Characterization of idempotent elements in a variant of the finite full transformation semigroup

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ABSTRACT (10 PT)

In this research work, we consider a finite set X with n elements and introduce a systematic discussion of the transformation semigroup on the finite set (namely the full transformation semigroup T_n) with the aim of studying the variants of this semigroup. There have been many approaches in studying the variant semigroup (i.e T_n^α for $\alpha \in T_n$), especially characterizing of Green's relation and classification of certain Subsemigroups. In this particular work we study and characterize the idempotent elements in a variant of the finite full transformation semigroup T_n , and present the findings in the form of results.

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1. INTRODUCTION:

Semigroup theory is a relatively young part of mathematics as a separate direction of algebra with its own objects, formulations of problems and methods of investigations. A semigroup is simply a set S which is closed under an associative binary operation. Obvious examples of semigroups are the natural numbers N under either addition or multiplication. If the semigroup S satisfies the property that for all $x, y \in S$ we have $xy = yx$, then we say that the semigroup S is a commutative semigroup. A semigroup S is called a monoid if it contains an identity; otherwise we can easily adjoin an extra identity 1 in order to make it monoid. Thus, we write S^1 and S^0 respectively to denote a semigroup S to which each 1 or 0 is adjoined.

That is: $$S^1 = \begin{cases} S & \textit{if S has identity element,} \\ S \cup \{1\} & \textit{othewise} \end{cases}$$

where $1x = x1 = x, \ \forall x \in S$.

And $$S^0 = \begin{cases} S & \textit{if S has identity element,} \\ S \cup \{0\} & \textit{othewise} \end{cases}$$

where $0x = x0 = 0, \ \forall x \in S$.

Many semigroups originally came to the interest of researchers as natural generalizations of groups, and the origin of group theory are in the study of permutations and the symmetric group S_n. The principal objects of interest in this work are finite sets and transformation of the finite sets. For instance: Let M be a finite set, say $M = \{m_1 \ m_2 \ . \ . \ . \ m_n\}$. A transformation of M is an array of the form $\alpha = \begin{pmatrix} m_1 & m_2 & . & . & . & m_n \\ k_1 & k_2 & . & . & . & k_n \end{pmatrix}$ where all $k_i \in M$.

There have been many approaches in studying classes of transformation semigroups. For instance, Dolinka and East (2015) investigated the full transformation semigroup T_n under an alternative binary operation and studied what is called the variant of T_n. The concept and the notion of variants of semigroup was first considered in the monograph of Lyapin 1960, and a 1967 paper of Magill that considered semigroups of functions $X \to Y$ under an operation defined by $f \cdot g = f * \alpha * g$, where α is some fixed function $Y \to X$;

Variants of finite full transformation semigroups have been studied in a variety of contexts. For example, Tsyaputa (2003, 2004a) classified the non-isomorphic variants, (2004b), he also characterized Green's relations in the variant of finite full transformations and symmetric inverse semigroup, and he also together with Mazorchuk (2008), classified the isolated subsemigroup in the variants of T_n, where similar problems were considered in the context of partial transformations.

Variants of arbitrary semigroups were first studied in 1983 by Hickey where (among other things) they were used to provide a novel characterization of Nambooripad's celebrated partial order on a regular semigroup. More generally, as noted by Khan and Lawson (2001), variants arise naturally in relation to Rees matrix semigroups, and also provide a useful alternative to the group of units in some classes of non-monoidal regular semigroups. Tsyaputa G. Y (2004/5) Classify a non-isomorphic variant of finite full transformation semigroups T_n , characterized the idempotents, Green's relations and analogous questions for PT_n.

2. PRELIMINARIES:

Here, we list some well-known definitions, lemmas and results, that will be used throughout this work and build on them to prove our assertion:

(i) A transformation is a map from { 1,2,…,n } to { 1,2,…,n } which can be written as $\alpha = \begin{pmatrix} 1 & 2 & 3 & . . . & n \\ 1\alpha & 2\alpha & 3\alpha & . . . & n \end{pmatrix}$.
For instance, if $M = \{1, 2, \}$ then the elements of the full transformation denoted by T_2 is as follows: $\left\{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\right\}$.

(ii) An element $e \in S$ where S is a semigroup, is called an idempotent if $e^2 = e$ and we denote the set of all idempotents in S by $E(S)$.
For instance: Let $e_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ & $e_2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ be elements in T_2,

Then by our definition, we have $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ & $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

Lemma 2.1: (The $E(T_X)$ Lemma). An element $\varepsilon \in T_n$ is an idempotent if and only if $\varepsilon/\text{im}\varepsilon = 1_{\text{im}\,\varepsilon}$.

Lemma 2.2: (Olexandr Ganyushkin**)** : Let $\alpha = \begin{pmatrix} A_1 & A_2 & . . . & A_k \\ a_1 & a_2 & . . . & a_k \end{pmatrix}$ be a sandwich element with rank

k > 0, and if $\beta = \begin{pmatrix} B_1 & B_2 & . . . & B_m \\ b_1 & b_2 & . . . & b_m \end{pmatrix} \in S$ be an element of rank m. Then β is an idempotent of S^α

if and only if there exists an injection f : {1 , . . . , m } → {1 , . . . , k} such that $b_i \in A_{f(i)}$ and $a_{f(i)} \in B_i$ for all i = 1 , . . . , m.

Theorem 2.3: (M. Tainiter): A mapping f in any symmetric groupoid is idempotent if and only if the restriction of f to its range set is the identity map.

3. VARIANT OF THE FINITE FULL TRANSFORMATION SEMIGROUP $(\mathbf{T}_n^\alpha)$:

The semigroup which will occur in this our discussion is the full transformation semigroup T_n on the set {1, . . . , n}, whose elements are all the maps from {1, . . . , n} to itself, and whose operation is a composition. The set T_n contains n^n elements and the set P_n contains $(n+1)^n$ elements.

For instance, if $X = \{1, 2, 3,4\}$ then the structure of the full transformation denoted by T_4 is as follows:

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\begin{pmatrix}1&2&3&4\\4&2&1&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&1&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&1&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&2&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&2&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&2&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&2&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&3&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&3&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&3&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&4&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&4&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&4&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&2&4&4\end{pmatrix}\\
\begin{pmatrix}1&2&3&4\\4&3&1&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&1&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&1&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&2&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&2&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&2&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&3&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&3&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&3&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&3&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&4&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&4&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&4&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&3&4&4\end{pmatrix}\\
\begin{pmatrix}1&2&3&4\\4&4&1&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&1&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&1&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&1&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&2&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&2&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&2&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&2&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&3&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&3&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&3&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&3&4\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&4&1\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&4&2\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&4&3\end{pmatrix}&\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}
\end{array}\right\}$$

We now focus on the main objects of our study which is the variant of the finite full transformation semigroup, that is (T_n^{α}) for $\alpha \in T_n$.

3.1. Definition Let S be a semigroup, for a fixed element $\alpha \in S$, we define a new operation $\star_\alpha$ on S by $x \star_\alpha y = x\,\alpha\,y, \ \forall\ x, y \in S$. Then $(S, \star_\alpha)$ is a semigroup called the variant of S with respect to α, and is denoted by S^{α}. This S together with the new binary operation $\star_{\alpha}$ is a semigroup called the variant of S with respect to α. Analogously, if we take $(N, \cdot)$ of all positive integers under multiplication which denoted by S, and pick the element 3 $\in N$, then $(N, \star_3)$ is the variant of S with respect to the fixed element 3, where $x \star_3 y = x \cdot 3 \cdot y$ for all $x, y \in S$, and is denoted by S^3. This N together with the new binary operation $\star_3$ is a semigroup called the variant of S with respect to the fixed element 3, and this will be denoted by S^3. That is, suppose we take $5 \ \& \ 8 \in (N, \cdot)$ then by the above example, we can vividly see that $5 \star_3 8 = 5 \cdot 3 \cdot 8 = 120$

Further, we study the variant of a finite full transformation semigroup which denote by T_n^{α}, and then explore the structure of it using the above definition.

3.2 Example: Let

$x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ be an element in T_3, then by the definition 3.5 we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \star_{\alpha} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$$

3.3 Example: Let $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ then by the above definition we have $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \star_{\alpha} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$

So we can now give the structure for the variant of a finite full transformation semigroup of the set $X = \{1, 2, 3,4\}$ which denoted by T_4^{α} for a fix element $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, and in order to explore the structure of the variant of a finite full transformation semigroup which denoted by T_4^{α}, we can summarize its details in a simple remark given below.

3.4 Remark: The elements of a variant of the finite full transformation semigroup (T_n^{α}) contain the same elements as that of the original finite full transformation semigroup (T_n). Now we proceed further by introducing the main objects of this work (which are idempotents of a variant of the finite full transformation semigroup), listening some relevant results upon them, and then consider the following results below:

3.5 Definition: Let S be a semigroup, for any sandwich element $\alpha \in S$, we define a new operation $\star_{\alpha}$ on S by $x \star_{\alpha} y = x \, \alpha \, y$, $\forall \ x, y \in S$. Then $(S, \star_{\alpha}) \Longrightarrow S^{\alpha} = \{x : x \, \alpha \, y = x\}$. Below are examples to shed more light on the above definition

(i) Let $x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ be a sandwich elememt, then by definition 3.5 we have $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{pmatrix} \star_{\alpha} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{pmatrix}$

(ii) Let $x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, then by by definition 4.0.3 we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \star_{\alpha} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$$

In order to explore the structure of the variant of a finite full transformation semigroup which denoted by T_n^{α}, we shall recall our remark in the preceding page that said " The elements of a variant of the finite full transformation semigroup are same as the elements of the original finite full transformation semigroup'' (Remark 3.4.)

4. CHARACTERISATION OF IDEMPOTENT ELEMENTS IN A VARIANT OF THE FINITE FULL TRANSFORMATION SEMIGROUP $E(\mathrm{T}_n^{\alpha})$

The idea on idempotent elements in the semigroups of full transformations was developed long time ago, and many researchers have worked in that direction, but less emphasis is shown on the idempotent of the variant semigroup. Therefore it is interesting to characterize the idempotent elements of a variant of the finite full transformation semigroup.

The method of research adopted in this work is to consult the necessary and relevant papers in the literature on variants of finite full transformations semigroups, and some analogous results related to idempotent generation, in order to obtain a background information for developing further the theory of how idempotent elements in the variant of a finite full transformations semigroups can be characterized.

In this section we first begin with a theoretic characterization of idempotent elements in T_n^{α} , using a theorem to enable us present the result of the finding.

4.1. RESULTS:

Theorem: Suppose $\alpha \in \mathrm{T}_n$ is a sandwich element, and let $\beta \in \mathrm{T}_n^{\alpha}$. Then the element β is an idempotent of T_n^{α} if and only if the range of β is equal to fix $(\alpha\beta)$ (that is $im(\beta) = \mathrm{fix}(\alpha\beta)$).

Proof: Suppose $im(\beta) = \mathrm{fix}(\alpha\beta)$ where α is a sandwich element, we wish to show that a mapping $\beta \in \mathrm{T}_n^{\alpha}$ is an idempotent. Now pick $y \in im(\beta)$, then $y(\alpha\beta) = y$ for any $x \in X$,
if $\alpha\beta = y$, we have $x(\beta\alpha\beta) = x\beta(\alpha\beta) = y(\alpha\beta) = y$,
Therefore, $x(\beta\alpha\beta) = x\beta$ or $\beta\alpha\beta = \beta$, which implies that β is an idempotent.

Conversely, suppose that β is an idempotent in T_n^{α}, then $\beta\alpha\beta = \beta \Longrightarrow im(\beta\alpha\beta) = im(\beta)$.
Let $y \in im(\beta)$, we shall show that y is a fixed element of $\alpha\beta$.
Now if $y \in im(\beta)$, then $\exists\ x(x, y) \in \beta$.

Suppose that $u, v \in X$ are such that $(x, y) \in \beta$, $(y, u) \in \alpha$ and $(u, v) \in \beta$,
$\Longrightarrow (x, v) \in \beta\alpha\beta = \beta$,
$\Longrightarrow (x, v) \in \beta$,

So $(x, v) \in \beta$ and $(x, y) \in \beta \Longrightarrow y = v$, Hence $(y, u) \in \alpha$ and $(u, v) \in \beta$
$\Longrightarrow (v, v) \in \alpha\beta$
$\Longrightarrow (y, y) \in \alpha\beta$ for all $y \in im(\beta)$.
$\Longrightarrow y \in \mathrm{fix}(\alpha\beta)$, therefore $im(\beta) \subseteq \mathrm{fix}(\alpha\beta)$.
On the other hand, if $y \in \mathrm{fix}(\alpha\beta)$, then $\mathrm{fix}(\alpha\beta) \subseteq im(\beta)$. Therefore, this suffices to show that $im(\beta) = \mathrm{fix}(\alpha\beta)$, and this proves the assertion.

4.2 Example: Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 2 \end{pmatrix}$ be a sandwich element and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix} \in \mathrm{T}_5^{\alpha}$. Then the element β in T_5^{α} is an idempotent in the variant of T_5^{α}, since the range of $\beta = (1, 2, 3, 4)$ is fixed in the product $\alpha\beta$. That is $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 2 \end{pmatrix}$

Thus the element $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix}$ is an idempotent for the sandwich element $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 2 \end{pmatrix}$, and it satisfied $\beta\alpha\beta = \beta$.

i.e $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 3 & 1 \end{pmatrix}$

4.3 Example: Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 1 & 5 & 2 & 7 \end{pmatrix}$ be a sandwich element and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 7 \end{pmatrix} \in T_7^{\alpha}$. Then the element $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 4 \end{pmatrix}$ in T_7^{α} is not an idempotent in the variant of T_7^{α}, since the range of $\beta = (1, 2, 3, 4)$ is doesn't fixed itself in the product $\alpha\beta$.

i.e $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 1 & 5 & 2 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 1 & 2 & 3 & 4 \end{pmatrix}$

Thus the element $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 4 \end{pmatrix}$ is not an idempotent for the sandwich element

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 1 & 5 & 2 & 7 \end{pmatrix}$, because it doesn't satisfy the condition $\beta\alpha\beta = \beta$.

That is :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 1 & 5 & 2 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 5 & 4 \end{pmatrix}$$

Generally, using the above theorem with the two attached examples 4.2 and 4.3, we can then provide the result of characterizing the idempotent elements for the variant of the finite full transformation semigroup, which denoted by $E(T_n^{\alpha})$ in Table 1 below:

Table 1: Result of characterizing the idempotent elements for the variant of the finite full transformation semigroup

S^{α}	Fixed element	List of idempotents	$E(T_n^{\alpha})$
T_3^{α}	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$	10
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}$	10
T_4^{α}	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \dots ,$	. . .
	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \dots,$	. . .
⋮	⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮
T_n^{α}		$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 2 & \cdots & 2 \end{pmatrix}, \quad . \quad . \quad . \quad , \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n & n & \cdots & n \end{pmatrix}$	N

5. CONCLUSION

In conclusion, idempotents have played an important role in semigroup theory. One reason for this is that they always exist in a finite semigroup. Thus, this led us to find the idea of investigating the kind of problems in generalizing the technique for determining the method of characterizing an idempotent element in the variant of the finite full transformation semigruoup. The results of this work suggest splitting the study into two different cases: (i) the case of studying the variant of a finite full transformation semigroup and (ii) the case of characterizing the idempotent elements in the variant of the finite full transformation semigroup.

The main structure and the result of this study presented a comprehensive description of the variant of a finite full transformation semigroup which is similar to the work of Dolinka & East on a finite set, where they explored the Green's relation on T_n^{α}. And in this study it was found that all the elements in the variant of a finite full transformation semigroup, i.e T_n^{α} are same as the original elements in the T_n and T_n^{α} is also isomorphic to the finite full transformation semigroup T_n. Finally, the study has revealed the method on how idempotent elements in the variant of a finite full transformation semigroup can be characterized (Theorem).

6. REFERENCES

[1] Balarabe, M. (2019), "Variant of finite full, partial and partial one-to-one transformation semigroups". Ph.D Thesis, Dept. of Mathematics, Ahmadu Bello University, Zaria, Kaduna, Nigeria.

[2] Clifford, A. and Preston, G. (1961), "The Algebraic Theory of Semigroups", Mathematical Survey, No. 7 (American Mathematical Society, Vol. 1).

[3] Clifford, A. and Preston G (1967), "The Algebraic Theory of Semigroups", Mathematical Surveys No. 7, (American Mathematical Society, vol. 2,).

[4] Devi, K.M. and Latha G. S. "Properties of Variants of Semigroups", Asian Journal of Current Engineering and Maths 2: 4 July – August (2013) 281 - 282.

[5] Dolinka I. and East. J. (2015), "Variants of finite full transformation Semigroups". International J. Algebra Computing, 25(8):1187–1222,

[6] E. S. Lyapin, (1960). Semigroups [in Russian], Gosudarstv. Izdat. Fiz-Mat. Lit., Moscow; English transl. (Transl. Math. Monogr. Vol.3), Amer. Math.

[7] Ganyushkin, O. and Mazorchuk, V. (2009), "Classical finite transformation semigroups", An introduction of Algebra and Applications, vol. 9. Springer-Verlag, London, Ltd,

[8] Garba, G. U., (1990), "Idempotents in partial transformation Semigroups", Proc. Roy. Soc. Edinburgh 116A, 359 - 366.

[9] Hickey, J. B. (1983), "Semigroups under a sandwich operation". Proc. Edinburgh Math. Soc. (2), 26(3): 371–382,

[10] Hickey. J. B. (1986), "On variants of a Semigroup". Bull. Austral. Math. Soc., 34(3):447–459,

[11] Hollings, C., (2009), "The early development of the algebraic theory of semigroups", Archive for History of Exact Science, (UK, University of Oxford), 63, 497 - 536.

[12] Howie, J. M., (1995), "Fundamentals of Semigroup theory", London Mathematical Society, New Series 12 (The Clarendon Press, Oxford University Press).

[13] Imam, A.T. (2013), "Subsemigroup generated by quasi-idempotents in certain finite semigroups of mappings", M.Sc Desertation, Dept. of Mathematics, A.B.U, Zaria, Kaduna. Nigeria.

[14] Kenneth D. Magill, Jr.(1967), "Semigroup structures for families of functions. I. Some homomorphism theorems". J. Austral. Math. Soc., 7:81–94, (14) Lyapin. E. S. (1960) Semigroups (in Russian). Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow,

[15] Khan, T. A and Lawson, M. V.(2001), "Variants of regular semigroups". Semigroup. Forum, 62(3): 358–374,

[16] Saito, T., (1989), "Products of idempotents in finite full transformation Semigroups", Semigroup forum, 39, 295 - 309.

[17] Tainiter, M. (1968), "A Characterization of idempotents in Semigroups". J. of Combinatorial Theory 5, 370–373, Brookhaven National Laboratory Upton, Long Island, New York.

[18] Tero. H. (1996) "Semigroup theory", Lecture Notes on semigroup, University of Turku, FIN-20014 Turku, Finland.