THE THEORY AND APPLICATION OF REGRESSION ANALYSIS AND THE LEAST-SQUARES PRINCIPLE

Maj. P. de Villiers*

Introduction

1. The theory and practice of regression analysis, and the principle of least-squares on which it is based, is frequently encountered in Mathematics and particularly Statistical Mathematics, but less well known are some very useful applications in a military environment. It is therefore the aim of this article to firstly give a general description of the theory of regression analyses, and secondly to highlight some military applications of the theory.

Theory of regression analysis

2. Let the relationship between the x- and y-values of a set of N (x, y)-data pairs be of the form

$$y = a_1 + a_2f_2(x) + a_3f_3(x) + a_4f_4(x) + \ldots + a_mf_m(x) \qquad (1)$$

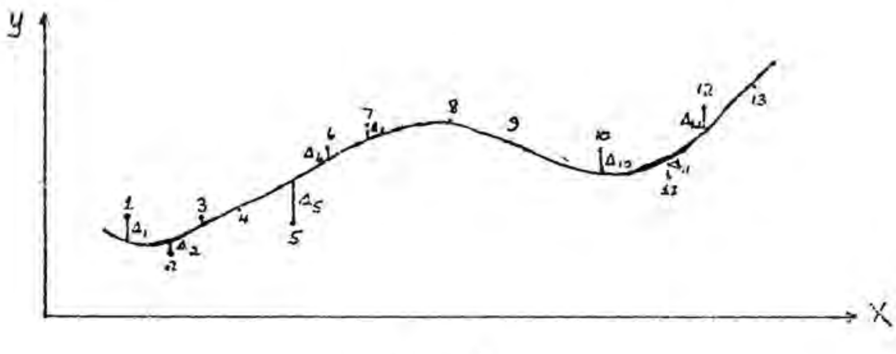

Figure 1

where the $f_k(x)$ are either known or assumed functions of the independent variable x. Then a regression analysis on a set of at least m+1 (x, y)-data pairs will determine the best values — in the "least-square" sense — of the m coefficients a_1, a_2, . . ., a_m in equation (1). The term "best values" above physically implies those values for which the smooth curve described by (1) comes closest to going through all the experimental points on the scatter diagram in figure 1, where for clarity the experimental point (x_5, y_5) is merely denoted by 5.

Quantitatively stated the condition to be satisfied by the choice of the coefficients a_1, a_2, . . ., a_m is that the sum of the squares of the vertical distances from the smooth curve to the experimental points must be a minimum, i.e. that

$$S = \sum_{i=1}^{N} \delta_i^2 = \sum_i [y_i - a_1 - a_2f_2(x_i) - a_3f_3(x_i) - \ldots - a_mf_m(x_i)]^2 \qquad (2)$$

be a minimum. Since $S = S(a_1, a_2, \ldots, a_m)$ a necessary, and in fact sufficient condition for S to be a minimum is that

$$\frac{\partial S}{\partial a_1} = \frac{\partial S}{\partial a_2} = \frac{\partial S}{\partial a_3} = \ldots\ldots = \frac{\partial S}{\partial a_m} = 0 \qquad (3)$$

Application of the conditions (3) to (2) gives rise to a system of m so-called *normal equations* in the m unknowns a_1, a_2, . . . a_m; and this system can then of course be solved for a_1, a_2, . . ., a_m bmo standard matrix methods. In writing down the normal equations the following simple, and obvious rule proves very helpful: to obtain the i-th normal equation, which is derived from $\frac{\partial S}{\partial a_i} = 0$, multiply both sides of (1) by the coefficient of a_i in (1) — $f_i(x)$ in the above case — and sum over N. Note that $f_i(x)$ is in fact the partial derivative of the right hand side of (1) wrt a_i, from which the truth of the mechanical rule immediately follows.

* Lecturer in Physics (Military) at the Military Academy.

3. If the system of normal equations is written in matrix form as

$$\begin{bmatrix} N & \Sigma f_2(x_i) & \Sigma f_3(x_i) & \dots & \Sigma f_m(x_i) \\ \Sigma f_2(x_i) & \Sigma f_2^2(x_i) & \Sigma f_2(x_i)f_3(x_i) & \dots & \Sigma f_2(x_i)f_m(x_i) \\ \Sigma f_3(x_i) & \Sigma f_3(x_i)f_2(x_i) & \Sigma f_3^2(x_i) & \dots & \Sigma f_3(x_i)f_m(x_i) \\ \Sigma f_4(x_i) & \Sigma f_4(x_i)f_2(x_i) & \Sigma f_4(x_i)f_3(x_i) & \dots & \Sigma f_4(x_i)f_m(x_i) \\ \dots & \dots & \dots & \dots & \dots \\ \Sigma f_m(x_i) & \Sigma f_m(x_i)f_2(x_i) & \Sigma f_m(x_i)f_3(x_i) & \dots & \Sigma f_m^2(x_i) \end{bmatrix} * \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \dots \\ a_m \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i f_2(x_i) \\ \Sigma y_i f_3(x_i) \\ \Sigma y_i f_4(x_i) \\ \dots \\ \Sigma y_i f_m(x_i) \end{bmatrix}$$

or in abbreviated form as

$$C * A = B \quad \dots\dots\dots\dots (4)$$

then

$$A = C^{-1} * B \quad \dots\dots\dots\dots (5)$$

where C^{-1} is the inverse of C. Note that C is a symmetrical matrix, a fact which considerably reduces the still lengthy calculation of C for large sets of (x, y)-data pairs. But as the accuracy of the regression analysis increases with the size of the data set, this greater computational effort is well worth the extra work.

4. A very important point to remember in connection with a regression analysis is that in spite of the fact that it does determine those values of the coefficients or parameters a_i which guarantee the best fit of the smooth curve (1) to the given or experimental data set, it does not in fact guarantee that (1) will fit the data very well. For example, if you try to fit a straight line to a quadratically related data set, the regression analysis *will* determine the best-fitting straight line, even though the straight line won't fit the data very well. The actual example in figure 2 illustrates this point quite clearly!

6. An alternative, and preferable approach, is to calculate the standard error of estimate of each of the regression coefficients a_i, as standard errors are easier to interpret quantitatively. It is in fact fairly simple since the variance (or square of the standard deviation) of the i-th regression coefficient a_i is given by $c_{ii}^{-1} * \sigma^2$, where c_{ii}^{-1} is the i-th diagonal element of the inverse matrix C^{-1} and σ^2 is the variance of the residuals $(y_i^{exp} - y_i^{th})$. As the best estimate of σ^2 is provided by

$$S^2 = \sum_i (y_i^{exp} - y_i^{th})^2/(N-1)$$

for one independent variable, it follows from the definition of standard error that the standard error of estimate of the regression coefficient a_i is given by the expression

$$s_m(a_i) = [c_{ii}^{-1} * \sum_i (y_i^{exp} - y_i^{th})^2/\{N(N-1)\}]^{\frac{1}{2}} \dots (6)$$

7. Fairly popular forms of the functions $f_k(x)$ in (1), which also work well in practice, are

$$x^k;\ \exp(\pm kx);\ \cos(kx);\ \sin(kx)$$

The trigonometric functions are particularly useful for periodic data, but then one must remember to stretch/compress the independent x-coordinates to

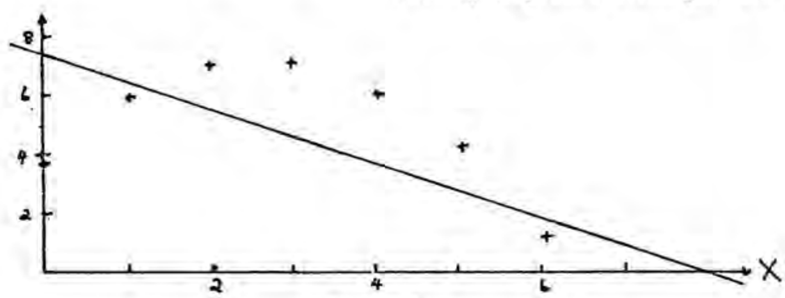

Figure 2

5. There are basically two ways of measuring the 'goodness of fit' quantitatively. Firstly, one can calculate the coefficient of correlation between the experimental y-values and those calculated from (1) with the regression coefficients — this is of course identical to calculating the correlation index between y and x — and use this coefficient as a quantitative estimate of the goodness of fit. But as it is possible to get an r-value very close to unity for a relatively poor fit, this is a fairly insensitive method which should only be used to choose between two different assumed relationships.

ensure that the period — be it 24 hours, 365 days of 560 meters — corresponds to 2π or 360°!

8. As a final remark it must be stressed that the theory as outlined above is applicable only to regression analyses of *y on x*, i.e. where the x-values are either exact or subject to errors negligible in comparison with the errors in y. Fortunately, this is normally the case, but even when both variables are subject to errors the same principle can still be applied by merely demanding that the sum of the squares of the *perpendicular* distances from the points on the scatter diagram to the regression

curve be a minimum. This approach will in fact be adopted in the first two applications dealt with in the sequel.

Application 1: MPP on a map from bearings to beacons

9. In any elementary course in Map Reading one is taught how to determine your position from the back-bearing lines from two or more beacons at known positions, and as the distances used are usually so small the error made in assuming your correct position to be at the midpoint of the (ideally) small error triangle or polygon is negligible. But for larger distances or greater accuracy this simple method will obviously not be good enough, and the method of least squares must then be used to determine the most probable position (MPP) mathematically. This theory is outlined below.

10. Since the observer's position should be somewhere on the back-bearing line from a beacon at known position, let us refer to these back-bearing lines as position lines (P/L's). Figure 3(a) shows a typical error triangle enclosed between three P/L's, and figure 3(b) illustrates exactly what is meant by the term MPP in the least-square sense, viz. that point for which the sum of the squares of the *perpendicular* distances δ_i from it to the various P/L's

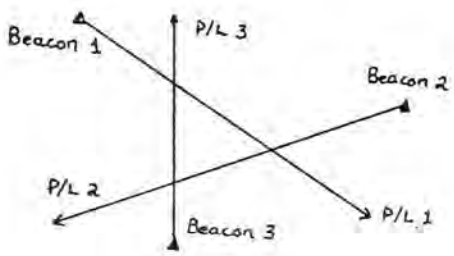

Figure 3(a)

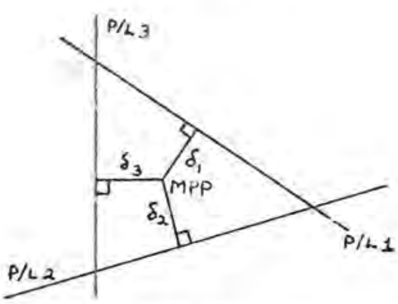

Figure 3(b)

is a minimum. For this statement to be translated into mathematical form we therefore require an expression for the perpendicular distance from a point (x, y) to a straight line, and the simplest expression for this can be expressed into the perpendicular distance p from the origin of a rectangular coordinate system to a straight line whose normal through the origin makes an angle α with the positive x-axis. From figure 4 it will be clear that

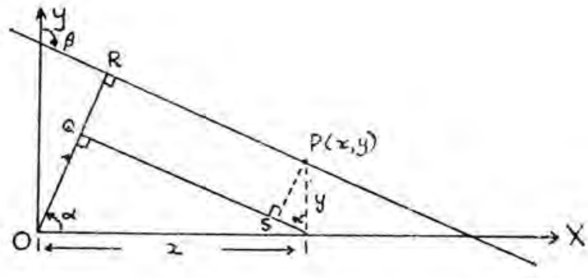

Figure 4

$$\begin{aligned} p &= OR \\ &= OQ + QR \\ &= OQ + SP \\ &= x \cos \alpha + y \sin \alpha \quad \ldots\ldots\ldots\ldots (7) \end{aligned}$$

where (x, y) is *any* point on the straight line. Equation (7) can now be used to express the perpendicular distance from the arbitrary point (x, y) to the line through (x_i, y_i) as the difference between the perpendicular distances from the origin of two parallel lines through the points (x, y) and (x_i, y_i) respectively, i.e.

$$\delta_i = (x_i \cos \alpha_i + y_i \sin \alpha_i - x \cos \alpha_i - y \sin \alpha_i)$$

11. At this stage the calculation of the x- and y-coordinates of a beacon from its 6- or 8-digit grid reference, and the calculation of α from the bearing to the beacon warrant some discussion. A grid reference of 562 147 implies that the x coordinate of the beacon is 56 200 (meters), and y is 14 700 — both being measured wrt the Universal Transverse Mercator grid which is superimposed on the Gauss Conformal Projection used for all Service maps. But with this projection, as with most others, a straight line on a map corresponds to a straight line on the earth's surface over only a limited distance, so one would be ill-advised to use an origin as far away from the area covered by the map as is the case in some South African maps where the origin is at 40°S, 12°E. To minimize this change in scale with distance from the origin an ideal origin must be somewhere in the vicinity of the beacons, and the logical choice appears to be the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \sum_i x_i'/N; \; \bar{y} = \sum_i y_i'/N$$

and x_i', y_i' are the (Mercator) coordinates of the i-th beacon. The coordinates in (8) are therefore

$$x_i = x_i' - \bar{x}; \; y_i = y_i' - \bar{y} \quad \ldots\ldots (9)$$

In connection with α_i it must be remembered that navigational bearings are measured clockwise from

North, whereas rectangular coordinate angles are measured counterclockwise from the positive x-axis, i.e. 'last', and the transformation between the two is

$$\theta^\circ_{RECT} = 90^\circ - \theta_{NAV}; \; _{NAV} = 90^\circ - \theta_{RECT} \quad (10)$$

And since OR is $\perp$ the P/L with bearings β, it follows that

$$\alpha_i = 90^\circ - (\beta_i \pm 180^\circ \pm 90^\circ)$$
$$= 180^\circ - \beta_i \text{ or } -\beta_i \quad \ldots\ldots\ldots\ldots (11)$$

Corrections for compass error and magnetic variation (or declination as it is sometimes called) can obviously be incorporated in (11) as well.

12. Finally, then, for the MPP at the point (x, y) the quantity

$$S = \sum_i \delta_i^2$$
$$= \sum_i (x_i \cos \alpha_i + y_i \sin \alpha_i - x \cos \alpha_i - y \sin \alpha_i)^2$$

must be a minimum. And since $S = S(x\,y)$, a necessary condition for this is that $\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0$. The condition

$$\frac{\partial S}{\partial x} = \sum_i -2 \cos \alpha_i (x_i \cos \alpha_i + y_i \sin \alpha_i - x \cos \alpha_i - y \sin \alpha_i) = 0$$

can be written as

$$x \sum \cos^2\alpha_i + y \sum \cos \alpha_i \sin \alpha_i = \sum (x_i \cos^2\alpha_i + y_i \cos \alpha_i \sin \alpha_i) \ldots (12(a))$$

Similarly the condition $\frac{\partial S}{\partial y} = 0$ can be rewritten in the form

$$x \sum \cos \alpha_i \sin \alpha_i + y \sum \sin^2\alpha_i = \sum (x_i \cos \alpha_i \sin \alpha_i + y_i \sin^2\alpha_i) \ldots (12(b))$$

Writing (12) in concise form as

$$c_{11}x + c_{12}y = b_1$$
$$c_{12}x + c_{22}y = b_2$$

the solutions are

$$x = (b_1c_{22} - b_2c_{12})/(c_{11}c_{22} - c_{12}^2)$$
$$y = (b_2c_{11} - b_1c_{12})/(c_{11}c_{22} - c_{12}^2) \quad \ldots (13)$$

where

$$c_{11} = \sum_i \cos^2\alpha_i; \; c_{12} = \sum_i \cos \alpha_i \sin \alpha_i;$$
$$c_{22} = \sum_i \sin^2\alpha_i$$
$$b_1 = \sum_i (x_i \cos^2\alpha_i + y_i \cos \alpha_i \sin \alpha_i);$$
$$b_2 = \sum_i (x_i \cos \alpha_i \sin \alpha_i + y_i \sin^2\alpha_i)$$

As these MPP coordinates are calculated wrt the transformed origin they must still be converted back to x' and y' via the inverse of (9), and hence into a 6- or 8-digit grid reference.

Application 2: MPP from a multi-sight fix in astro-navigation

13. A very similar application, which is in fact currently being introduced in the SAN, is to determine the MPP from a multi-sight fix as accurately as possible. Using the Marc St-Hilaire method (which will *not* be described here) each individual star sight yields a P/L on which that particular sight indicates the observer to be, and the MPP is again obtained from the solution of the error triangle or polygon ('cocked hat' as it is frequently referred to) enclosed by the various P/L's. As in paragraph 10 the MPP is once more defined to be that point for which $\sum_i \delta_i^2$ is a minimum, where δ_i has the same meaning as before. So the only difference between this case and the previous one is in the expression for δ_i. Since each P/L is specified ito an azimuth (AZ), i.e. the bearing towards the star from the observer, and an intercept (IC) which indicates the distance of the P/L from the dead reckoning position (DRP), the δ's must obviously be expressed ito these two variables as well. From figure 5 it is clear that

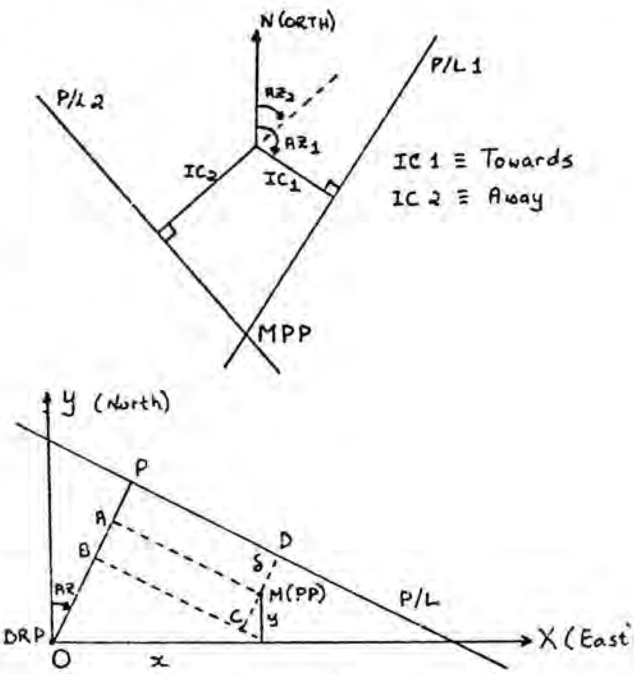

Figure 5

$$\delta = MD$$
$$= OP - OB - BA$$
$$= OP - OB - CM$$
$$= IC - x \sin (AZ) - y \cos (AZ)$$
$$\therefore S = \sum_i \delta_i^2$$
$$= \sum_i [IC_i - x \sin (AZ_i) - y \cos (AZ_i)]^2$$

In identical fashion to the previous case the conditions $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$ can be written in the form

$$x \sum \sin^2 (AZ_i) + y \sum \cos (AZ_i) \sin (AZ_i) = \sum IC_i * \sin (AZ_i)$$
$$x \sum \cos (AZ_i) \sin (AZ_i) + y \sum \cos^2 (AZ_i) = \sum IC_i * \cos (AZ_i)$$

or when expressed as

$$c_{11}\,x + c_{12}\,y = b_1$$
$$c_{12}\,x + c_{22}\,y = b_2$$

the solutions are once more

$$\text{D LAT (n m)} \equiv y = (b_2c_{11} - b_1c_{12})/(c_{11}c_{12} - c_{12}^2)$$
$$\text{DEP (n m)} \equiv x = (b_1c_{22} - b_2c_{12})/(c_{11}c_{12} - c_{12}^2)$$

so that finally

$$\text{MPP LAT} = \text{DR LAT} + \text{D LAT}/60$$
$$\text{MPP LONG} = \text{DR LONG} + \text{DEP}/60/\cos(\text{DR LAT})$$

where D LAT and Departure (both in nautical miles since the intercepts are specified in these units) are the differences in Latitude and 'Longitude' between the DRP and MPP. Although it will not be described here it is fairly simple to compensate for the motion of the navigation officer's own ship between individual sights. This correction, obviously calculated from the ship's course and speed, is described in detail in the NAV PAC notes issued by the Military Academy to all students doing navigation courses there.

The role of calculators

14. In both the applications dealt with so far the computational effort required is so large as to make these methods completely impractical for manual, logarithmic table or even slide rule calculations. But with modern hand-held, pocket-sized electronic calculators with the additional facilities of storage and even programmability, computational complexity ceases to be an obstacle. Instead, the price of such calculators — ±R550 at the time of writing, and decreasing! — becomes the limiting factor. So ultimately the accuracy and/or speed required must be weighed against the inevitable calculator cost entailed by the mathematically correct methods described above.

Application 3: Regression analysis on tabular data

15. Conventionally a plotting board is used to convert a correction from the observation post into an updated bearing and range for an 81-mm mortar, and then the weapon's range tables are used to obtain a charge-elevation combination which gives the required range. If one has access to a programmable calculator, particularly one with facilities for converting from rectangular to polar coordinates and vice versa, it is fairly simple to calculate updated bearings and distances from OP corrections, but if the range tables are then again used to convert distances to charge-elevation combinations the capability of the (expensive) calculator is not fully utilized. The problem is thus to derive an expression for charge and elevation as function of range from the range tables, and this is where a regression analysis is called for.

16. Table 1 in Appendix A below is an extract from the range tables of 81-mm mortar for charges 1 and 4, which are representative of all 7 charges. The ranges are still in yards, but the metric equivalent will be very similar. Before discussing the regression analysis on this data it is perhaps necessary to point out that no simple, closed-form expression for the elevation angle as function of range and muzzle velocity (i.e. charge) does exist. As a brief motivation of this statement the following: If the frictional resistance of a mortar in flight is neglected its range can easily be shown to be

$$\text{Range } R = v_0^2 \sin 2\alpha/g$$

Where v_0 is the muzzle velocity, α the elevation angle and g the acceleration of a freely falling body (9,8 ms^{-2}). But if a realistic expression for the friction or drag is included (it is roughly proportional to the square of velocity for non-compressible flow) no closed-form expression for the range ito v_0 and α can be found. The range or firing tables from which table 1 is extracted are in fact drawn up from numerical solutions of the second order differential equations for the x- and y- coordinates separately, and not from actual firings — not only would the cost of the experimental approach be prohibitive, but actual measurements of all the parameters normally included in a firing table would also be extremely difficult. With the knowledge that no closed-form expression for elevation ito range and muzzle velocity exists one is therefore left with complete freedom to find *any* function that successfully reproduces the data in table 1 bmo regression analyses. The two different charges will be discussed separately.

Charge 1

17. A plot of elevation vs range for charge 1 shows that the data from $R = 400^x$ to $R = 1\,100^x$ lies exactly along the straight line

$$\alpha = 90{,}5 - 0{,}025R \quad [400 \leq R \leq 1\,100] \ldots . (14)$$

Above $1\,100^x$ the deviation of the plotted curve from the straight line (14) increases roughly quadratically or cubically with range. Assuming the relationships between these deviations δ and R to be

$$\delta = a_0 + a_1 * R + a_2 * R^2 \ldots . . (15(a))$$

a regression analysis on these 9 data pairs yields

$$a_0 = -40{,}32762;\ a_1 = 0{,}07216;$$
$$a_2 = -3{,}24675.10^{-5} \ldots \ldots (15(b))$$

Combining (14) and (15) then gives the formula applicable for ranges larger than 1 100 yards:

$$\alpha = 50{,}17238 + 0{,}04716R - 3{,}24675.10^{-5}R^2 \,. . (16)$$

A comparison of the elevations predicted by (16) with those in table 1 shows that (16) is in error by a maximum angle of 0,25° for $R = 1\,450^x$; from the third column in table 1 this means that the use of (16) instead of the range tables will introduce a maximum range error of $6{,}26^x$, which compares favourably with the probable error of 6^x in the fourth column, and (16) is thus an acceptable formula to use. It is interesting to note that if (15(a)) is replaced by the cubic or third degree relation

$$\delta = a_0 + a_1R + a_2R^2 + a_3R^3$$

Then a regression analysis on the deviations gives the following equivalent of (16), viz.

$$\alpha = 217{,}20646 - 0{,}32820R + 2{,}47085.10^{-4}\,R^2 - 6{,}90252R^3 \quad \ldots (17)$$

Predictions from (17) have a maximum error of 0,12° for R = 1 350^x, which corresponds to a maximum range error of 3^x arising from the use of (17) instead of the range tables. Equation (17) is thus more accurate than (16), but will require a longer programme if α is to be computed on a programmable calculator. For completeness it must be added that if the relationship between α and R is assumed to be linear or a polynomial of degree higher than 3, one obtains a poorer fit than with (16) or (17).

Charge 4

18. In this case a graph of elevation vs range has an approximately parabolic shape, so it once more seems reasonable to assume a polynomial relationship between α and R. Regression analyses on the data show that polynomials of degree 2, 3 and 4 give acceptable fits, while any higher degree does not. The respective regression formulae are:

$$\alpha = 50{,}26638 + 0{,}02396R - 7{,}11766.10^{-6}R^2 \quad \ldots (18)$$

$$\alpha = 101{,}24140 - 0{,}02829R + 1{,}05910.10^{-5}R^2 - 1{,}98420.10^{-9}R^3 \quad \ldots (19)$$

$$\alpha = -874{,}07 + 1{,}30625R - 6{,}70150.10^{-4}R^2 + 1{,}51455.10^{-7}R^3 - 1{,}28949.10^{-11}R^4 \quad \ldots (20)$$

For these 3 formulae the maximum errors are respectively 0,63° at 3 550^x; 0,16° at 2 400^x and 0,34° at 3 200^x, which correspond to range errors of 15,60^x; 17,00^x and 17,16^x. All three of these range errors are at least smaller than the probable range errors of 19,5^x; 18^x and 19^x as shown in the fourth column of the charge 4 data, and fortunately the simplest equation (18) is the best one to use! As a final remark concerning the regression analyses on these 2 data sets it must be noted that the input data is fairly inaccurate as all elevations are only given to the nearest quarter degree. If more exact data is used the regression formulae will of course be correspondingly more accurate.

Two or more independent variables

19. Although the theory in the first section was only developed for one independent variable x, its extension to 2 or more independants is straightforward. For example, if there are 4 independent variables x_1, x_2, x_3, x_4 then the generalized theory starts from the following equivalent of equation (1):

$$y = a_1 + a_2f_2(x_1) + a_3f_3(x_2) + a_4f_4(x_3) + a_5f_5(x_4) \quad \ldots (21)$$

and is identical to the single-variable case in all other respects. But for best results the y-dependence on the 4 x's must be fairly simple, such as linear or at most quadratic or exponential. An illustrative example of this point is the fuel consumption of an aircraft which obviously depends on the aircraft's weight, speed and altitude. But as the fuel consumption is such a complicated function of weight ($f = a_1 + a_2e^w$), height ($f = a_1 + a_2e^{-h} + a_3e^{-2h}$) and particularly velocity or mach number ($f = a_1 + a_2m^2 + a_3m^4 + a_4m^6 + a_5m^8 + a_6m^{10}$) that a regression analysis on all 3 variables simultaneously doesn't give very good results.

Conclusion

20. As can be seen from the preceding paragraph and the first section the theory of regression analysis, and the least-squares principle on which it is based, is in fact very simple, but lends itself to many extremely useful applications in the military sphere.

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Appendix A

Table 1: Extract from an 81-mm mortar's Range Tables

Charge 1				Charge 4			
Range (yards)	Elevation (degrees)	Variation for 50^x in range (degrees)	Probable range error (yards)	Range (yards)	Elevation (degrees)	Variation for 50^x in range (degrees)	Probable range error (yards)
350	81,50	—	3,0	2 400	66,75	—	18,0
400	80,50	1,00	3,0	2 450	66,25	0,50	18,0
450	79,25	1,25	3,0	2 500	65,75	0,50	18,0
500	78,00	1,25	3,5	2 550	65,25	0,50	18,0
550	76,75	1,25	3,5	2 600	64,50	0,75	18,5
600	75,50	1,25	3,5	2 650	63,75	0,75	18,5
650	74,25	1,25	3,5	2 700	63,00	0,75	18,5
700	73,00	1,25	4,0	2 750	62,25	0,75	18,5
750	71,75	1,25	4,0	2 800	61,50	0,75	19,0
800	70,50	1,25	4,0	2 850	60,75	0,75	19,0
850	69,25	1,25	4,0	2 900	60,00	0,75	19,0
900	68,00	1,25	4,5	2 950	59,00	1,00	19,0
950	66,75	1,25	4,5	3 000	58,00	1,00	19,0
1 000	65,50	1,25	4,5	3 050	57 00	1,00	19,0
1 050	64,25	1,50	4,5	3 100	56,00	1,00	19,0
1 100	63,00	1,50	5 0	3 150	55,00	1,00	19,0
1 150	61,50	1,50	5,0	3 200	54,00	1,00	19,0
1 200	60,00	1,50	5,0	3 250	53,00	1,00	19,0
1 250	58,25	1,75	5,0	3 300	52,00	1,00	19,5
1 300	56,50	1,75	5,5	3 350	51,00	1,25	19,5
1 350	54,50	2,00	5,5	3 400	49,75	1,25	19,5
1 400	52,50	2,00	5,5	3 450	48,50	1,25	19,5
1 450	50,50	2,00	6,0	3 500	47,00	1,50	19,5
1 500	48,00	2,50	6,0	3 550	45,00	2,00	19,5
1 550	45,00	3,00	6,5				