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Iterative Method of Analysis of Single Queue, Multi-Server with Limited System Capacity

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Abstract

In this paper, analysis of single queue, multi-server with limited system capacity under first come first served discipline was carried out using iterative method. The arrivals of customers and service times of customers are assumed poisson and exponential distributions respectively. This queuing model is an extension of single queue, single server with limited system capacity. Performance measures of the model, such as the expected number of customers in the queue and in the system, the expected waiting times of customers in the queue and in the system respectively were derived. The performance measures so derived were compared with that of single queue, single server with limited capacity {M/M/1:(N/FCFS)} model. The numerical illustration indicates that single queue, multi-server with limited capacity {M/M/c:(N/FCFS)} model is more effective and efficient in handling congestions.

Key words: Iterative method, Single queue, Multi-Server queue model, Limited capacity, Performance measures.

Introduction

A common situation that occurs in everyday life is that of queuing or waiting in a line. Queues are usually seen at bus stops, ticket booths, restaurants, supermarkets, doctors' clinics, bank counters, traffic light and so on. In general, a queue is formed at a queuing system when either the customers requiring service wait due to the fact that the number of customers exceeds the number of service facilities, or facilities do not work efficiently and take more time than prescribed to serve a customer. The principal actors in a queuing system are the customers (those demanding for service) and the servers (those providing the service at the service facility). The two most important characteristics of a queuing system are the queue length and the waiting time of customers-what should be done to reduce queue length and waiting time of customers. There are cases where one can be in a queue without physically being visible to other individual members of the queue. An example is the case of callers trying to communicate through a telephone switchboard. In other cases, non-human commodities can also be in a queue, for example, a computer system manages queues of computer programs. Jobs can queue waiting to be attended to etc. In this paper, we study the iterative method of analysing a queuing system where there is a single queue with many service points and the system has a limited capacity. This queuing model is an extension of single queue, single server with limited capacity. Single queue, multi-serve with limited capacity N can be seen in hair dressing salon, Barbers' shops, vehicle parking area, and production facility, (Sharma,2011). In this queuing model, the traffic intensity ρ need not be less than one, (Gupta, 2008). Consequently, this paper uses iterative method to obtain the probability of x customers, p_x in the system, the probability of no customers, p_0 in the system, and the expected number of customers in the system and queue respectively in a single queue, multi-server with limited system capacity. In the study, the arrival process is poisson and the service time exponential. The queuing model is represented as {M/M/C:(N/FCFS)}, where M is poisson arrival, M is exponential service time, N is the system capacity, and FCFS is first come first served discipline.

Charan and Madhu (2011) studied single server queuing model with N-policy and removable server. They investigated $M/E_K/1$ queuing system under N-policy, by using the method of entropy maximization. The inter-arrival times and service times were assumed to follow negative exponential distribution and Erlang k distribution respectively. Sultan et al (2005) analysed a multiple server bulk arrival with two modes of server breakdown. Oualide and Alex (2014) analysed a multiple priority, multi-server queues with impatience. They considered Markovian multi-server queues with two types of impatient customers: High and low priority ones, where the first type of customers has a non-pre-emptive priority over the other type. They also considered two cases where the discipline of service within each customer type is FCFS or LCFS. Mor et al (2003) analysed multi-server queuing system with multiple priority classes, where they presented the first near-exact analysis of an M/PH/K queue with $m > 2$ pre-emptive-resume priority classes. Their analysis introduced a new technique, which they referred to as Recursive Dimensionality Reduction (RDR). The key idea in RDR is that m-dimensionally infinite Markov chain, representing the m class state space, is recursively reduced to a 1-dimensionally infinite Markov chain that is easily and quickly solved. Marcel et al (2005) considered multi- server tandem queues with finite buffers and blocking. The service times were generally distributed. They developed an efficient approximation method for determining performance characteristics such as the throughput and mean sojourn times. The method was based on decomposition into two-station subsystems, the parameters of which are determined by iteration. Ezeliora et al (2014) analysed single line, multi- server system using Shoprite Plaza, Enugu State, Nigeria as a case study. The technique used for the analysis was an infinite single-line multiple channel technique. Ekpenyong and Udoh (2011) extended and improved on the performance measures of the single server, single queue system with multiple phases. The extension resulted in a new queuing system of multi-server with multiple phases under the condition of first come first served, infinite population source, poisson arrival and Erlang service time.

Methodology

The model is an extension of single queue, single server with N limited capacity model. This model is made up of single queue, multi-server with N limited capacity. Here, customers arrive in a single queue. If the waiting space is filled to the capacity, the further arriving customers are turned away. The arrivals follow a poisson distribution and the service times of customers follow an exponential distribution. It is symbolically represented as M/M/c:(N/FCFS).

Derivation of Performance Measures of M/M/c: (N/FCFS)

Queue can be seen as a form of birth and death process where arrival can be likened to birth and departure likened to death. Consider $p_x(t)$ to be the probability that there are x customers in the system at time t. We recall that, under stationary or equilibrium queuing models, the following assumptions are made.

(i) $p_x(t) = p_x$

(ii) $p_x^1(t) = 0$

We also note here that, in general birth and death process where birth and death parameters are λ_x and μ_x respectively, the differential equation is

$$p_x^1(t) = -(\lambda_x + \mu_x)p_x(t) + \lambda_{x-1}p_x(t) + \mu_{x+1}p_{x+1}(t)$$

Using the assumptions, we have

$$0 = -(\lambda_x + \mu_x)p_x + \lambda_{x-1}p_{x-1} + \mu_{x+1}p_{x+1}$$

$$\Rightarrow (\lambda_x + \mu_x)p_x - \lambda_{x-1}p_{x-1} = \mu_{x+1}p_{x+1} \qquad (1)$$

We note here that λ_{-1} and μ_0 are undefined.

For $x = 0$, equation (1) gives

$$\lambda_0 p_0 = \mu_1 p_1$$

$$\Rightarrow p_1 = \frac{\lambda_0}{\mu_1} p_0$$

For $x = 1$, $(\lambda_1 + \mu_1)p_1 - \lambda_0 p_0 = \mu_2 p_2$

$$p_2 = \left(\frac{\lambda_1 + \mu_1}{\mu_2}\right)p_1 - \lambda_0 p_0$$

$$\Rightarrow p_2 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1} p_0$$

For $x = 2$, $(\lambda_2 + \mu_2)p_2 - \lambda_1 p_1 = \mu_3 p_3$

$$p_3 = \left(\frac{\lambda_2 + \mu_2}{\mu_3}\right)p_2 - \lambda_1 p_1$$

$$\Rightarrow p_3 = \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1} p_0$$

In general,

$$p_x = \frac{\lambda_{x-1}\lambda_{x-2}.......\lambda_0}{\mu_x\mu_{x-1}........\mu_1} p_0 \quad (2)$$

For queue of {M/M/c:(N/FCFS)}

$$\lambda_x = \begin{cases} \lambda; x \le N \\ 0; x > N \end{cases}$$

$$\mu_x = \begin{cases} x\mu; x < N \\ c\mu; c \le x \le N \end{cases}$$

Here, we obtain the probability of x customers, (p_x) in the system.

For $\lambda_x = \lambda$ and $\mu_x = x\mu$, we have,

$$p_x = \frac{\lambda.\lambda.\lambda.....\lambda}{x\mu(x-1)\mu......\mu} p_0 = \frac{\lambda.\lambda.\lambda..........\lambda}{x(x-1).(x-2)........1.\mu.\mu.......\mu} p_0$$

$$= \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x$$

For $\lambda_x = \lambda$ and $\mu_x = c\mu$, we have,

$$p_x = \frac{\lambda.\lambda.\lambda..........\lambda}{\prod_{i=1}^{c}(i\mu)(c\mu)^{x-c}} p_0 = \frac{\lambda^x}{c!c^{x-c}\mu^x} p_0 = \frac{1}{c!c^{x-c}}\left(\frac{\lambda}{\mu}\right)^x p_0$$

Therefore, equation (3.2) becomes

$$p_x = \begin{cases} \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x p_0; 0 \le x \le c \\ \frac{1}{c!c^{x-c}}\left(\frac{\lambda}{\mu}\right)^x p_0; c < x \le N \\ 0; x \succ N \end{cases} \quad (3)$$

Here, the interest is to obtain the value of p_0 (probability of no customer in the system).

Recall $\sum_{x=0}^{\infty} p(x) = 1$; for discrete distribution.

Then, $$\sum_{x=0}^{c-1} p_x + \sum_{x=c}^{N} p_x = 1 \quad (4)$$

Putting equation (3.3) into equation (3.4), we have,

$$\sum_{x=0}^{c-1} \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x p_0 + \sum_{x=c}^{N} \frac{1}{c!c^{x-c}}\left(\frac{\lambda}{\mu}\right)^x p_0 = 1 \quad (5)$$

$$p_0\left[\sum_{x=0}^{c-1} \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x + \sum_{x=c}^{N} \frac{1}{c!c^{x-c}}\left(\frac{\lambda}{\mu}\right)^x\right] = 1$$

$$\Rightarrow p_0 = \left[\sum_{x=0}^{c-1} \frac{1}{x!}\left(\frac{\lambda}{\mu}\right)^x + \sum_{x=c}^{N} \frac{1}{c!c^{x-c}}\left(\frac{\lambda}{\mu}\right)^x\right]^{-1}$$

$$= \left[\sum_{x=0}^{c-1} \frac{c^x}{x!} \left(\frac{\lambda}{c\mu} \right)^x + \sum_{x=c}^{N} \frac{c^x}{c!c^{x-c}} \left(\frac{\lambda}{c\mu} \right)^x \right]^{-1}$$

$$= \left[\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!} \sum_{x=0}^{N-c} \rho^x \right]^{-1} ; \rho = \frac{\lambda}{c\mu} \qquad (6)$$

$$\sum_{x=0}^{N} \rho^x = \left[1 + \rho + \rho^2 + \rho^3 + \ldots.. + \rho^{N-c} \right]$$

$$= \begin{cases} \dfrac{1-\rho^{N-c+1}}{1-\rho} ; \rho = \dfrac{\lambda}{c\mu} \neq 1 \quad \text{(sum of N-c+1 terms of GP)} \\ N - c + 1; \rho = 1 \end{cases} \qquad (7)$$

Putting equation (3.7) into (3.6), we have,

$$p_0 = \left[\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!} \left\{ \frac{1-\rho^{N-c+1}}{1-\rho} \right\} \right]^{-1} \qquad (8)$$

Then, putting equation (3.8) into (3.3), we have,

$$p_x = \begin{cases} \dfrac{1}{x!} \left(\dfrac{\lambda}{\mu} \right)^x \left[\displaystyle\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!} \left\{ \frac{1-\rho^{N-c+1}}{1-\rho} \right\} \right]^{-1} ; 0 \leq x < c \\ \dfrac{1}{c!c^{x-c}} \left(\dfrac{\lambda}{\mu} \right)^x \left[\displaystyle\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!} \left\{ \frac{1-\rho^{N-c+1}}{1-\rho} \right\} \right]^{-1} ; c \leq x \leq N \end{cases}$$

Performance Measures of {M/M/c:(N/FCFS)}

To determine the expected number of customers in the system (L_s), the expected waiting times in the system (W_s) and queue (W_q) respectively, we define λ_{lost} as $\lambda_{lost} = \lambda\, p_N$ and λ_e as $\lambda_e = \lambda - \lambda\, p_N = \lambda(1 - p_N)$, where λ_e is the effective arrival rate.

The effective traffic intensity, $\rho_e = \dfrac{\lambda_e}{\mu}$

(1) Expected Number of Customers in the Queue, (L_q)

To do this, we recall that

$$E(X) = \sum_{x=0}^{\infty} xP(x)\text{; for discrete distribution} \quad (9)$$

Therefore,

$$L_q = \sum_{x=c}^{N}(x-c)\, p_x = \sum_{x=c}^{N}(x-c)\frac{c^x}{c!\,c^{x-c}}\left(\frac{\lambda}{c\mu}\right)^x p_0 \quad (10)$$

$$= \sum_{x=c}^{N}(x-c)\frac{(c\rho)^x}{c!\,c^{x-c}}\, p_0$$

$$= \frac{(c\rho)^c \rho\, p_0}{c!}\sum_{y=0}^{N-c} y\,\rho^{y-1};\ y = x-c,\ \rho = \frac{\lambda}{c\mu}$$

$$L_q = \frac{(c\rho)^c \rho\, p_0}{c!}\sum_{y=0}^{N-c}\frac{d}{d\rho}\left(\rho^y\right) \qquad ;\ \frac{d}{d\rho}\rho^y = y\,\rho^{y-1}$$

$$L_q = \frac{(c\rho)^c \rho}{c!}\frac{d}{d\rho}\left[\sum_{y=0}^{N-c}\rho^y\right] p_0 \quad (11)$$

$$\sum_{y=0}^{N-c}\rho^y = \left[1 + \rho + \rho^2 + \ldots\ldots + \rho^{N-c}\right]$$

$$= \frac{1-\rho^{N-c+1}}{1-\rho} \text{; sum of N-c+1 terms of GP} \quad (12)$$

Putting equation (4.4) into (4.3), we have,

$$L_q = \frac{(c\rho)^c \rho}{c!} \frac{d}{d\rho}\left[\frac{1-\rho^{N-c+1}}{1-\rho}\right] p_0 \quad (13)$$

Differentiating $\frac{1-\rho^{N-c+1}}{1-\rho}$ w.r.t ρ, we have,

$$L_q = \frac{(c\rho)^c \rho}{c!(1-\rho)^2}\left[1-\rho^{N-c+1} - (1-\rho)(N-c+1)\rho^{N-c}\right] p_0 \quad (14)$$

But

$$p_0 = \left[\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!}\left\{\frac{1-\rho^{N-c+1}}{1-\rho}\right\}\right]^{-1}$$

Therefore,

$$L_q = \frac{(c\rho)^c \rho}{c(1-\rho)^2}\left[1-\rho^{N-c+1} - (1-\rho)(N-c+1)\rho^{N-c}\right]\left[\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!}\left\{\frac{1-\rho^{N-c+1}}{1-\rho}\right\}\right]^{-1}$$

(2) Expected Number of Customers in the System (L_s)

$$L_s = L_q + \rho_e = L_q + \left(\frac{\lambda}{\mu}\right)(1-p_N)$$

$$= L_q + c - p_0 \sum_{x=0}^{c-1} \frac{c-x}{x!}\left(\frac{\lambda}{\mu}\right)^x \quad (15)$$

(3) Expected Waiting Time in the System (W_s)

$$W_s = \frac{L_s}{\lambda_e} = \frac{L_s}{\lambda(1 - p_N)} \quad (16)$$

(4) Expected Waiting Time in the Queue (W_q)

$$W_q = W_s - \frac{1}{\mu} = \frac{L_s}{\lambda(1 - p_N)} - \frac{1}{\mu}$$

$$= \frac{L_q + \left(\frac{\lambda}{\mu}\right)(1 - p_N)}{\lambda(1 - p_N)} - \frac{1}{\mu} = \frac{\mu L_q + \lambda - \lambda p_N - \lambda + \lambda p_N}{\lambda\mu(1 - p_N)}$$

$$W_q = \frac{L_q}{\lambda(1 - p_N)} \quad (17)$$

(5) The Fraction of Server Idle Time

$$I = \frac{L_s - L_q}{c} = 1 - \frac{\rho_e}{c}; \quad \text{where } \rho_e = \frac{\lambda_e}{\mu}$$

Therefore,

$$I = 1 - \frac{\lambda_e}{c\mu} = 1 - \frac{\lambda(1 - p_N)}{c\mu} = 1 - \rho(1 - p_N); \rho = \frac{\lambda}{c\mu} \quad (18)$$

Application

Illustration

Consider a hair dressing salon with three service points. The hair dressing Salon can only contain seven customers waiting (ten in the system) at one time. Assume that the arrival pattern of customers is poisson with mean of one customer every fifteen minutes. Assume also that the service time of customers is exponential with one hour. The queue discipline is first come first served and the capacity of the system is limited. We then obtain the performance measures as

$$\lambda = 1 per 15 \min s = \frac{1}{15} per \min s, \quad \mu = \frac{1}{60} per \min s$$

$$\rho = \frac{\lambda}{c\mu} = \frac{4}{3}$$

$$p_0 = \left[\sum_{x=0}^{c-1} \frac{(c\rho)^x}{x!} + \frac{(c\rho)^c}{c!} \left\{ \frac{1-\rho^{N-c+1}}{1-\rho} \right\} \right]^{-1}$$

$$= \left[\sum_{x=0}^{3-1} \frac{4^x}{x!} + \frac{4^3}{3!} \left\{ \frac{1-\left(\frac{4}{3}\right)^8}{1-\frac{4}{3}} \right\} \right]^{-1} = \frac{1}{300.65}$$

$$L_q = \frac{(c\rho)^c \cdot \rho}{c!(1-\rho)^2} \left[1-\rho^{N-c+1} - (1-\rho)(N-c+1)\rho^{N-c} \right] p_0$$

$$\frac{4^3 \cdot \frac{4}{3}}{3!\left(1-\frac{4}{3}\right)^2} \left[1-\left(\frac{4}{3}\right)^8 - \left(1-\frac{4}{3}\right)(8)\left(\frac{4}{3}\right)^7 \right] \cdot \frac{1}{300.65} \approx 5 customers$$

$$L_s = L_q + c - p_0 \sum_{x=0}^{c-1} \frac{(c-x)}{x!} \left(\frac{\lambda}{\mu}\right)^x$$

$$= 5 + 3 - \frac{1}{300.65} \sum_{x=0}^{2} \frac{(3-x)}{x!} \cdot 4^x \approx 8 customers$$

$$W_s = \frac{L_s}{\lambda(1-p_N)} = \frac{8}{1/15 \left(1-4^{10}/3!3^7\right)\left(1/300.65\right)} = 163 \min s$$

Where $p_N = \frac{1}{c!c^{x-c}} \left(\frac{\lambda}{\mu}\right)^x p_0 ; c \prec x \leq N$

$$W_q = \frac{L_q}{\lambda(1-p_N)} = \frac{5}{\frac{1}{15}\left(1-\frac{4^{10}}{3!3^7}\right)\left(\frac{1}{300.65}\right)} = 102\min s$$

$$I = 1-\frac{\lambda_e}{c\mu} = 1-\frac{\lambda(1-P_N)}{c\mu} = 1-\frac{4(1-p_{10})}{3} = 0.03$$; **the** fraction of server idle time.

Comparison of {M/M/1: (N/FCFS)} and {M/M/c:(N/FCFS)}

Here, we compare the two queuing models to find out which of the two models performs better.

$\lambda = 1/15 per \min$ and $\mu = 1/60$

For M/M/1: N/FCFS, $\rho = \frac{\lambda}{\mu} = 4$

$$p_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-4}{1-4^{11}} = 0.00072$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} = -\frac{4}{3} - \frac{(11)4^{11}}{1-4^{11}} \approx 10 customers$$

$$L_q = L_s - \frac{\lambda(1-p_N)}{\mu} = 10 - \frac{\frac{1}{15}(1-p_{10})}{\frac{1}{60}} = \frac{\frac{1}{15}\left\{1-\left(\frac{1-4}{1-4^{11}}\right)4^{10}\right\}}{\frac{1}{60}} \approx 9 customers$$

$$W_s = \frac{L_s}{\lambda(1-p_N)} = \frac{10}{\frac{1}{15}(1-p_{10})} = 600\min s$$

$$W_q = \frac{L_q}{\lambda(1-p_N)} = \frac{9}{\frac{1}{15}(1-p_{10})} = 560\min s$$

The results obtained from the two models when $\lambda = \frac{1}{15}$ and $\mu = \frac{1}{60}$, showed that M/M/3:10/FCFS performs better in handling congestion through queuing model. For example, for queue of M/M/3:10/FCFS, $L_s = 8, L_q = 5$, while for

{M/M/1:(10/FCFS)}, $L_S \approx 10, L_q \approx 9.$ Also, the waiting times in the system for {M/M/3:(10/FCFS)} and M/M/1:10/FCFS are 163mins and 600mins respectively. Since we are talking about reduction of waiting time of customers in the system, {M/M/c:(N/FCFS)} performs better.

Conclusion

In this paper, we derived the {M/M/c:(N/FCFS)} models which is an extension of {M/M/I:(N/FCFS)} model using iterative method. The performance measures of the model were obtained and the numerical illustrations showed that the model when compared to M/M/I:N/FCFS model performs better. For example, in {M/M/3:(10/FCFS)}, an average of 5 customers would wait in the queue for service and each customer would spend 102mins waiting for service. In contrast, an average of 9 customers would wait in the queue for service in {M/M/1:(10/FCFS)} and each customer would spend 560mins waiting for service. This shows that {M/M/c:(N/FCFS)} is better in handling congestion especially at peak period.

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