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APPLICATION OF NORMAL FAMILY TO THE SPREAD INEQUALITY AND THE PALEY TYPE INEQUALITY

Seid Mohammed

Department of Mathematics, College of Natural Sciences, Addis Ababa University
PO Box1176, Addis Ababa, Ethiopia. E-mail: Seid_mt@yahoo.com

ABSTRACT: In this paper we derive a Paley type inequality for subharmonic functions of order $\lambda, 0 < \lambda \le \frac{1}{2}$ and describe the asymptotic behaviour of the extremal functions near Pòlya peaks. We also give an alternative proof for the spread inequality using a non-asymptotic method via - a normal family of δ -subharmonic functions.

Key words/phrases: **Characteristic function, δ -subharmonic function, Pòlya peaks, star-function,**

INTRODUCTION

Let u = $u_1 - u_2$ be a subharmonic function in the complex plane, where u_1, u_2 are subharmonic functions. We write

$$N(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta$$

$$B(r, u) = \sup_{|z|=r} u(z).$$

The Nevanlinna characteristic T(r, u) of u is defined by,

$$T(r, u) = N(r, u^+) + N(r, u_2)$$

and the order λ of u by

$$\lambda = \lim_{r\to\infty}\sup \frac{\log T(r,u)}{T(r,u)}.$$

If λ is finite, T(r, u) has sequence of Pòlya peaks $\{r_n\}$ of order λ, i.e,

$$T(r, u) \le (1+\epsilon_n)\left(\frac{r}{r_n}\right)^{\lambda} T(r, u), \quad (\epsilon_n r_n \le r \le \frac{r_n}{\epsilon_n})$$

for some sequence $\epsilon_n \to 0$ and $\epsilon_n r_n \to \infty$ as $n \to \infty$ (for the proof see Edrei (1965)).

Let $\{r_n\}$ be a sequence of Pòlya peaks for T(r,u) of order λ, and set

$$\beta_n = \frac{1}{2} m\{\theta \in (-\pi, \pi]: u(r_n e^{i\theta}) > 0\}, n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(1)}$$

where m is the Lebesgue measure on the real line. Put

$$\beta_0 = \underline{\lim}\beta_n, \quad 0 \le \beta_o \le \pi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(2)}$$

There is a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $\beta_{n_k} \to \beta_0$ as $k \to \infty$. Since a subsequence of $\{r_n\}$ is also a sequence of Pòlya peaks for T(r, u), we assume that $\beta_n \to \beta_0$ as $n \to \infty$.

Let β be the smallest non-negative number such that

$$\cos\lambda\beta = 1 - \delta, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3)}$$

where $\delta = \delta(\infty, u) = 1 - \lim_{r\to\infty}\sup \frac{N(r,u_2)}{T(r,u)}$, called the Nevalinna deficiency of u. We note that $\beta \le \frac{\pi}{2\lambda}$.

The Spread inequality asserts that $\beta_0 \ge \min\{\pi, \beta\}$. It describes the size of the set for which a $\delta-$subharmonic grows on $|z| = r$ as $r \to \infty$. It was conjectured by Edrei and proved by Baernstein (1973) and also by Rossi and Weitsman (1983), when u = log $|f(z)|$, f a mermorphic function. Their proofs mainly depend on the star function and on the formulation of the theory of indicators introduced by Edrei (1970). In this paper we do not use the theory involving indicators instead we use a non-asymptotic method via normal family of δ-subharmonic functions to yield simpler proof for the spread relation. As we shall see the non-asymptotic approach provides a method to obtain sharp upper bound and to describe the asymptotic behavior of certain class of subharmonic functions.

Let u be a subharmonic function of order $\lambda, 0<\lambda\le\frac{1}{2}$ and $\{r_n\}$ a sequence of Pòlya peaks of order λ.

The main objective of the paper is to show that

$$\lim_{n\to\infty}\frac{B(rr_n,u)}{T(r_n,u)}\le\frac{\pi\lambda}{\sin\pi\lambda} \quad \text{……………………………}\textbf{(4)}$$

for each r, ($0<r<\infty$) and to describe the global asymptotic behavior of the extremal functions, i.e, functions for which equality holds in (4). Inequality (4) was proved by Rossi and Weitsman (1983) for the case u = $\log|f(z)|$, f an entire function and r = 1. I refer to the inequality in (4) as Paley type inequality since Paley (1932) had conjectured the inequality

$$\lim_{r\to\infty}\sup\frac{B(r,u)}{T(r,u)}\le\frac{\pi\lambda}{\sin\pi\lambda},(0<\lambda\le\frac{1}{2})$$

The inequality was proved by Govorov (1969) for u = $\log|f(z)|$, where f is an entire function.

The subharmonic function

$$u\,(re^{i\theta})=\frac{\pi\lambda r^{\lambda}}{\sin\pi\lambda}\cos\lambda\theta,(|\theta|\le\pi) \quad \text{…………………}\textbf{(5)}$$

is a typical example extremal to the Paley type inequality (4) showing that the inequality is sharp.

We will show that the extremal functions for the Paley type inequality behave asymptotically as rotations of the subharmonic function in (5). We use again a non-asymptotic method via a normal family of δ-subharmonic function to obtain the desired result. More over the functions extremal to the Paley inequality are also in some sense extremal (Theorem I′) to the well known inequality of the classical $\cos\pi\rho$ theorem of Valiron (1914) and Wiman (1915), and to the inequality proved by Ostrovskii (1963) and also by Edrei (1970).

Statement of main result

Theorem I: Let u be a sub harmonic function of order λ, $0<\lambda\le\frac{1}{2}$ and $\{r_n\}$ a sequence of Pòlya peaks of order λ. Then

a) $\lim_{r\to\infty}\frac{B(rr_n,u)}{T(r_n,u)}\le\frac{\pi\lambda r^{\lambda}}{\sin\pi\lambda}$, $\quad(0<r<\infty)$

b) If equality holds in (a) for some r > 0, then it holds for all r > 0. In this case, we have

i) $\lim_{n\to\infty}\frac{B(rr_n,u)}{T(r_n,u)}=\frac{\pi\lambda r^{\lambda}}{\sin\pi\lambda}$, $\quad(0<r<\infty)$,

ii) $\lim_{n\to\infty}\frac{T(rr_n,u)}{T(r_n,u)}=r^{\lambda}=\lim_{n\to\infty}\frac{N(rr_n,u)}{T(r_n,u)}$, and

iii) there is a subsequence I of the positive integers such that

$$u(rr_ne^{i\theta})=(1+o(1))T(r_n)\frac{\pi\lambda r^{\lambda}}{\sin\pi\lambda}\cos(\lambda(\theta-\alpha))$$

as $n\to\infty$, $n\in I$ for almost all $\theta,|\theta-\alpha|\le\pi$ and for some $\alpha\in(-\pi,\pi]$.

Statement of the Spread inequality

Theorem II: (The Spread inequality)

If u is a δ- subharmonic function of order $\lambda, 0<\lambda<\infty$, then

$$\beta_0\ge\min\{\pi,\beta\},$$

where β_0 and β are given in (2) and (3) respectively. If $0<\lambda\le\frac{1}{2}$ we have $\beta_0=\pi$.

Preliminaries

1. The star-function. Let u = u_1 - u_2 be a δ-subharmonic function in the complex plane. Following Baernstein (1973), we define the star - function of u by:

$$u^*\,(re^{i\theta})\;=\;\sup\frac{1}{2\pi}\int_E u(re^{i\varphi})d\varphi\;+N(r,u_2) \quad \text{…….}\textbf{(6)}$$

where the supremum is taken over all measurable set E, with mE = 2θ. It is proved that

$$u^*(re^{i\theta})=\frac{1}{\pi}\int_0^{\theta}\tilde{u}(re^{i\varphi})d\varphi+N(r,u_2) \quad \text{……..……}\textbf{(7)}$$

where $\varphi\to\tilde{u}(re^{i\varphi})$ is the symmetric decreasing rearrangement of u ($re^{i\varphi}$) on [$-\pi,\pi$]. Baernstein (1973) proved that u^* is subharmonic in the upper half plane π^+ and continuous on the closure of π^+ except possibly at the origin, and that the supremum in (6) is attained in some set E $\subseteq$ [$-\pi,\pi$].

From the definition of T (r, u) we have

$$T(r,u)\;=\;\max_{\theta}u^*(re^{i\theta})\;(0\le\theta\le\pi) \quad \text{………………}\textbf{(8)}$$

$$N(r,u_1) = u^*(re^{i\pi}),\ N(r,u_2) = u^*(r) \text{ …………}\mathbf{(9)}$$

$$B(r,u) = \sup_{|z|=r} u(z) = \pi \frac{\partial}{\partial\theta} u^*(re^{i\theta})|\theta = 0 \text{, and}$$

$$A(r,u) = \inf_{|z|=r} u(z) = \pi \frac{\partial}{\partial\theta} u(re^{i\theta})|_{\theta=\pi} \text{ ……………}\mathbf{(10)}$$

We also need the following result due to Anderson and Baerstein (1978). Let u = $u_1 - u_2$ a be δ - subharmonic function in the plane of order $0 \le \lambda < \infty$, and $\{r_n\}$ a sequence of Pòlya peaks for T(r,u) of order λ. Set

$$u_n(z) = \frac{u(zr_n)}{T(r_n,u)}, \quad n = 1,2,3\ldots, \text{ and}$$

$$= u_1^{(n)}(z) - u_2^{(n)}(z)$$

where

$$u_i^{(n)}(z) = \frac{u_i(r_n z)}{T(r_n,u)} \qquad (i = 1,2)$$

Here we have $T(r,u_n) = \dfrac{T(rr_n,u)}{T(r_n,u)}$ and

$$B(r,u_n) = \frac{B(rr_n,u)}{T(r_n,u)}.$$

2. Anderson and Baerstein (1978) have proved that there is a δ - subharmonic function v = $v_1 - v_2$ and a subsequence I = $\{n_k\}$ of the positive integers such that the following statements hold as $n \to \infty$ in I

a. $\lim_{n\to\infty} \int_{-\pi}^{\pi} |u_n(re^{i\theta}) - v(re^{i\theta})| d\theta = \lim_{n\to\infty} N(r, |u_n - v|)$

$= 0, \quad 0 < r < \infty$ ………………………………….**(11)**

b. $\lim_{n\to\infty} T(r,u_n) = T(r,v) \le r^{\lambda}$ ……………..…..**(12)**

c. $\lim_{n\to\infty} N(r, u_1^{(n)}) = N(r, v_1)$ ………………..……**(13)**

d. $\lim_{n\to\infty} N(r, u_2^{(n)}) = N(r, v_2) \le (\cos\lambda\beta)\ r^{\lambda}$, where β as in (3)…………………………………………**(14)**

Since $T(1, u_n) = 1$, it follows from (b) that T(1, v) = 1. We refer to the δ - subharmonic function v as the limit function of u. We restate Theorem I in terms of the limit function v of u.

Theorem I: Let u be a sub harmonic function of order λ, $0 < \lambda \le \frac{1}{2}$ and $\{r_n\}$ a sequence of Pòlya peaks of order λ. If v is a limit function of u then

i) $B(r,v) \le \dfrac{\pi\lambda r^{\lambda}}{\sin\lambda\pi}$, $(0 < r < \infty)$.

If equality holds in (a) of Theorem I for some r > 0, then for all r > 0, we have

ii) $B(r,v) = \dfrac{\pi\lambda r^{\lambda}}{\sin\lambda\pi}$

iii) $A(r,v) = \pi\lambda r^{\lambda} \dfrac{\cos\pi\lambda}{\sin\pi\lambda}$

iv) $N(r,v) = r^{\lambda} = T(r,v)$, $(0 < r < \infty)$

v) $v(re^{i\theta}) = \dfrac{\pi\lambda r^{\lambda}}{\sin\pi\lambda} \cos\lambda(\theta - \alpha), |\theta - \alpha| \le \pi$, for some $\alpha \in [-\pi, \pi)$

Thus, if equality holds in the Paley type inequality for some r > 0, then the limit function v satisfies:

$\dfrac{B(r,v)}{T(r,v)} = \dfrac{\pi\lambda}{\sin\pi\lambda}$, that is, v is extremal to Paley inequality,

$$\limsup_{r\to\infty} \frac{B(r,u)}{T(r,u)} \le \frac{\pi\lambda}{\sin\pi\lambda}, (0 < \lambda \le \frac{1}{2})$$

$\dfrac{A(r,v)}{B(r,v)} = \cos\pi\lambda$, that is, v is extremal to the inequality

$$\limsup_{r\to\infty} \frac{A(r,u)}{B(r,u)} \ge \cos\pi\lambda$$

due to Valiron (1914) and Wiman (1915), and

$$\frac{A(r,v)}{T(r,v)} = \pi\lambda \frac{\cos\pi\lambda}{\sin\pi\lambda}$$

that is, v is extremal to the inequality

$$\limsup_{r\to\infty} \frac{A(r,u)}{T(r,u)} \ge \pi\lambda \frac{\cos\pi\lambda}{\sin\pi\lambda}$$

due to Ostrovisklii and Edei (1963).

To prove Theorem II we need

Lemma 1: Let u = $u_1 - u_2$ be a δ - subharmonic function of order $\lambda > 0$, $\{r_n\}$ a sequence Pòlya peaks for T(r,u) and I = $\{n_k\}$ be a sequence of positive integers associated with v = $v_1 - v_2$. Then $v^*(e^{i\beta_0}) = 1$. where β_0 is given by (2).

Proof Let $A_n = \{\theta : u_n(e^{i\theta}) > 0\}$, $m(A_n) = 2\beta_n$ by (1), and

$$\alpha = \frac{1}{2} m\{\theta \in [-\pi, \pi] : v(e^{i\theta}) > 0\}$$

We have, using (11),

$$\left|\int_{A_n} (u_n(e^{i\theta}) - v(e^{i\theta}))d\theta\right| \le \int_{A_n} \left|u_n(e^{i\theta}) - v(e^{i\theta})\right| d\theta \le$$

$$\int_0^{2\pi} |u_n - v| \, d\theta \to 0 \text{ as } n \to \infty \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(15)}$$

Since by (13)

$N(1, u_2^{(n)}) \to N(1, v_2)$ as $n \to \infty$ and $1 = \frac{1}{2\pi}\int_{A_n} u_n(e^{i\theta})\, d\theta + N(1, u_2^n)$ we conclude from (13)

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{A_n} u_n(e^{i\theta})\, d\theta = 1 - N(1, v_2) = \lim_{n\to\infty} \frac{1}{2\pi}\int_{A_n} v(e^{i\theta}) d\theta$$

But by (7) and the definition of T(r, v), we have

$$1 = v^*(e^{i\alpha}) = \frac{1}{\pi}\int_0^{\alpha} \tilde{v}(e^{i\theta}) d\theta + N(1, v_2).$$

Hence

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{A_n} v(e^{i\theta}) d\theta = \frac{1}{\pi}\int_0^{\alpha} \tilde{v}(e^{i\theta}) d\theta$$

Since

$$v^*(e^{i\beta_n}) \ge \frac{1}{2\pi}\int_{A_n} v(e^{i\theta}) d\theta + N(1, v_2)$$

for each n, letting $n \to \infty$, and by the continuity of v^*, we have

$$v^*(e^{i\beta_0}) \ge v^*(e^{i\alpha}) = 1$$

But by (8),

$$v^*(e^{i\beta_0}) \le T(1, v) = 1$$

Thus $v^*(e^{i\beta_0}) = 1$

We first give a proof of theorem II (The Spread inequality). Proof Theorem II we consider two cases:

Case 1. $\beta \ge \pi$. Here we have $\lambda \le \frac{1}{2}$, since $\lambda\pi \le \lambda\beta \le \frac{\pi}{2}$ (note that we always have $\beta \le \frac{\pi}{2\lambda}$). Thus if v is a limit function of u then by (12) and (13) we have $v^*(r) \le r^{\lambda}\cos\lambda\beta \le r^{\lambda}\cos\lambda\pi$, $v^*(re^{i\pi}) \le r^{\lambda}$ and, hence by Phragme'n Lindelöf principle

$$v^*(re^{i\theta}) \le r^{\lambda}\cos\lambda(\pi - \theta), \ (0 \le \theta \le \pi) \ \ldots\ldots\mathbf{(16)}$$

Thus, if $\beta_0 < \pi$, then using the above Lemma and applying the inequality in (16) with r = 1, $\theta = \beta_0$ we get (by lemma 1)

$$1 = v^*(e^{i\beta_0}) \le \cos\lambda(\beta_o - \pi),$$

which is a contradiction since in this case $0 < \lambda(\pi - \beta_0) \le \frac{\pi}{2}$. Thus $\beta_0 \ge \pi$. Since $\beta_0 \le \pi$, we conclude that $\beta_o = \pi$. Consequently

$$\beta_0 \ge \min\{\pi, \beta\}.$$

Case 2. $\beta < \pi$. Since $v^*(re^{i\beta}) \le T(r,v) \le r^{\lambda}$ and $v^*(r) = N(r, v_2) \le r^{\lambda}\cos\lambda\beta$ by (12) and 12(d), we have by Phragme'n Lindelo''f principle

$$v^*(re^{i\theta}) \le r^{\lambda}\cos\lambda(\theta - \beta), \ (0 \le \theta \le \beta)$$

If $\beta_0 < \beta$, then, with r = 1, $\theta = \beta_0$ and using Lemma 1 we get

$$1 = v^*(e^{i\beta_0}) \le \cos\lambda(\beta_0 - \beta),$$

which is a contradiction. Hence $\beta_0 \ge \beta$; and $\beta_0 \ge \min\{\pi, \beta\}$. This completes the proof of theorem II.

Proof of Theorem I. We need the following well known lemma due to Petrenko (1969) Lemma 2. Suppose u is subharmonic in the plane. Fix γ, $0 < \gamma \le 1$, and let

$$k(t,\gamma) = \frac{\gamma^{-2} t^{\frac{1}{\gamma}}}{(t^{\frac{1}{\gamma}} + 1)}$$

Then

$$B(r,u) \le \int_0^R u(te^{i\pi\gamma})\, k\left(\frac{r}{t}, \gamma\right)\frac{dt}{t} + c\left(\frac{r}{R}\right)^{\frac{1}{\gamma}} T(2R, u), (0 < r < \frac{R}{2})$$

for an absolute constant c.

Proof of the above lemma are given by Essén (1975) and by Edrei and Fuchs (1976), lemma11.1) where it was shown that

$$\hat{k}(s,\gamma)=\int_0^\infty k(t,\gamma)\frac{dt}{t^{1+s}}=\frac{\pi}{\sin(\pi\gamma s)},(0<s<\frac{1}{\gamma})$$

Proof of assertion (a). Since $v^*(r)=0$ and $v^*(re^{i\pi})\le r^\lambda$ by Phragme'n Lindel$\ddot{o}$f principle, we have

$$v^*(re^{i\theta})\le r^\lambda\frac{\sin\lambda\theta}{\sin\pi\lambda},(0\le\theta\le\pi)\text{ ....}\mathbf{(17)}$$

This implies

$$\frac{\partial v^*}{\partial\theta}(re^{i\theta})\Big|_{\theta=0}=r^\lambda\frac{\lambda r^\lambda}{\sin\pi\lambda}$$

Thus by (8)

$$B(r,v)\le\frac{\pi\lambda r^\lambda}{\sin\pi\lambda},(0<r<\infty)\text{ ,.........................}\mathbf{(18)}$$

Now fix r > 0 and put $B(r,u_n)=u_n(re^{i\alpha_n})$, $\alpha_n\in(-\pi,\pi]$, n = 1, 2, 3... Assume $\alpha_n\to\alpha_o$ as $n\to\infty$. Then for s > r, we have

$$B(r,u_n)\le\frac{1}{2\pi}\int_{-\pi}^{\pi}u_n(se^{i\theta})P_{\frac{r}{s}}(\theta-\alpha_n)d\theta$$

where $p_s(\theta)=\dfrac{1-s^2}{1+s^2+2s\cos\theta}$ is the Poisson kernel. By (9) and since $P_{\frac{r}{s}}(\theta)\le\dfrac{s+r}{s-r}$ we have, dominated convergence theorem,

$$\limsup_{n\to\infty}B(r,u_n)\le\frac{1}{2\pi}\int_{-\pi}^{\pi}v(se^{i\theta})P_{\frac{r}{s}}(\theta-\alpha_o)d\theta\le B(s,v)$$

where v is the limit function of u. Since this holds for any s > r and B(s, v) is a continuous function of s, we have, letting s approach r and by (16) we get

$$\limsup_{n\to\infty}B(r,u_n)\le B(r,v)\le\frac{\pi\lambda r^\lambda}{\sin\pi\lambda},\qquad(0<r<\infty)\text{ .....}\mathbf{(19)}$$

For 0<s<r, let $\alpha\in(-\pi,\pi]$, such that $v(se^{i\alpha})=B(s,v)$. Then we have,

$$B(s,v)\le\frac{1}{2\pi}\int_{-\pi}^{\pi}v(re^{i\theta})P_{\frac{s}{r}}(\theta-\alpha)d\theta=\lim\frac{1}{2\pi}\int_0^{2\pi}u_n(re^{i\theta})P_{\frac{s}{r}}(\theta-\alpha)d\theta$$

$$=\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}u_n(re^{i\theta})P_{\frac{s}{r}}(\theta-\alpha)d\theta\le\liminf_{n\to\infty}B(r,u_n)$$

Thus B(s, v) $\le\liminf_{n\to\infty}B(r,u_n)$. Letting s$\to r$ we get

$$B(r,v)\le\liminf_{n\to\infty}B(r,u_n)\text{ ....}\mathbf{(20)}$$

From (19) and (20) we get

$$\lim_{n\to\infty}B(r,u_n)=B(r,v)\le\frac{\pi\lambda r^\lambda}{\sin\pi\lambda}\text{ ....}\mathbf{(21)}$$

which proves assertion (a).

To prove assertion (b) we assume equality holds for r = r_1 > 0, i.e.,

$$\lim_{n\to\infty}B(r_1,u_n)=r_1^\lambda\frac{\pi\lambda}{\sin\pi\lambda}\text{ ....}\mathbf{(22)}$$

Setting r = r_1 in (21) and together with (22) we get

$$B(r_1,v)\;=\;r_1^{\lambda 1}\frac{\pi\lambda}{\sin\pi\lambda}.$$

We now apply Lemma 2 with $\gamma=\frac{\alpha}{\pi},(0<\alpha<\pi)$ and (17) to the limit function v to obtain

$$B(r_1,v)\;=\;r_1^\lambda\frac{\pi\lambda}{\sin\pi\lambda}$$

$$\le\int_0^\infty v^*(te^{i\alpha})k(\frac{r_1}{t},\gamma)\frac{dt}{t}\le\frac{\sin\lambda\alpha}{\sin\pi\lambda}\int_0^\infty t^\lambda k(\frac{r_1}{t},\gamma)\frac{dt}{t}=r_1^\lambda\frac{\pi\lambda}{\sin\pi\lambda}$$

Thus equality holds through out. Using (15), the continuity of v^*, and basic fact in Lebesgue integral we conclude that

$$v^*(te^{i\alpha})=t^\lambda\frac{\sin\lambda\alpha}{\sin\pi\lambda}$$

Hence by (17) and the maximum principle for subharmonic function, we have

$$v^*(re^{i\theta})=r^\lambda\frac{\sin\lambda\theta}{\sin\pi\lambda},(0\le\theta\le\pi,0<r<\infty)\text{,......}\mathbf{(23)}$$

which implies by (8) and (12)

$$B(r,v)=r^\lambda\frac{\pi\lambda}{\sin\pi\lambda}\text{ for all r > 0, }T(r,v)=r^\lambda=N(r,v)$$

....**(24)**

Thus from (21) it follows

$$\lim_{n\to\infty} B(r,u_n) = r^{\lambda}\frac{\pi\lambda}{\sin\pi\lambda}$$

for all r > 0, which proves assertion (i) of (b). Moreover assertion (ii) of (b) follows from (12), (13) and (24).

Proof of assertion (iii) of (b). Let $\tilde{v}$ be the symmetric decreasing rearrangement of v, so that

$$v^*(re^{i\theta}) = \frac{1}{\pi}\int_0^{\theta}\tilde{v}(re^{i\alpha})d\alpha, (0 \le \theta \le \pi)$$

Thus using (23) we have

$$\tilde{v}(re^{i\theta}) = \lambda\pi r^{\lambda}\frac{\cos\lambda\theta}{\sin\pi\lambda}, (|\theta| \le \pi) \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(23)}$$

and $\tilde{v}$ is harmonic in {z: -π < arg (z) < π}. A well known result of Essén and Shea (1978/79) shows that

$v(ze^{i\alpha}) = \tilde{v}(z)$, $(|\arg z| \le \pi)$, for some $\alpha \in (-\pi,\pi])$

Thus setting z = $re^{i(\theta-\alpha)}$, where $|\theta-\alpha| \le \pi$ and using (21) we get

$$v(re^{i\theta}) = \frac{\pi\lambda r^{\lambda}}{\sin\pi\lambda}\cos(\theta-\alpha), \ (|\theta - \alpha| \le \pi)$$

A standard result in the theory of integration and (9) shows that there is subsequence I of positive integers such that

$$u_n(re^{i\theta}) = (o(1)+1)v(re^{i\theta})$$

as $n \to \infty$ (n$\in I$) for almost all θ, $(|\theta - \alpha| \le \pi)$. This completes the proof of Theorem I.

CONCLUSION

The study shows that the functions which are extremal for the Paley type inequality are completely characterized by the fact that they behave asymptotically as the function

$$u(re^{i\theta}) = \frac{\pi\lambda r^{\lambda}}{\sin\pi\lambda}\cos\lambda\theta, (|\theta| \le \pi)$$

and are in some way extremal to other inequalities arising in function theory.

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