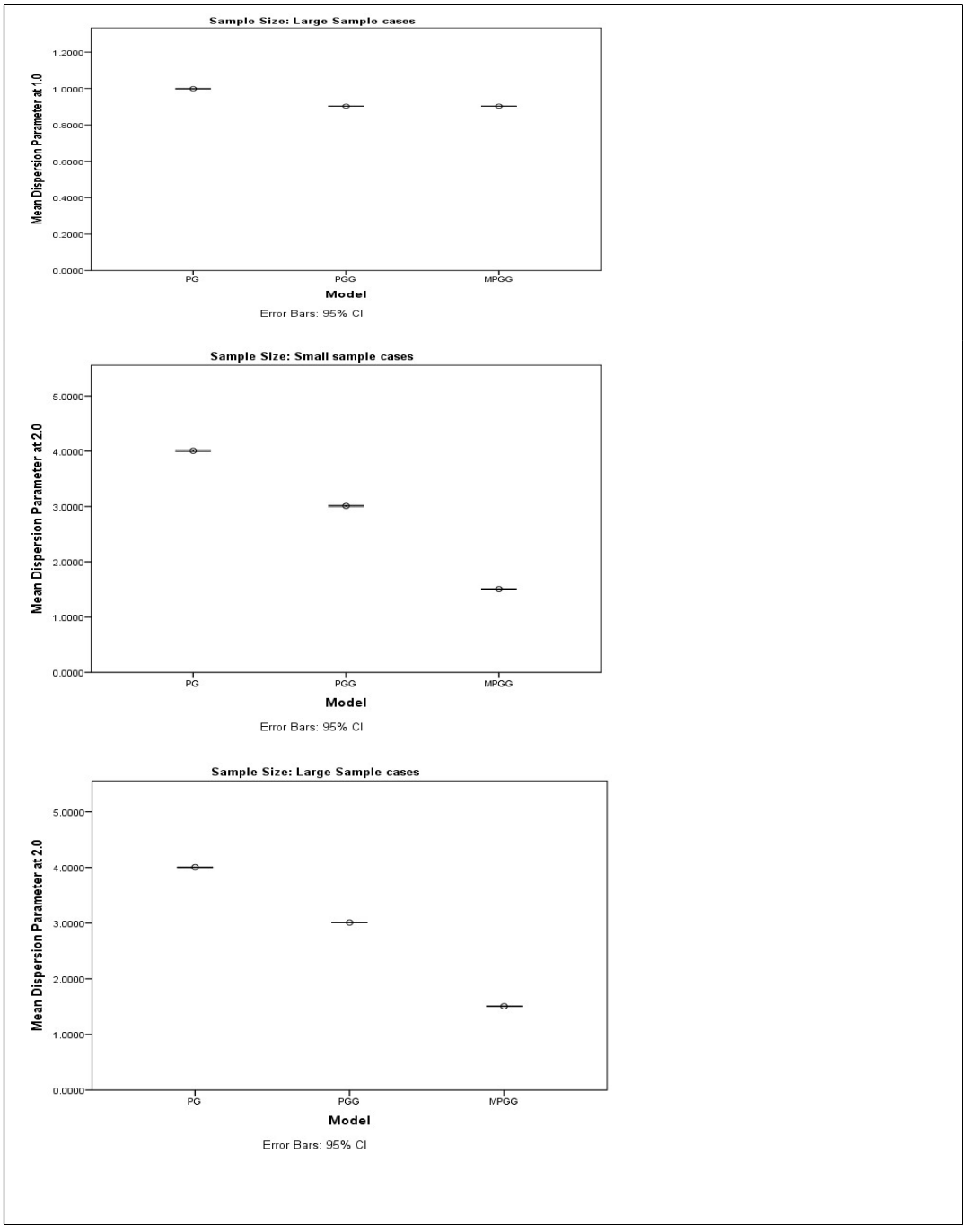


Table 2: Simulation results for dispersion parameter values in contaminated data

		0.5			1.0			2.0		
		PG	PGG	MPGG	PG	PGG	MPGG	PG	PGG	MPGG
n = 5										
0.5		0.6867	0.5045	0.8989	1.5329	1.1996	1.1996	3.5828	3.0688	1.6862
	SD	0.3678	0.1703	0.3167	0.5997	0.2761	0.2761	0.7762	0.4366	0.2500
1.0		0.5809	0.5374	0.8135	1.2022	1.0614	1.0614	2.6834	2.5138	1.5354
	SD	0.2979	0.1637	0.2637	0.4042	0.2454	0.2454	0.8552	0.4375	0.2050
10		0.1827	0.2011	0.2747	0.3826	0.3984	0.3984	1.0646	0.9483	0.6284
	SD	0.0713	0.0636	0.1502	0.3171	0.2486	0.2486	1.4686	1.0324	0.4597
n = 10										
0.5		0.5403	0.5034	0.7818	1.3473	1.0985	1.0985	3.6031	2.9679	1.5924
	SD	0.3266	0.1641	0.2995	0.4798	0.2424	0.2424	0.7108	0.2891	0.1866
1.0		0.4703	0.5278	0.6975	1.0963	1.0867	1.0867	2.9477	2.5707	1.5129
	SD	0.2638	0.1382	0.2184	0.2913	0.1806	0.1806	0.9769	0.4467	0.1451
10		0.1941	0.2055	0.3251	0.5306	0.5117	0.5117	1.8022	1.4618	0.8297
	SD	0.0708	0.0621	0.1700	0.3950	0.3167	0.3167	1.8226	1.2766	0.5426
n = 20										
0.5		0.6597	0.5028	0.8719	1.5383	1.1837	1.1837	3.9101	3.1053	1.7008
	SD	0.5031	0.1634	0.4211	0.7662	0.3615	0.3615	0.9431	0.4773	0.3074
1.0		0.5672	0.5199	0.7843	1.2460	1.0712	1.0712	3.1327	2.7310	1.5562
	SD	0.4075	0.1090	0.3484	0.5084	0.2910	0.2910	0.9439	0.4768	0.2512
10		0.2186	0.2297	0.3519	0.5573	0.5421	0.5421	1.8299	1.4864	0.8694
	SD	0.0841	0.0748	0.1638	0.3743	0.3039	0.3039	1.7957	1.2527	0.5358
n = 50										
0.5		0.6398	0.5110	0.8632	1.4711	1.1632	1.1632	3.6547	3.0171	1.6648
	SD	0.3947	0.1860	0.3302	0.5956	0.2929	0.2929	0.7988	0.3807	0.2340
1.0		0.5451	0.5325	0.7570	1.1795	1.0378	1.0378	2.8445	2.5800	1.5099
	SD	0.3153	0.1682	0.2768	0.4000	0.2274	0.2274	0.9255	0.4401	0.1985
10		0.1920	0.2084	0.3040	0.4609	0.4661	0.4661	1.4352	1.2089	0.7366
	SD	0.0731	0.0680	0.1583	0.3635	0.2962	0.2962	1.6835	1.1881	0.5067
n = 100										
0.5		0.6263	0.5084	0.8569	1.4606	1.1571	1.1571	3.7129	3.0488	1.6688
	SD	0.3719	0.1839	0.3240	0.5396	0.2714	0.2714	0.6621	0.3252	0.2107
1.0		0.5369	0.5288	0.7582	1.1866	1.0440	1.0440	2.9476	2.6242	1.5197
	SD	0.2965	0.1617	0.2572	0.3424	0.2033	0.2033	0.8858	0.3862	0.1652
10		0.2004	0.2162	0.3225	0.5016	0.4999	0.4999	1.6270	1.3430	0.7950
	SD	0.0649	0.0596	0.1581	0.3698	0.3033	0.3033	1.7562	1.2325	0.5229
n = 1000										
0.5		0.6243	0.5058	0.8540	1.4660	1.1566	1.1566	3.7653	3.0591	1.6665
	SD	0.3849	0.1818	0.3339	0.5553	0.2770	0.2770	0.6547	0.3271	0.2161
1.0		0.5356	0.5168	0.7594	1.1939	1.0450	1.0450	3.0129	2.6572	1.5303
	SD	0.3052	0.1566	0.2639	0.3500	0.2094	0.2094	0.8779	0.3878	0.1653
10		0.2045	0.2197	0.3337	0.5248	0.5179	0.5179	1.7181	1.4098	0.8274
	SD	0.0634	0.0585	0.1583	0.3735	0.3038	0.3038	1.7704	1.2448	0.5309

Note: θ = Fixed Mean. ϕ = Theoretical Dispersion parameter, $\tilde{\phi}$ = Posterior estimate, n = Sample size

The simulation results of dispersion parameter for the different EB models have shown the following characteristics under contaminated data (Table 2):

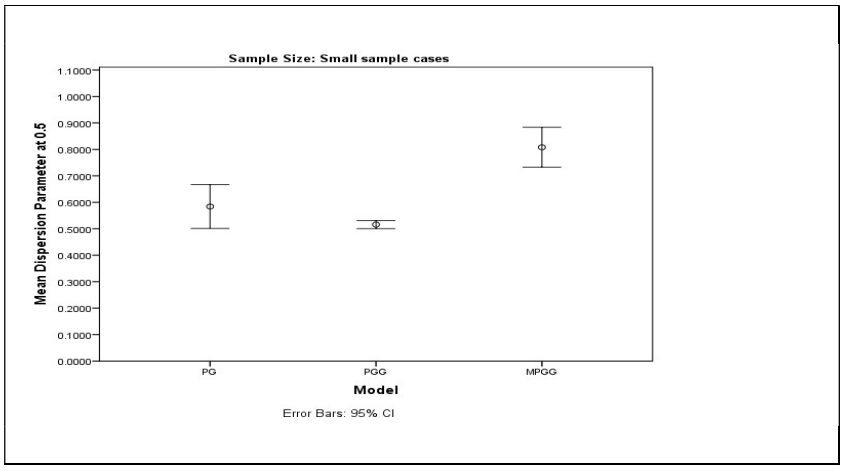
1. For large and small sample sizes of a contaminated data characterized by a low sample mean bias, PGG model provided an unbiased estimate of the theoretical value of the dispersion parameter at $\phi = 0.5$ while for contaminated data characterized by a

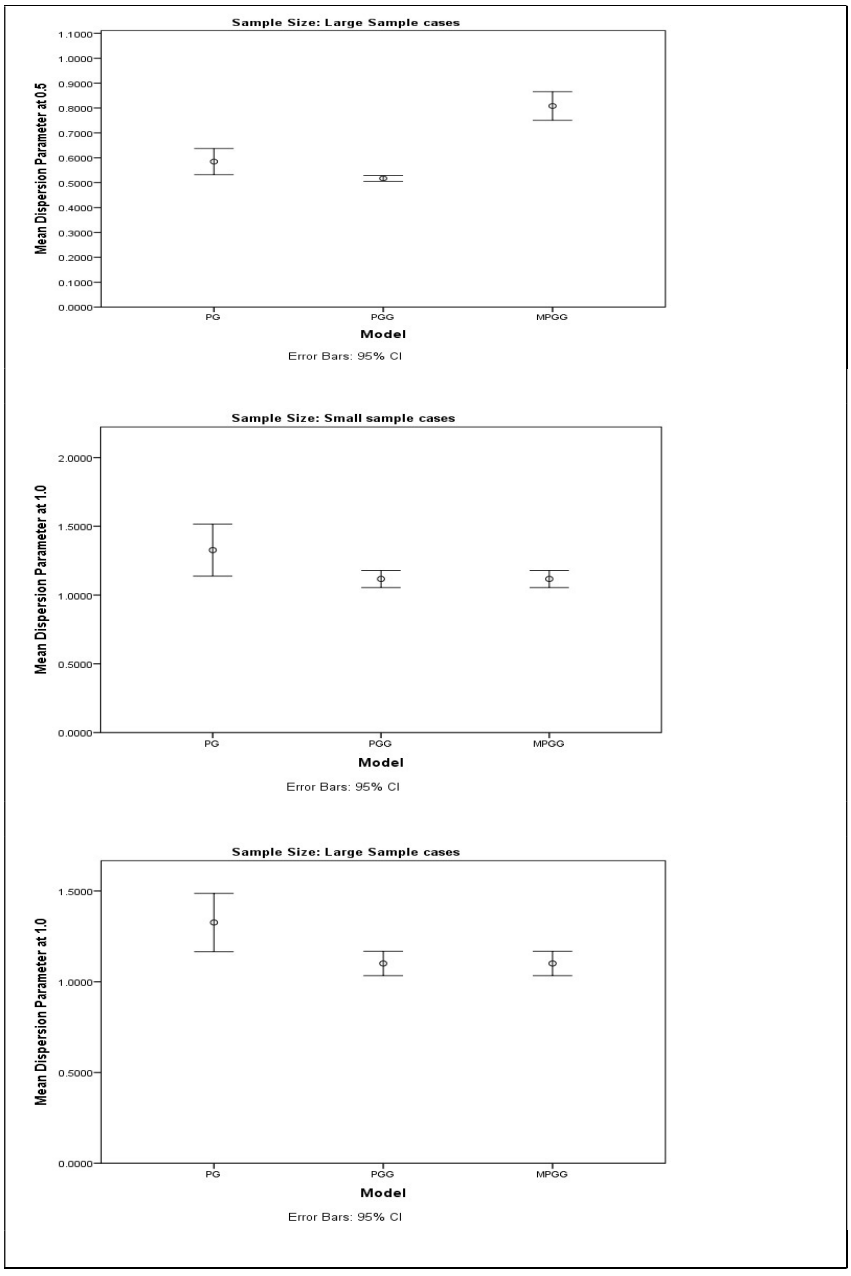
high sample mean bias, PGG model, including MPGG and PG models, is likely to provide unreliable estimates of the theoretical value of the dispersion parameter.

2. For large and small sample sizes of a contaminated data characterized by a low sample mean, PGG and MPGG models provided a better estimate of the dispersion parameter at = 1.0 while for contaminated data characterized by a high sample mean bias, the three models are likely to provide unreliable estimates of the theoretical value of the dispersion parameter.
3. For large and small sample sizes of a contaminated data characterized by a high sample mean bias (under-dispersion), the three EB models are likely unreliable to handle dispersionat = 2.0. However, estimate of the predicted dispersion parameter value from MPGG model are much closer with lesser standard deviation to the theoretical value of the dispersion parameter.
4. For large and small sample sizes contaminated data characterized by a low sample mean (over-dispersion), PGG EB models are equally likely efficient in the estimation of relative risk parameter, since the standard deviations of the PGG models are generally less than the PG model. Although the models may show the dispersion parameter is either underestimated or overestimated. The simulation results of the posterior estimates of dispersion parameter are reliable, since there is convergence of MCMC sampling, as shown in the series plots (figure 5) for each model.

The summary analysis using diagnostic error bar plots presented in figure 2 depicted that under contaminated data (both small and large sample sizes) characterize with low sample mean bias, PGG model provided reliable estimates of theoretical values of dispersion parameter, which implied that PGG model has a better model efficiency in handling over-dispersion in the estimation of relative risk parameter for disease mapping. In the other hand, contaminated data (both small and large sample sizes) characterize with high sample mean bias, MPGG model provided much more reliable estimates of theoretical value of dispersion parameter, which implied that MPGG has a better model efficiency in handling under-dispersion in the estimation of relative risk parameter for disease mapping.

Figure 2: Diagnostic Error Bar Plots for Contaminated Data





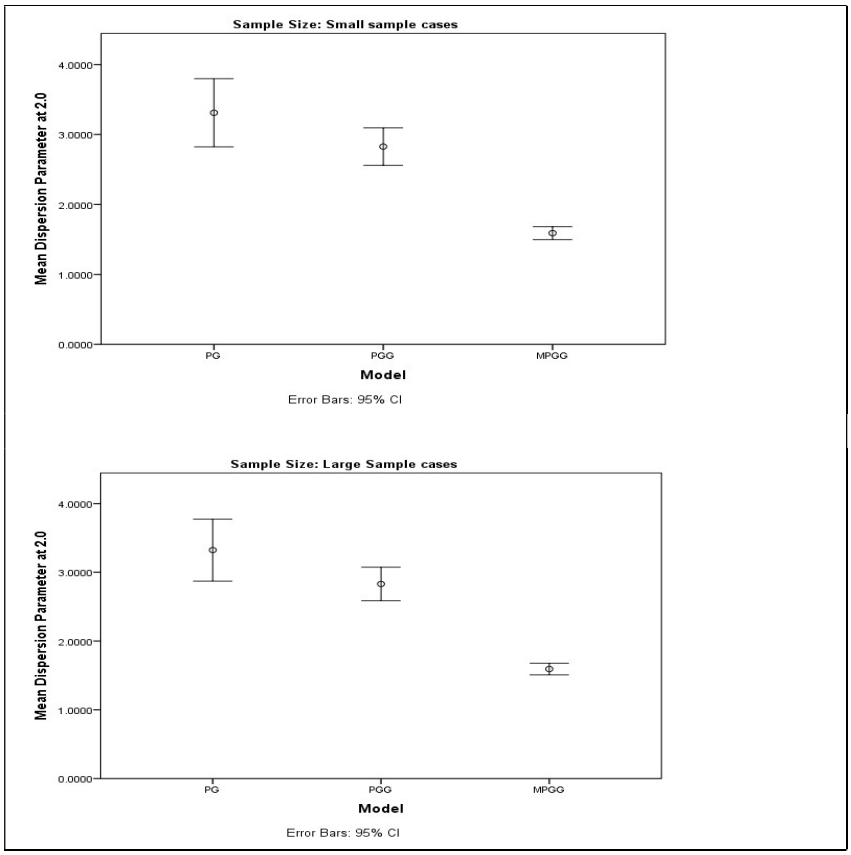


Table 3: Simulation results for dispersion parameter values in Zero-Inflated data

		0.5			1.0			2.0		
		PG	PGG	MPGG	PG	PGG	MPGG	PG	PGG	MPGG
n = 5										
0.5		0.4266	0.4022	0.6399	1.2032	1.0463	1.0463	2.8150	2.1792	1.5150
	SD	0.3048	0.1556	0.2973	0.5016	0.2601	0.2601	0.7505	0.1926	0.2071
1.0		0.4389	0.3681	0.6313	0.9026	0.8787	0.8787	1.8762	2.0858	1.3148
	SD	0.2541	0.1538	0.2613	0.3764	0.2443	0.2443	0.5002	0.1358	0.2169
10		0.1097	0.1302	0.1446	0.1641	0.2091	0.2091	0.2681	0.3676	0.3473
	SD	0.0635	0.0578	0.0720	0.0684	0.0674	0.0674	0.0715	0.0628	0.0646
n = 10										
0.5		0.3602	0.3188	0.5702	0.9374	0.8821	0.8821	2.4160	2.1759	1.3944
	SD	0.2625	0.1378	0.2771	0.4454	0.2338	0.2338	0.6699	0.1740	0.1980
1.0		0.3002	0.2864	0.4830	0.7030	0.7459	0.7459	1.6108	2.0761	1.2173
	SD	0.2188	0.1341	0.2420	0.3340	0.2147	0.2147	0.4467	0.1031	0.1902
10		0.0750	0.0937	0.1023	0.1278	0.1766	0.1766	0.2301	0.3291	0.3082
	SD	0.0547	0.0513	0.0633	0.0607	0.0605	0.0605	0.0638	0.0576	0.0587
n = 20										
0.5		0.4013	0.3357	0.6166	0.9963	0.9177	0.9177	2.5009	2.1727	1.4276
	SD	0.3453	0.1748	0.3343	0.5752	0.2945	0.2945	0.8558	0.1605	0.2526
1.0		0.3344	0.3047	0.5149	0.7471	0.7779	0.7779	1.6672	2.0272	1.2444
	SD	0.2876	0.1622	0.2922	0.4312	0.2680	0.2680	0.5704	0.1933	0.2326
10		0.0836	0.1058	0.1162	0.1359	0.1837	0.1837	0.2382	0.3386	0.3173
	SD	0.0719	0.0678	0.0808	0.0784	0.0775	0.0775	0.0815	0.0750	0.0755

n = 50										
0.5	$\tilde{\phi}$	0.4329	0.3577	0.6524	1.0551	0.9479	0.9479	2.5814	2.1201	1.4485
	SD	0.3219	0.1631	0.3083	0.5386	0.2764	0.2764	0.7998	0.1356	0.2372
1.0	$\tilde{\phi}$	0.3607	0.3230	0.5457	0.7913	0.8102	0.8102	1.7209	2.0643	1.2566
	SD	0.2682	0.1527	0.2760	0.4039	0.2429	0.2429	0.5332	0.1598	0.2160
10	$\tilde{\phi}$	0.0902	0.1117	0.1239	0.1439	0.1906	0.1906	0.2459	0.3460	0.3256
	SD	0.0671	0.0630	0.0736	0.0734	0.0721	0.0721	0.0762	0.0669	0.0696
n = 100										
0.5	$\tilde{\phi}$	0.4163	0.3488	0.6420	1.0262	0.9374	0.9374	2.5399	2.1083	1.4437
	SD	0.3006	0.1570	0.2955	0.5013	0.2654	0.2654	0.7517	0.1059	0.2259
1.0	$\tilde{\phi}$	0.3469	0.3182	0.5343	0.7696	0.7959	0.7959	1.6933	2.0485	1.2517
	SD	0.2505	0.1470	0.2576	0.3760	0.2369	0.2369	0.5011	0.1481	0.2055
10	$\tilde{\phi}$	0.0867	0.1083	0.1204	0.1399	0.1893	0.1893	0.2419	0.3428	0.3245
	SD	0.0626	0.0606	0.0683	0.0684	0.0680	0.0680	0.0716	0.0634	0.0659
n = 1000										
0.5	$\tilde{\phi}$	0.4025	0.3428	0.6236	1.0037	0.9226	0.9226	2.5063	2.0846	1.4320
	SD	0.2818	0.1489	0.2791	0.4702	0.2475	0.2475	0.7059	0.1880	0.2142
1.0	$\tilde{\phi}$	0.3355	0.3116	0.5224	0.7528	0.7847	0.7847	1.6708	2.0375	1.2425
	SD	0.2348	0.1394	0.2447	0.3526	0.2205	0.2205	0.4706	0.1246	0.1934
10	$\tilde{\phi}$	0.0839	0.1057	0.1173	0.1369	0.1856	0.1856	0.2387	0.3386	0.3202
	SD	0.0587	0.0564	0.0644	0.0641	0.0631	0.0631	0.0672	0.0576	0.0623

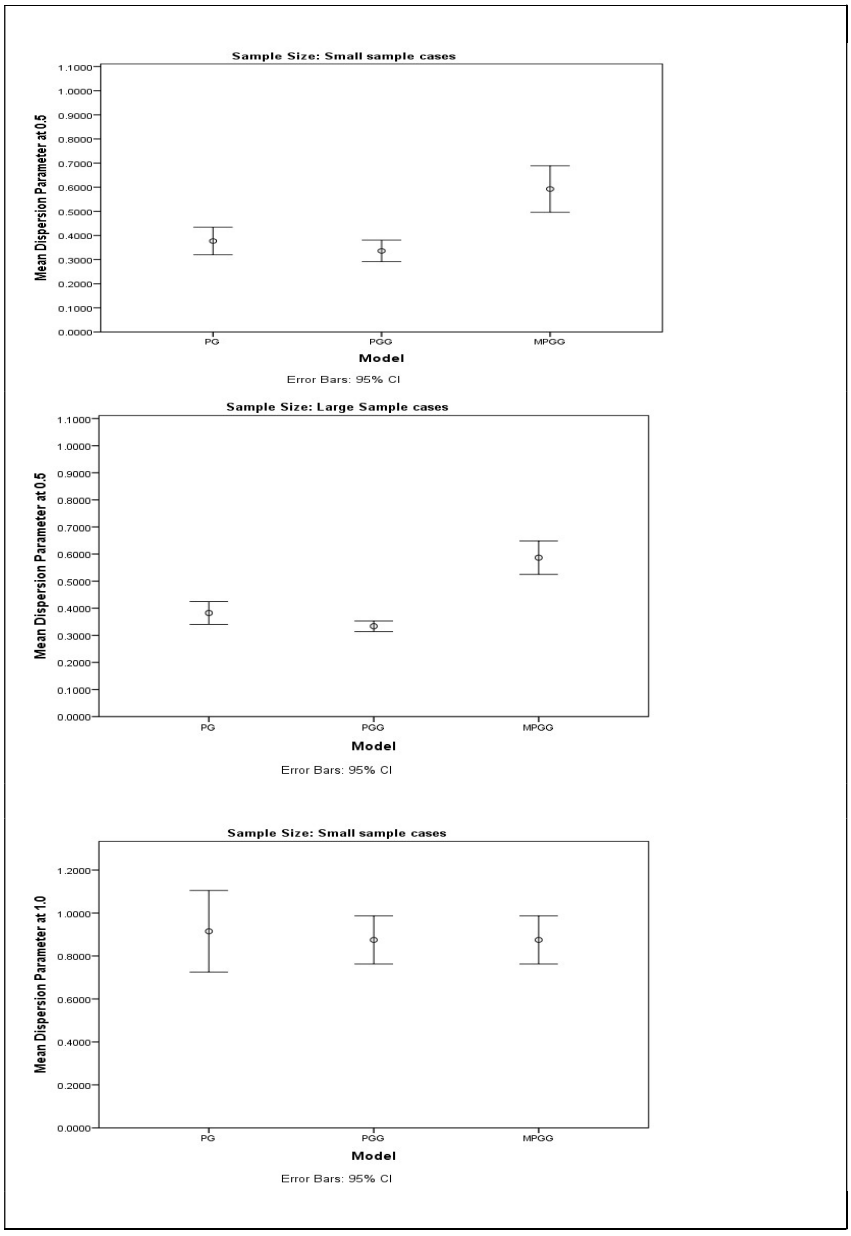
Note: θ = Fixed Mean. ϕ = Theoretical Dispersion parameter, $\tilde{\phi}$ = Posterior estimate, n = Sample size

The simulation results of dispersion parameter for the different EB models have shown the following characteristics under zero-inflated data (Table 3):

1. For large and small sample sizes of a zero-inflated data characterized by a low sample mean bias, the estimates from MPGG model are much closer to the theoretical dispersion parameter than PG model and PGG model at ϕ = 0.5.
2. For large and small sample sizes of a zero-inflated data characterized by a low sample mean, the three EB models provided a good estimate of the dispersion parameter at ϕ = 1.0. But estimates from PGG models have smaller standard deviations.
3. For large and small sample sizes of a zero-inflated data characterized by a high sample mean bias (under-dispersion), PGG model provided reliable estimates of theoretical dispersion parameter value at ϕ = 2.0.
4. The simulation results of the posterior estimates of dispersion parameter are reliable, since there is convergence of MCMC sampling, as shown in the series plots (figure 6) for each model.

The summary analysis using diagnostic error bar plots presented in figure 4.3 depicted that under zero-inflated data (both small and large sample sizes) characterize with low sample mean bias, MPGG model provided a much better prediction of the theoretical value of dispersion parameter than PG model and PGG model. In the other hand, zero-inflated data (both small and large sample sizes) characterize with high sample mean bias, PGG model provided much more reliable estimates of theoretical value of dispersion parameter. Hence, it implied that MPGG has a better model efficiency in handling over-dispersion while PGG has a better model efficiency in handling under-dispersion in the estimation of relative risk parameter for disease mapping.

Figure 3: Diagnostic Error Bar Plots for Zero-Inflated Data



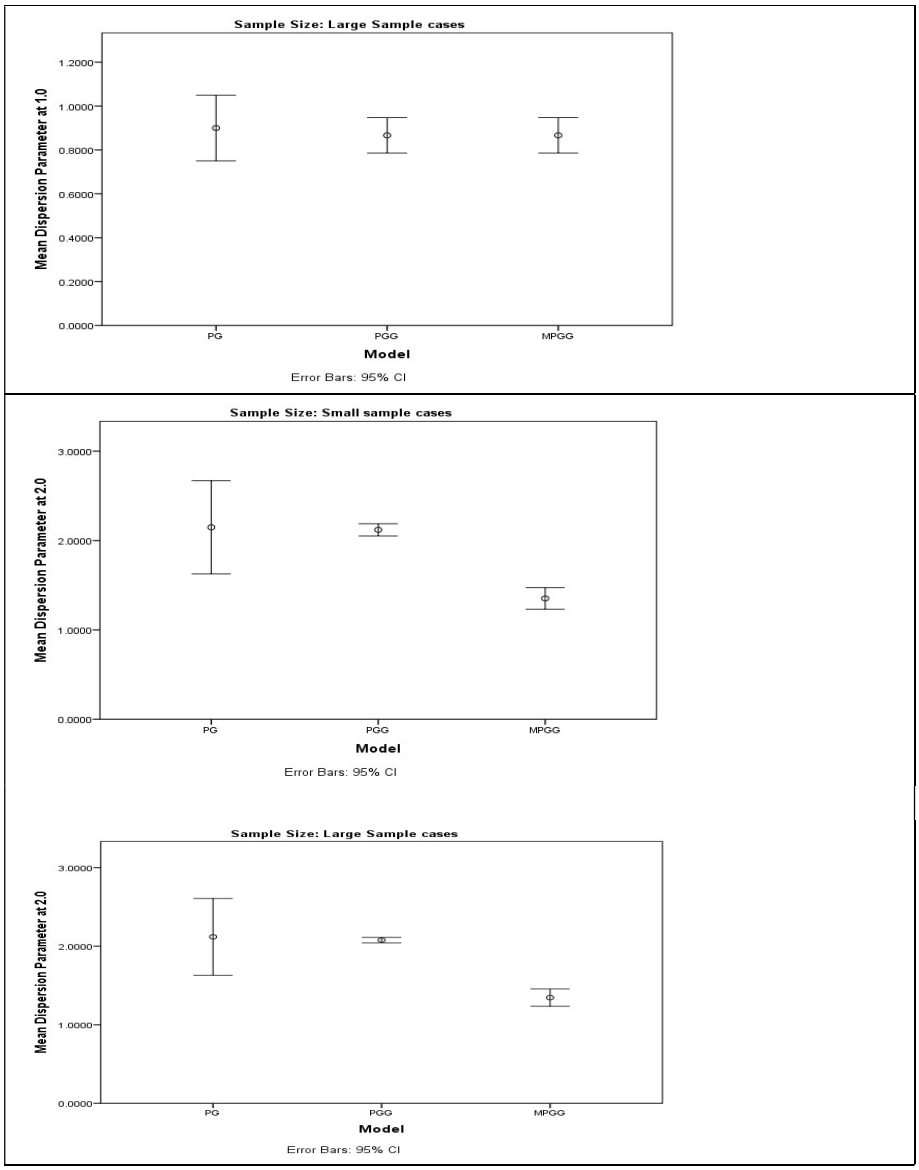
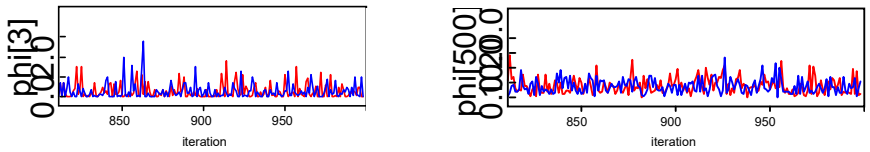
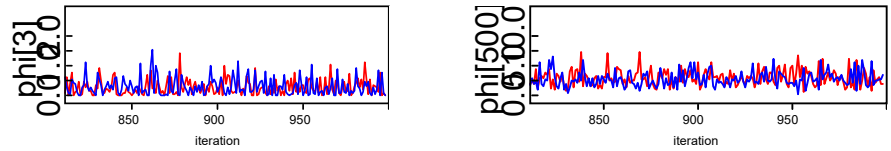


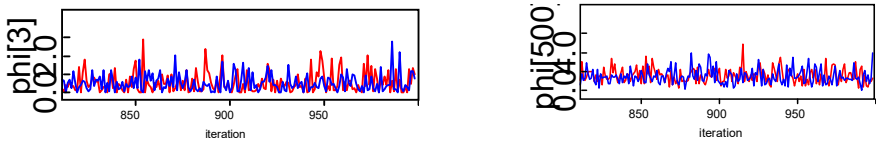
Figure 4: Posterior Convergence of the Different EB Models for Small and Large Sample Sizes Under Zero-deflated Data



PG Model

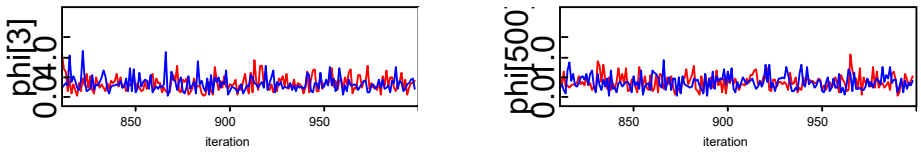


PGG Model

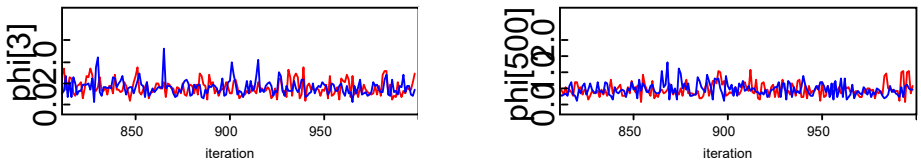


MPGG Model

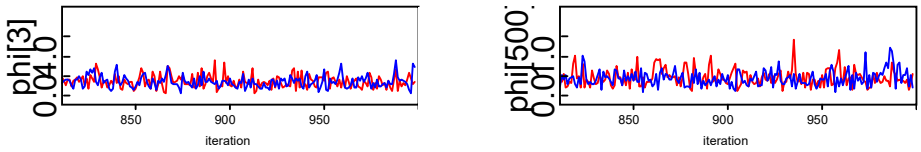
Figure 5: Posterior Convergence of the Different EB Models for Small and Large Sample Sizes under Contaminated Data



PG Model

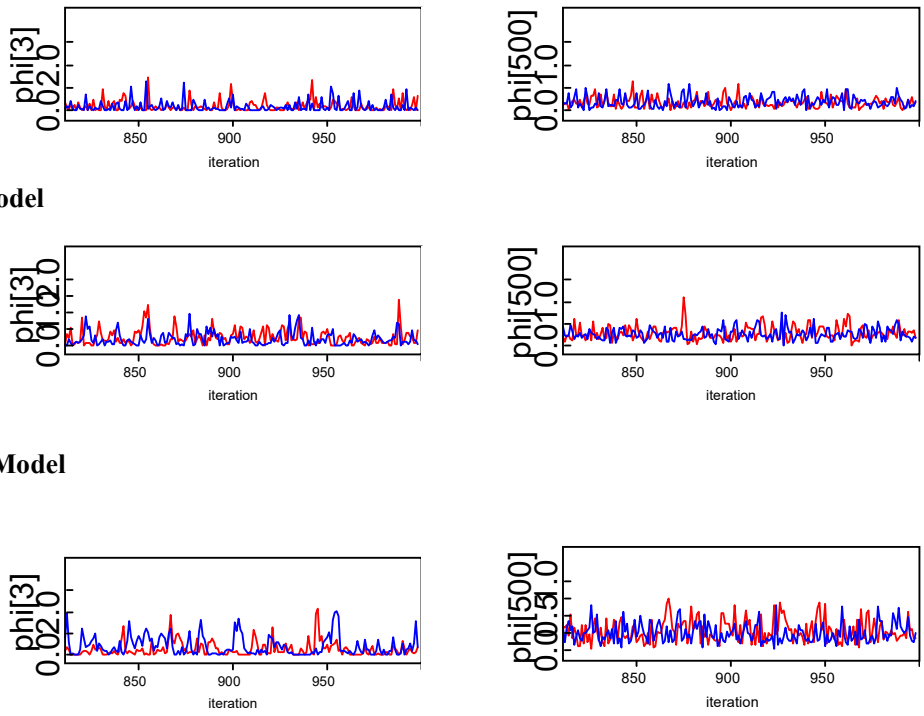


PGG Model



MPGG Model

Figure 6: Posterior Convergence of the Different EB Models for Small and Large Sample Sizes under Zero-Inflated Data



PG Model

PGG Model

MPGG Model

SUMMARY OF RESULTS

- The paper has been able to show that there is conjugacy in the use of generalized gamma distribution as prior with respect to Poisson likelihood, hence, through this conjugacy, the models: Poisson–Generalized Gamma distribution (PGG) and modified Poisson–Generalized Gamma distribution (MPGG) are established for Bayesian disease mapping.
- The paper has shown that the use of generalized gamma distribution as Bayesian conjugate prior with respect to Poisson likelihood is more likely to improve precision in handling over-dispersion (low sample mean bias) and under-dispersion (high sample mean bias) than the existing Poisson–Gamma model, under zero-deflated data, contaminated data and zero inflated data for small and large sample data.
- The proposed EB models PGG and MPGG are highly likely to improve model efficiency in relative risk parameter estimation for disease mapping.

CONCLUSION

The study aimed at enhancing the efficiency of relative risk parameter estimation for spatial disease mapping using generalized gamma distribution as Bayesian conjugate prior with respect to Poisson likelihood, for health risk assessment. This study is as a result of recent needs for development of Bayesian priors and modeling using generalized distributions. The

study has shown that the use of generalized gamma distribution as Bayesian conjugate prior with respect to Poisson likelihood is equally likely to improve efficiency in relative risk parameter estimation in small and large sample data under a zero-deflated data, a contaminated data, and a zero-inflated data characterized with either over-dispersion or under-dispersion. This implied that the posterior distribution of standardized incidence ratios (SIR) derived from the new EB models have smaller variance compared to the existing PG model.However, under-dispersion is usually rare in count data. When under-dispersion is present PG EB model lack the capacity to handle such situation, as indicated in the results. Hence, EB modelling of generalized gamma distributions with respect to Poisson likelihood are equally useful and highly competitive in terms of efficiency in relative risk parameter estimation for disease mapping.

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