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CONVOLUTION THEOREMS FOR HYPERGEOMETRIC BERNOULLI NUMBERS AND POLYNOMIALS

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ABSTRACT: In this paper, we study hypergeometric Bernoulli numbers and polynomials and use recurrence relations to generate higher order convolution identities involving the sums of these Bernoulli numbers. We consider cases where the Bernoulli number/polynomial are of various order.

Key words/phrases: Recurrence, Bernoulli Numbers, Bernoulli Polynomials, Hypergeometric Bernoulli Numbers and Polynomials

INTRODUCTION

In this paper, we give a simple proof of the sums of products of hypergeometric Bernoulli numbers, $B_n(N)$ defined by the generating function

$$G(x) = \frac{x^N/N!}{e^x - 1 - x - x^2/2! - \cdots - x^{N-1}/(N-1)!} = \sum_{n=0}^{\infty} \frac{B_n(N)}{n!} x^n \qquad (1.1)$$

Throughout this paper we assume that N is a positive integer. Observe that when $N = 1$, this reduces to the classical Bernoulli numbers, that is $B_n(1) = B_n$. In his papers[5], [6], [7], F. T. Howard commenced the study of these numbers when$N = 2$. K. Dilcher in [1] and [2] has studied these numbers and their sums and products. In recent years, H. D. Nguyen and the second author also studied these numbers and the corresponding polynomials in [3] and [4]. In 2014, H. D. Nguyen and C. L. Cheong in [9] have established some convolution results involving these numbers. Our results are similar to theirs but we arrive at our results from a different approach.

Theorem 1.1. Let m be a positive integer. Then

$$\sum_{k_0+k_1+\cdots+k_m=n} \binom{n}{k_0, k_1 \ldots, k_m} B_{k_0}(N) \ldots B_{k_n}(N) = \frac{1}{m!\,(-N)^m} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{n}{k} a_{k-j,m}(N,k) B_{n-k}(N) k!,$$

where $a_{k,m}(N, j)$ are defined by

$$P_{k,m}(N,x) = \sum_{j=k}^{m} a_{k,m}(N,j) x^j$$

and $P_{k,m}(N, x)$ are polynomials given by

$$(-1)^{m-1}(m-1)!\, N^{m-1}(G(x))^m = \sum_{k=0}^{m} P_{k,m}(N,x) G^{(k)}(x)$$

Here $G^{(k)}(x)$ is the k^{th} derivative of $G(x)$.

PRELIMINARIES

In this section, we shall prove some preliminary results needed to prove Theorem 1.1. We begin with

Lemma 2.1. The function $G(x)$ given in (1.1) satisfies the equations

$$N\big(G(x)\big)^2 = (N-x)G(x) - xG'(x) \qquad (2.1)$$

And

$$2N^2\big(G(x)\big)^3 = [2x^2 + (1-4N)x + 2N^2]G(x) + [3x^2(13N)x]G'(x)x^2G''(x) \qquad (2.2)$$

Proof. To simplify notation, let

$$T_{N-1}(x) = \sum_{k=0}^{N-1} \frac{x^k}{k!}$$

Then from the quotient rule, the fact that $T'_{N-1}(x) = T_{N-2}(x)$ and algebraic manipulations, we get

$$G'(x) = \frac{\frac{x^{N-1}}{(N-1)!}[e^x - T_{N-1}(x)] - \frac{x^N}{N!}[e^x - T_{N-2}(x)]}{[e^x - T_{N-1}(x)]^2}$$

$$= \frac{N}{x}G(x) - G(x)\left[\frac{e^x - T_{N-2}(x)}{e^x - T_{N-1}(x)}\right]$$

$$= \frac{N}{x}G(x) - G(x)\left[1 - \frac{N}{x}G(x)\right],$$

where we have used

$$\frac{e^x - T_{N-2}(x)}{e^x - T_{N-1}(x)} = \frac{e^x - T_{N-1}(x) + x^{N-1}/(N-1)!}{e^x - T_{N-1}(x)}$$
$$= 1 + \frac{N}{x}G(x)$$

Rearranging the last line of the previous equation, we get

$$G'(x) = \left(\frac{N}{x} - 1\right)G(x) - \frac{N}{x}[G(x)]^2.$$

Multiplying through by x and solving for $N[G(x)]^2$ yields (2.1).

Differentiate both sides of (2.1) and multiplying the resulting equation by $-x$, we get

$$-2NxG'(x)G(x) = xG'(x) - (N - x - 1)xG'(x) + x^2G''(x). \tag{2.3}$$

From (2.1), we have

$$-xG'(x) = N[G(x)]^2 - (N-x)G(x)$$

Now we use this in (2.3) too obtain

$$2NG(x)[NG(x)^2 - (N-x)G(x)] = xG'(x) - (N-x-1)xG'(x) + x^2G''(x)$$

Solving for $2N^2G(x)^3$ from the last equation and some algebraic manipulations yield (2.2) and completes the proof of the lemma.

We have the following generalization of the above Lemma

Lemma 2.2. For the function G(x) given in (1.1) and any positive integer m, we have

$$C_m[G(x)]^{m+1} = \sum_{k=0}^{m} P_{k,m}(N,x)G^{(k)}(x), \tag{2.4}$$

where

$$C_m = (-1)^m m!\, N^m$$

and $P_{k,m}(N,x)$ satisfies the recurrence equation:

$$P_{k,m}(N,x)\ m(x-N)P_{k,m-1}(N,x) + x\left[P'_{k,m-1}(N,x)P_{k-1,m-1}(N,x)\right] \tag{2.5}$$

With initial conditions $P_{0,0}(N,x) = 1$ and $P_{k,m}(N,x) = 0$ if $k > m$.

Proof. We prove the lemma by induction. First observe that the recurrence relation (2.5) and the initial conditions specified, we have

$$P_{1,1}(N,x) = x, \quad P_{0,1}(N,x) = x - N$$

$$P_{2,2}(N,x) = x^2 \quad P_{1,2}(N,x) = (1-3N)x + 3x^2$$
$$P_{0,2}(N,x) = 2x^2 + (1-4N)x + 2N^2$$

Thus, for $m = 1, m = 2$, the assertions are proved in Lemma 2.1.

Suppose that (2.4) holds for some $m \geq 1$. Then differentiating both sides of the equation, we have

$$(m+1)C_m[G(x)]^m G'(x) = \sum_{k=0}^{m}\left(P_{k,m}(N,x)G^{(k+1)}(x)P'_{k,m}(N,x)G^{(k)}(x)\right)$$

We now multiply both sides of this equation by x and the fact that

$$xG'(x) = (N-x)G(x) - N\left(G(x)\right)^2$$

(which is a restatement of (2.1)) to rewrite the above equation as

$$-(m+1)C_mN\left(G(x)\right)^{m+2}$$
$$= (m+1)C_m(x-N)\left(G(x)\right)^{m+1}$$
$$+ \sum_{k=0}^{m}\left(P_{k,m}(N,x)G^{(k+1)}(x) + P'_{k,m}(N,x)G^{(k)}(x)\right)$$

Since $C_{m+1} = -(m+1)NC_m$ and since by the induction assumption

$$C_m\left(G(x)\right)^{m+1} = \sum_{k=0}^{m} P_{k,m}(N,x)G^{(k)}(x)$$

we obtain

$$C_{m+1}\left(G(x)\right)^{m+2}$$
$$= (m+1)(x-N)\sum_{k=0}^{m} P_{k,m}(N,x)G^{(k)}(x) + \sum_{k=0}^{m}\left(P_{k,m}(N,x)G^{(k+1)}(x) + P'_{k,m}(N,x)G^{(k)}(x)\right)$$

This can be expressed as

$$C_{m+1}\big(G(x)\big)^{m+2} = \sum_{k=0}^{m}\Big((m+1)(x-N)P_{k,m}(N,x) + xP'_{k,m}(N,x)xP_{k-1,m}(N,x)\Big)G^{(k)}(x) + xP_{m,m}(N,x)G^{(m+1)}(x)$$

Where we have used the initial condition $P_{-1,m}(N,x) = 0$. Finally, from the recurrence relation (2.5) of $P_{k,m}(N,x)$, it is immediate that $P_{m+1,m+1}(N,x) = xP_{m,m}(N,x)$. Thus

$$C_{m+1}\big(G(x)\big)^{m+2} = \sum_{k=1}^{m+1} P_{k,m+1}(N,x)\,G^{(k)}(x)$$

and this completes the proof of the lemma.

Lemma 2.3. If $P_{k,m}$ are as given by (2.5), then

$$P_{k,m}(N,x) = \sum_{j=k}^{m} a_{k,m}(N,j)x^j \qquad (2.6)$$

where $a_{k,m}(N,j) = 0$ if

$$k > m, k > j, \ j > m, k < 0, j < 0 \qquad (2.7)$$

and for $0 \le k \le j \le m$,

$$a_{k,m}(N,j) = ma_{k,m-1}(N,j-1) + (j-mN)a_{k,m-1}(N,j) + a_{k-1,m-1}(N,j-1) \qquad (2.8)$$

Proof. We shall prove the lemma by induction. For $m = 0$ and $m = 1$, the lemma follows from (2.5) and assumptions following that equation. So suppose the lemma is true for $m-1$. From equation (2.5) and induction assumption, we have

$$\begin{aligned} P_{k,m}(N,x) &= m(x-N)P_{k,m-1}(N,x)x\big(P'_{k,m-1}(N,x) + P_{k-1,m-1}(N,x)\big) \\ &= m(x-N)\textstyle\sum_{j=k}^{m-1} a_{k,m-1}(N,j)x^j + x\displaystyle\sum_{j=k}^{m-1} ja_{k,m-1}(N,j)x^{j-1} + x\sum_{j=k-1}^{m-1} a_{k-1,m-1}(N,j)x^j \\ &= \sum_{j=k+1}^{m} ma_{k,m-1}(N,j-1)x^j + \sum_{j=k}^{m-1}(-mN)a_{k,m-1}(N,j)x^j + \sum_{j=k}^{m-1} ja_{k,m-1}(N,j)x^j + \sum_{j=k}^{m} a_{k-1,m-1}(N,j-1)x^j \end{aligned}$$

By (2,7), we have $a_{k,m}(N,k-1) = 0$ and $a_{k,m-1}(N,m) = 0$. We use this in the first and second sums of the above to get

$$P_{k,m}(N,x) = \textstyle\sum_{j=k}^{m}\big[ma_{k,m-1}(N,j-1) + (j-mN)a_{k,m-1}(N,j) + a_{k-1,m-1}(N,j-1)\big]x^j$$

From (2.8), we see that $P_{k,m}(N,x)$ has the form stated in (2,6) and our lemma is proved.

A CONVOLUTION THEOREM FOR HYPERGEOMETRIC BERNOULLI POLYNOMIALS

The hypergeometric Bernoulli polynomials are defined by

$$G(x,z) = \frac{e^{xz}x^N/N!}{e^x-1-x-x^2/2!-\cdots-x^{N-1}/(N-1)!} = \frac{e^{xz}\,x^N/N!}{T_{N-1}(x)} = \sum_{n=0}^{\infty}\frac{B_n(N,z)}{n!}x^n \qquad (3.1)$$

Lemma 3.1. The function $G(x,z)$ satisfies the following differential equations

$$N\big(G(x,z)\big)^2 = (N-x+xz)e^{xz}G(x,z) - xe^{xz}\frac{\partial G(x,z)}{\partial x} \qquad (3.2)$$

$$\begin{aligned} 2N^2G(x,z)^3 = {} & [x + 2(N-x+xz)^2 - xz(N+1-x+xz)]e^{2xz}G(x,z) \\ & + [3x^2 + (1-3N)x - 2x^2z]e^{2xz}\frac{\partial G(x,z)}{\partial x} \\ & + x^2e^{2xz}\frac{\partial^2 G(x,z)}{\partial x^2} \end{aligned} \qquad (3.3)$$

Proof. Differentiate (3.1) with respect to x to get:

Multiplying (3.4) by xe^{xz} and rearranging terms yields (3.2)

To prove (3.3), we differentiate (3.2) with respect to x to get

$$\begin{aligned} 2NG(x,z)\frac{\partial G(x,z)}{\partial x} = {} & [-x + z(N+1-x+xz)]e^{xz}G(x) \\ & + (N-1-x)e^{xz}\frac{\partial G(x,z)}{\partial x} \\ & - xe^{xz}\frac{\partial^2 G(x,z)}{\partial x^2} \end{aligned} \qquad (3.5)$$

Multiplying (3.2) by $2NG(x,z)$, we get

$$2N^2G(x,z)^3 = 2(N-x+xz)e^{xz}[NG(x,z)^2] - xe^{xz}\left[2NG(x,z)\,\frac{\partial G(x,z)}{\partial x}\right]$$

In this equation, we replace $NG(x,z)^2$ by the right hand side of (3.2), $2NG(x,z)\frac{\partial G(x,z)}{\partial x}$ by the right hand side of (3.5), and simplify the resulting equation to get (3.3), thereby completing the proof of the lemma.

Remark 3.1. We observe that if we define $C_k(N,z)$ by

$$C_k(N,z) = \left(1-\frac{k}{N}\right)B_k(N,z) + \frac{k}{N}(z-1)B_{k-1}(N,z)$$

then (3.2) can be expressed as

$$N(G(x,z))^2 = \left(\sum_{n=0}^{\infty} C_k(N,z)\frac{x^n}{n!}\right)e^{xz}$$

It was shown by Nguyen and Cheong in ([10]) that the $\{C_n(N,z)\}$ is an Appell sequence. They used this to prove the following theorem. But we shall give a proof based on Lemma 3.1.

Theorem 3.1. The polynomials $\mathrm{B_n(N,z)}$ satisfy the convolutions

$$\sum_{k=0}^{n}\binom{n}{k}B_k(N,z)B_{n-k}(N,z) = \sum_{k=0}^{n}\binom{n}{k}\left[\left(1-\frac{k}{N}\right)B_k(N,z)+\frac{k}{N}(z-1)B_{k-1}(N,z)\right]z^{n-k} \tag{3.6}$$

and

$$2N^2\sum_{k_1+k_2+k_3=n}\binom{n}{k_1,k_2,k_3}B_{k_1}(N,z)B_{k_2}(N,z)B_{k_3}(N,z) = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}z^{n-k}A_k(N,z) \tag{3.7}$$

$$A_k(N,k) = k(k-1)(z^2-3z+2)B_{k-2}(N,z) + [(k(3N+1)-2k^2)z+3k^2-2k(2N+1)]B_{k-1}(N,k) + (2N^2-3kN+k^2)B_k(N,z) \tag{3.8}$$

Here we define $B_k(N,z) = 0$ *for* $k < 0$.

Proof. We use the series representation of $G(x,z)$ given in (3.1) to get

$$N\big(G(x,z)\big)^2 = N\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k(N,z)B_{n-k}(N,z)\right)\frac{x^n}{n!} \tag{3.9}$$

On the other hand, using

$$\frac{\partial G(x,z)}{\partial x} = \sum_{n=0}^{\infty}\frac{B_{n+1}(N,z)}{n!}x^n \quad \text{and}$$

$$e^{xz} = \sum_{n=0}^{\infty}\frac{z^n}{n!}x^n$$

and applying the Cauchy product formula for product of series and some rearrangement we get

$$\left[(N-x+xz)G(x,z) - x\frac{\partial G(x,z)}{\partial x}\right] = \left(\sum_{n=0}^{\infty}[n(z-1)B_{n-1}(N,z)+(N-n)B_n(N,z)]\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}z^{n-k}[k(z-1)B_{k-1}(N,z)+(N-k)B_k(N,z)]\right)\frac{x^n}{n!} \tag{3.10}$$

The recurrence in (3.6) follows from (3.2), (3.9), and (3.10).

To prove (3.7), we rewrite (3.3) as

$$2N^2G(x,z)^3 = \left[(ax^2+bx+c)G(x,z) + (dx^2+ex)\frac{\partial G(x,z)}{\partial x} + x^2\frac{\partial^2 G(x,z)}{\partial x^2}\right]e^{2xz} \tag{3.11}$$

where

$$a = z^2-3z+2,\ b=(3N-1)z+1-4N,\quad c = 2N^2,\quad d = 3-2z,\quad e = 1-3N \tag{3.12}$$

Substituting the power series representation for e^{2xz}, $G(x,z)$, and its derivatives in (3.11), we get

$$2N^2\sum_{n=0}^{\infty}\left(\sum_{k_1+k_2+k_3=n}\binom{n}{k_1,k_2,k_3}B_{k_1}(N,z)B_{k_2}(N,z)B_{k_3}(N,z)\right)\frac{x^n}{n!} = \left(\sum_{n=0}^{\infty}d_n\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}2^nz^n\frac{x^n}{n!}\right) \tag{3.13}$$

where

$$d_n = n(n-1)aB_{n-2}(N,z) = [nb+n(n-1)d]B_{n-1}(N,z) + [c+ne+n(n-1)]B_n(N,z) \tag{3.14}$$

We now carry out the power series multiplication in (3.13), use (3.12) and (3.14) in the resulting equation, and compare coefficients (after some simplifications) to obtain (3.7). This completes the proof.

A CONVOLUTION THEOREM FOR HYPERGEOMETRIC BERNOULLI NUMBERS

We begin this section by stating a consequence of Lemma 3.1. First note that for $N = 1$, $B_n(1) = B_n$ is the nth Bernoulli number and from (3.1) we obtain

$$\sum_{k=2}^{n-1} \binom{n}{k} B_k B_{n-k} = -(n+1)B_n$$

which is Euler's Identity.

For $N = 2$, we have the following theorem, which was proved by Kamano in [8].

Theorem 4.1. The hypergeometric Bernoulli numbers $\mathrm{B_n(2)}$ given by(1.1), satisfy the convolution formula

$$\sum_{k=2}^{n-2} \binom{n}{k} B_k(2) B_{n-k}(2) = -n\left[B_n(2) + \tfrac{1}{3} B_{n-1}(2)\right] \quad (4.1)$$

Proof. This follows from applying the Cauchy product formula for infinite series,(3.1), and the fact that $B_0(2) = 1$ and $B_1(2) = -1/3$.

Theorem 4.2. The hypergeometric Bernoulli numbers of order N satisfy the relation

$$\sum_{k_0+\cdots+k_m=n} \binom{n}{k_0,\ldots,k_m} \frac{\prod_{j=0}^{m} B_{k_j}(N)}{n!} = \frac{(-1)^m}{m!N^m} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{a_{k-j,m}(N,k)}{(n-k)!} B_{n-j}(N) \quad (4.2)$$

Proof. We begin by substituting the series representation of $G(x)$:

$$G(x) = \sum_{n=0}^{\infty} B_n(N) \frac{x^n}{n!}$$

and use this in (2.4) to get

$$\left(\sum_{n=0}^{\infty} B_n(N) \frac{x^n}{n!}\right)^{m+1} = \sum_{k=0}^{m}\left(\sum_{j=k}^{m} a_{k,m}(N,k)x^j\right) \sum_{n=0}^{\infty} B_{n+k}(N) \frac{x^n}{n!} \quad (4.3)$$

Dividing by $m!\,(-N)^m$ and expanding the left hand side of (4.3) gives

$$\left(\sum_{n=0}^{\infty} B_n(N) \frac{x^n}{n!}\right)^{m+1} = \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_m=n} \frac{B_{k_0}(N)\ldots B_{k_m}(N)}{k_0!\cdot\ldots k_m!}\right) x^n = \sum_{n=0}^{\infty} \left[\sum_{k_0+\cdots+k_m=n} \binom{n}{k_0,\ldots,k_m} \frac{B_{k_0}(N)\ldots B_{k_m}(N)}{n!}\right] x^n \quad (4.4)$$

For the right hand side of (4.3), we have

$$\frac{(-1)^m}{m!\,N^m} \sum_{k=0}^{m} \left(\sum_{j=k}^{m} a_{k,m}(N,k)x^j\right) \sum_{n=0}^{\infty} B_{n+k}(N) \frac{x^n}{n!}$$
$$= \frac{(-1)^m}{m!N^m} \sum_{n=0}^{\infty} \sum_{k=0}^{m} \sum_{j=k}^{m} a_{k,m}(N,j) \frac{B_{n+k}(N)}{n!} x^{n+j}$$
$$= \frac{(-1)^m}{m!\,N^m} \sum_{n=0}^{\infty} \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{a_{k,m}(N,j) B_{n+k}(N)}{n!} x^{n+j}$$
$$= \frac{(-1)^m}{m!\,N^m} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \frac{a_{k-j,m}(N,k) B_{n-j}(N)}{(n-k)!}\right) x^n$$

Now, by (2.7), we have that $a_{k,m}(N,j) = 0$ for $j < k$. Then we note that the above sum can be expressed as

$$\sum_{k=0}^{m}\left(\sum_{j=k}^{m} a_{k,m}(N,j)x^j\right) \sum_{n=0}^{\infty} \frac{B_{n+k}(N)}{n!} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{j=0}^{k} \frac{a_{j,m}(N,n-k) B_{n+j}(N)}{(n-k)!}\right) x^n \quad (4.5)$$

The theorem follows from comparing the coefficients in (4.4) and (4.5).

We now present the rest of the proof of the main result, Theorem 1.1.

Proof. Looking at the right hand side of (4.2), we have

$$\sum_{k=0}^{n} \sum_{j=0}^{k} \frac{a_{k-j}(N,k)}{(n-k)!} B_{n-j}(N) = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!\,k!\,a_{k-j}(N,k)}{n!\,k!\,(n-k)!} B_{n-j}(N) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \frac{k!\,a_{k-j}(N,k)}{n!} B_{n-j}(N)$$

Removing the common factor of $\frac{1}{n!}$ from both sides yields the theorem.

A CONVOLUTION THEOREM FOR HYPERGEOMETRIC BERNOULLI NUMBERS OF DIFFERENT ORDERS

In this section we give a convolution formula for hypergeometric Bernoulli numbers of different orders. We begin with an important lemma.

Lemma 5.1. Let

$$G_N(x) = \frac{x^N/N!}{e^x - T_{N-1}(x)} = \sum_{n=0}^{\infty} \frac{B_n(N)}{n!} x^n \tag{5.1}$$

If $N_1 < N_2 < \cdots < N_m$, then

$$D_m(x) \prod_{k=1}^{m} G_{N_k}(x) = \sum_{k=1}^{m} p_k(x) G_{N_k}(x), \tag{5.2}$$

where

$$D_m(x) = \prod_{1 \le i < j \le m} \left[T_{N_j-1}(x) - T_{N_i-1}(x)\right] \tag{5.3}$$

and

$$p_k(x) = (-1)^{m-k} \left(\frac{x^{\Sigma' N_i}}{\Pi' N_i!}\right) \prod_{\substack{1 \le i < j \le m \\ i,j \ne k}} \left[T_{N_j-1}(x) - T_{N_i-1}(x)\right] \tag{5.4}$$

Here the primes in Σ' and Π' indicate that the sum and product are from $l = 1$ to $l = m$ with $l \ne k$.

Proof. Let w, y_k, z_k and A_k $(k = 1, \dots, m)$ be complex numbers. If

$$\prod_{k=1}^{m} \frac{y_k}{w - z_k} = \sum_{k=1}^{m} \frac{A_k}{w - z_k}$$

then a simple calculation shows that

$$A_k = \frac{\prod_{k=1}^{m} y_k}{\prod_{j \ne k}(z_k - z_j)}.$$

Now, assume that $N_1 < N_2 < \cdots < N_m$. Let $y_k = \frac{x^k}{N_k!}$, $w = e^x$, and $z_k = T_{N_k-1}(x)$. Then

$$A_k(x) := A_k = \left(\prod_{j=1}^{m} \frac{x^j}{N_j!}\right)\left[\frac{1}{\prod_{j \ne k}\left(T_{N_k-1}(x) - T_{N_j-1}(x)\right)}\right]$$

On the other hand, with $D_m(x)$ as in (5.3), we have

$$D_m(x) A_k(x) = \left(\prod_{j \ne k} \frac{x^j}{N_j!}\right)\left[(-1)^{m-k} \prod_{\substack{1 \le j < k \le m \\ i,j \ne k}} \left(T_{N_k-1}(x) - T_{N_j-1}(x)\right)\right]$$

But this is $p_k(x)$ as defined in (5.4). The lemma follows from the fact that

$$\prod_{k=1}^{m} \frac{y_k}{w - z_k} = \prod_{k=1}^{m} G_k(x).$$

The cases for $m = 2$ and $m = 3$ are the contents of the following two corollaries.

Corollary 5.1. If $G_N(x)$ is given by (5.1) and $N_1 < N_2$, then

$$\left[T_{N_2-1}(x) - T_{N_1-1}(x)\right] G_{N_1}(x) G_{N_2}(x) = \frac{x^{N_1}}{N_1!} G_{N_2}(x) - \frac{x^{N_2}}{N_2!} G_{N_1}(x) \tag{5.5}$$

Corollary 5.2. If $G_N(x)$ is given by (5.1) and $N_1 < N_2 < N_3$, then

$$D_3(x) G_{N_1}(x) G_{N_2}(x) G_{N_3}(x) = p_1(x) G_{N_1}(x) + p_2(x) G_{N_2}(x) + p_3(x) G_{N_3}(x),$$

Where

$$D_3(x) = \left(T_{N_3-1}(x) - T_{N_1-1}(x)\right)\left(T_{N_3-1}(x) - T_{N_2-1}(x)\right)\left(T_{N_2-1}(x) - T_{N_1-1}(x)\right)$$

$$p_1(x) = \frac{x^{N_2+N_3}}{N_2! N_3!}\left(T_{N_3-1}(x) - T_{N_2-1}(x)\right)$$

$$p_2(x) = \frac{x^{N_1+N_3}}{N_1! N_3!}\left(T_{N_3-1}(x) - T_{N_1-1}(x)\right)$$

$$p_3(x) = \frac{x^{N_1+N_2}}{N_1! N_2!}\left(T_{N_2-1}(x) - T_{N_1-1}(x)\right)$$

The following theorem was proved by Nyugen and Cheong in [9] by a method different from the one given here. It is easy to see that this follows from Corollary 5.1.

Theorem 5.1. The polynomials $B_n(N, z)$ satisfy the convolutions
If $N_1 < N_2$, then for $N_1 \le n \le N_2 - 1$, we have

$$\sum_{k=N_1}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} B_j(N_1) B_{n-k-j}(N_2)$$

$$= \binom{n}{N_1} B_{n-N_1}(N_2) \quad (5.6)$$

and for $n \geq N_2$, we have

$$\sum_{k=N_1}^{N_2-1} \sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j} B_j(N_1)B_{n-k-j}(N_2) = \binom{n}{N_1} B_{n-N_1}(N_2) - \binom{n}{N_2} B_{n-N_2}(N_1) \quad (5.7)$$

Proof. We use the series representation in (5.1) and the fact that

$$T_{N_2-1}(x) - T_{N_1-1}(x) = \sum_{k=N_1}^{N_2-1} \frac{x^k}{k!}$$

to write

$$\begin{aligned}[T_{N_2-1}(x) - T_{N_1-1}(x)]G_{N_1}(x)G_{N_2}(x) &= \sum_{k=N_1}^{N_2-1}\sum_{n=0}^{\infty} \frac{C_n x^{n+k}}{k!\,n!} \\ &= \sum_{k=N_1}^{N_2-1}\sum_{n=k}^{\infty} \binom{n}{k} C_{n-k} \frac{x^{n+k}}{n!} \\ &= \sum_{n=N_1}^{N_2-1}\left(\sum_{k=N_1}^{n}\binom{n}{k} C_{n-k}\right)\frac{x^n}{n!} \\ &\quad + \sum_{n=N_2}^{\infty}\left(\sum_{k=N_1}^{N_2-1}\binom{n}{k} C_{n-k}\right)\frac{x^n}{n!} \quad (5.8)\end{aligned}$$

where

$$C_n = \sum_{k=0}^{n} \binom{n}{k} B_k(N_1)B_{n-k}(N_2).$$

On the other hand

$$\begin{aligned}&\frac{x^{N_1}}{N_1!}G_{N_2}(x) - \frac{x^{N_2}}{N_2!}G_{N_1}(x) \\ &= \sum_{n=0}^{\infty} B_n(N_2)\frac{x^{n+N_1}}{N_1!\,n!} - \sum_{n=0}^{\infty} B_n(N_1)\frac{x^{n+N_2}}{N_2!\,n!} \\ &= \sum_{n=N_1}^{\infty}\binom{n}{N_1}B_{n-N_1}(N_2)\frac{x^n}{n!} - \sum_{n=N_2}^{\infty}\binom{n}{N_2}B_{n-N_2}(N_1)\frac{x^n}{n!} \\ &= \sum_{n=N_2}^{N_2-1}\binom{n}{N_1}B_{n-N_1}(N_2)\frac{x^n}{n!} + \sum_{n=N_2}^{\infty}\left[\binom{n}{N_1}B_{n-N_1}(N_2) - \binom{n}{N_2}B_{n-N_2}(N_1)\right]\frac{x^n}{n!} \quad (5.9)\end{aligned}$$

It is now clear that (5.6) and (5.7) both follow from (5.2), (5.5), (5.8), and (5.9).

A similar theorem can be shown using similar methods for Bernoulli polynomials of different order. Lemma 5.1 can be extended the same, proof and all. The difference comes when recognizing that

$$p_k(x,z) = (-1)^{m-k} e^{xz}\left(\frac{x^{\Sigma' N_l}}{\Pi' N_l!}\right)\prod_{\substack{1\leq i<j\leq m,\\ i,j\neq k}}\left[T_{N_j-1}(x) - T_{N_i-1}(x)\right].$$

When $m = 2$, this yields

$$D_2(x)G_{N_1}(x,z)G_{N_2}(x,z) = \frac{e^{xz}x^{N_1}}{N_1!}G_{N_2}(x,z) - \frac{e^{xz}x^2}{N_2!}G_{N_1}(x,z) \quad (5.10)$$

Here is a consequence of (5.10).

Theorem 5.2. If $N_1 < N_2$, then for $N_1 \leq n \leq N_2 - 1$, we have

$$\sum_{k=N_1}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}B_j(N_1,z)B_{n-k-j}(N_2,z) = \sum_{k=N_1}^{n}\binom{n}{k}\binom{k}{N_1}B_{k-N_1}(N_2,z)z^{n-k} \quad (5.11)$$

and for $n \geq N_2$, we have

$$\begin{aligned}&\sum_{k=N_1}^{N_2-1}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}B_j(N_1,z)B_{n-k-j}(N_2,z) \\ &= \sum_{k=N_1}^{n}\binom{n}{k}\binom{k}{N_1}B_{k-N_1}(N_2,z)z^{n-k} - \sum_{k=N_2}^{n}\binom{n}{k}\binom{k}{N_2}B_{k-N_2}(N_1,z)z^{n-k} \quad (5.12)\end{aligned}$$

Proof. First note that

$$\begin{aligned}&\frac{e^{xz}x^{N_1}}{N_1!}G_{N_2}(x,z) - \frac{e^{xz}x^{N_2}}{N_2!}G_{N_1}(x,z) \\ &= e^{xz}\sum_{n=0}^{\infty}B_n(N_2,z)\frac{x^{n+N_1}}{N_1!\,n!} - e^{xz}\sum_{n=0}^{\infty}B_n(N_2,z)\frac{x^{n+N_1}}{N_1!\,n!} \\ &= \sum_{n=0}^{\infty} z^n\frac{x^n}{n!}\sum_{n=N_1}^{\infty}\binom{n}{N_1}B_{n-N_1}(N_2,z)\frac{x^n}{n!} - \sum_{n=0}^{\infty} z^n\frac{x^n}{n!}\sum_{n=N_2}^{\infty}\binom{n}{N_2}B_{n-N_2}(N_1,z)\frac{x^n}{n!}\end{aligned}$$

$$= \sum_{n=N_1}^{N_2-1} \sum_{k=N_1}^{n} \binom{n}{k}\binom{k}{N_1} B_{n-N_1}(N_2,z) z^{n-k} \frac{x^n}{n!}$$
$$+ \sum_{n=N_2}^{\infty} \left[\sum_{k=N_1}^{n} \binom{n}{k}\binom{k}{N_1} B_{k-N_1}(N_2,z) z^{n-k} - \sum_{k=N_2}^{n} \binom{n}{k}\binom{k}{N_2} B_{k-N_2}(N_1,z) z^{n-k} \right] \frac{x^n}{n!}$$

On the other hand, from Lemma 5.1, (with $m = 2$ in (5.2)), we have

$$D_2(x) G_{N_1}(x) G_{N_2}(x) = p_1(x) G_{N_1}(x) - p_2(x) G_{N_2}(x).$$

We now use (5.2), (5.3), (5.4), and (5.10), simple but tedious algebraic manipulations, and comparison of coefficients of powers of x of the resulting power series yield the theorem.

In exactly the same manner, we can prove the following consequence of Corollary 5.2.

Theorem 5.3. Suppose $N_1 < N_2 < N_3$. Define

$$\alpha_k = \begin{cases} \frac{k!}{(N_1+k)!} & \text{if } 0 \le k \le N_3 - N_1 - 1 \\ 0 & \text{if } k > N_3 - N_1 - 1 \end{cases}$$
$$\beta_k = \begin{cases} \frac{k!}{(N_2+k)!} & \text{if } 0 \le k \le N_3 - N_2 - 1 \\ 0 & \text{if } k > N_3 - N_2 - 1 \end{cases}$$
$$\gamma_k = \begin{cases} \frac{k!}{(N_1+k)!} & \text{if } 0 \le k \le N_2 - N_1 - 1 \\ 0 & \text{if } k > N_2 - N_1 - 1 \end{cases}$$

Let a, b, c, d, e be functions of n defined by

$$a(n) = \begin{cases} n & \text{if } 0 \le n \le 2N_3 - 2N_1 - 3 \\ 2N_3 - 2N_1 - 3 & \text{if } n > 2N_3 - 2N_1 - 3 \end{cases}$$
$$b(n) = \begin{cases} n & \text{if } 0 \le n \le N^* \\ N^* & \text{if } n > N^* \end{cases}$$
$$c(n) = \begin{cases} n & \text{if } 0 \le n \le N_3 - N_2 - 1 \\ N_3 - N_2 - 1 & \text{if } n > N_3 - N_2 - 1 \end{cases}$$
$$d(n) = \begin{cases} n & \text{if } 0 \le n \le N_3 - N_1 - 1 \\ N_3 - N_1 - 1 & \text{if } n > N_3 - N_1 - 1 \end{cases}$$
$$e(n) = \begin{cases} n & \text{if } 0 \le n \le N_2 - N_1 - 1 \\ N_2 - N_1 - 1 & \text{if } n > N_2 - N_1 - 1 \end{cases}$$

where
$$N^* = \min\{N_2 - N_1 - 1, N_3 - N_2 - 1\}.$$

Finally, define I, J, and K by

$$I(n) = \frac{1}{N_1!\,N_2!} \sum_{k=0}^{e(n)} \binom{n}{k} \frac{k!}{(N_2+k)!} B_{n-k}(N_1),$$

$$J(n) = \frac{n!}{(n - N_3 + N_2)!\,N_1!\,N_3!} \sum_{k=0}^{d(n)} \binom{n}{k} \frac{k!}{(N_1+k)!} B_{n-k}(N_2) \quad n \ge N_3 - N_2,$$

$$K(n) = \frac{n!}{(n - N_3 - N_2 + 2N_1) N_1!\,N_2!} \sum_{k=0}^{c(n)} \binom{n}{k} \frac{k!}{(N_1+k)!} B_{n-k}(N_3) \quad n \ge N_3 + N_2 - 2N_1.$$

Then

$$\sum_{k=0}^{a(n)} \sum_{l=0}^{b(m)} \sum_{k=0}^{l} \sum_{j=0}^{n-m} \sum_{i=0}^{j} \binom{n}{i,j,k,l,m} \gamma_k \beta_{l-k} \alpha_{m-l} B_i(N_1) B_j(N_2) B_{n-m-j}(N_3)$$
$$= \begin{cases} I(n) & \text{if } 0 \le n \le N_3 - N_2 - 1 \\ I(n) - J(n) & \text{if } N_3 - N_2 \le n \le N_3 + N_2 - N_1 - 1 \\ I(n) - J(n) + K(n) & \text{if } n \ge N_3 + N_2 - 2N_1 \end{cases} \tag{5.13}$$

Remark 5.1. One can exploit the results of Lemma 5.1 to get more general convolution theorems. But we found this to be cumbersome but routine and elaborate calculation similar to the ones given in Theorems 5.1 and 5.2.

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Appendix

Here are the polynomials $P_{k,m}(N,x)$ for $m = 1, \ldots, 6$. Note that by (2.6) the associated $a_{k,m}(N,j)$ are here as well. For example, for $P_{1,3}(N,x)$, $a_{1,3}(N,2) = 10 - 22$

$m = 1$	$P_{0,1}$	=	$-N + x$
	$P_{1,1}$	=	x
$m = 2$	$P_{0,2}$	=	$2N^2 + (1-4N)x + 2x^2$
	$P_{1,2}$	=	$(1-3N)x + 3x^2$
	$P_{2,2}$	=	x^2
$m = 3$	$P_{0,3}$	=	$-6N^3 + (1 - 7N + 18N^2)x + (7 - 18N)x^2 + 6x^2$
	$P_{1,3}$	=	$(1 - 6N + 11N^2)x + (10 - 22N)x^2 + 11x^3$
	$P_{2,3}$	=	$(3 - 6N)x^2 + 6x^3$
	$P_{3,3}$	=	x^3
$m = 4$	$P_{0,4}$	=	$24N^4 + (1 - 11N + 46N^2 - 96N^3)x + (18 - 92N + 144N^2)x^2 + (46 - 96N)x^3$ $+24x^4$
	$P_{1,4}$	=	$(1 - 10N + 35N^2 - 50N^3)x + (25 - 115N + 150N^2)x^2 + (80 - 150N)x^3 + 50x^4$
	$P_{2,4}$	=	$(7 - 30N + 35N^2)x^2 + (40 - 70N)x^3 + 35x^4$
	$P_{3,4}$	=	$(6 - 10N)x^3 + 10x^4$
	$P_{4,4}$	=	x^4
$m = 5$	$P_{0,5}$	=	$-120N^5 + (1 - 16N + 101N^2 - 326N^3 + 600N^4)x + (41 - 329N + 987N^2 - 1200)x^2$ $+(228 - 978N + 1200N^2)x^3 + (326 - 600N)x^4 + 120x^5$
	$P_{1,5}$	=	$(1 - 15N + 85N^2 - 225N^3 + 274N^4)x + (56 - 416N + 1096N^2 - 1096N^3)x^2$ $+(383 - 1517N + 1644N^2)x^3 + (646 - 1096N)x^4 + 274x^5$
	$P_{2,5}$	=	$(15 - 105N + 255N^2 - 225N^3)x^2 + (180 - 675N + 675N^2)x^3 + (420 - 675N)x^4 + 274x^5$
	$P_{3,5}$	=	$(25 - 90N + 85N^2)x^3 + (110 - 170N)x^4 + 85x^5$
	$P_{4,5}$	=	$(10 - 15N)x^4 + 15x^5$
	$P_{5,5}$	=	x^5

$m=6$	$P_{0,6}$	$=$	$720N^6+(1-22N+197N^2-932N^3+2556N^4-4320N^5)x$ $+(88-1000N+4536N^2-10224N^3+10800N^4)x^2$ $+(930-6276N+15336N^2+10444N^3)x^3+(2672-10224N+10800N^2)x^4$ $+(2556-4320N)x^5+720x^6$
	$P_{1,6}$	$=$	$(1-21N+175N^2-735N^3+1624N^4-1764N^5)x$ $+(119-1274N+5299N^2-10444N^3+8820N^4)x^2$ $+(1526-9674N+21588N^2-17640N^3)x^3+(5110-18340N+17640N^2)x^4$ $+(5572-8820N)x^5+1764x^6$
	$P_{2,6}$	$=$	$(31-315N+1225N^2-2205N^3+1624N^4)x^2+(686-4151N+8701N^2-6496N^3)x^3$ $+(3143-10787N+9744N^2)x^4+(4291-6496N)x^5+1624x^6$
	$P_{3,6}$	$=$	$(90-525N+1050N^2-735N^3)x^3+(770-2555N+2205N^2)x^4+(1505-2205N)x^5$ $+1624x^6$
	$P_{4,6}$	$=$	$(65-210N+175N^2)x^4+(245-350N)x^5+175x^6$
	$P_{5,6}$	$=$	$(15-21N)x^5+21x^6$
	$P_{6,6}$	$=$	x^6