SINET: Ethiop. J. Sci., 47(3): 251-257, 2024 ISSN: 0379-2897 (PRINT)

eISSN: 2520-7997

Date received: November 17, 2023; Date revised: December 20, 2024; Date accepted: December 22, 2024
DOI: https://dx.doi.org/10.4314/sinet.v47i3.7

FUZZY POINTS IN FUZZY GEOMETRY REDEFINED

Tesfaye Degife Weldetekle[1], ***Berhanu Bekele Belayneh***[2] **and** ***Zelalem Teshome Wale***[3*]

Department of Mathematics, Addis Ababa University, P.O.Box 1176, Addis Ababa, Ethiopia.
E-mail: [1] tesfayedegife72@gmail.com; E-mail: [2] berhanu.bekele@aau.edu.et and E-mail: [3] zelalem.teshome@aau.edu.et

ABSTRACT

In this paper, we introduce a new approach to study several notions of fuzzy geometry. This approach uses a modified definition of fuzzy points using connected and simply connected instead of convexity on the study of fuzzy geometry. We have shown that the center of a fuzzy circle is a fuzzy point.

Keywords: Fuzzy subset; Fuzzy number; Fuzzy point; Fuzzy distance.

Introduction

Zadeh in 1965 [21] initiated the theory of fuzzy sets as a new mathematical tool for dealing with uncertainties. The ideas on fuzzy geometrical notions have been proposed by many researchers. In 1997, Buckley and Eslami [3,4] introduced some ideas on the construction of the basic fuzzy geometrical entities in a mathematical framework. An overview of fuzzy geometry and some topological properties have been introduced later in 1984 [17] and modified in [1,18]. In [3,4] the authors introduced the idea by conforming to Zadeh's extension principle [20], using sup-min combination of fuzzy sets. This concept was further extended to define fuzzy space geometry by Qiu and Zhang [5].

Fuzzy geometry was further studied by Ghosh and Chakraborty [9] in 2012. A new concept of same and inverse points have been introduced in this regard. Definitions of Fuzzy line segment and fuzzy line, which is obtained by extending fuzzy line segment are introduced in four different forms by Chakraborty and Ghosh in [6]. The idea of same points also redefined with a parametric expression in [5]. A detailed study on the parametric and general form of fuzzy lines can be found in [10].

Recently, Chakraborty and Das [8] have shown that the intersection of two perpendicular fuzzy lines is a fuzzy point where the fuzzy lines are obtained by joining fuzzy points. Seçil O zekinci and Cansel Aycan [15,16] studied about a detailed analysis of fuzzy hyperbolas and ellipses in 2022. One of the most significant steps for the construction of fuzzy geometry was fuzzy points. However, in the former definition of fuzzy points the constraint that α-cuts of fuzzy points must be convex subsets of $\mathbb{R}^2$. In this paper, we propose a new definition for fuzzy points by replacing the concept of convexity by connected and simply connected using algebraic topology. We also apply this concept to study fuzzy distances, fuzzy lines, fuzzy circles and their properties. Further, in Ghosh and Chakraborty [11] the center of a fuzzy circle is not in general a fuzzy point but we have also shown that the center of a fuzzy circle is a fuzzy point.

Now let us introduce the notation that will be used in the rest of this paper. We will place a "bar" over a capital letter to denote a fuzzy subset of $\mathbb{R}$ or $\mathbb{R}^2$. So $\bar{A}$, $\bar{B}$, $\bar{C}$,... all represent fuzzy subsets of $\mathbb{R}^n$, n=1,2. Any fuzzy subset is defined by its membership function. If $\bar{A}$ is a fuzzy subset of $\mathbb{R}$, we write its membership function as $\mu(x|\bar{A})$, $x \in \mathbb{R}$, with $\mu(x|\bar{A})$ in $[0,1]$ for all $x \in \mathbb{R}$. If $\bar{P}$ is a fuzzy subset of $\mathbb{R}^2$ we write $\mu((x,y)|\bar{P})$ for its membership function with (x,y) in $\mathbb{R}^2$. Unless and otherwise α-cut of any fuzzy subset

*Author to whom correspondence should be addressed.

$\bar{A}$ of $\mathbb{R}$, written $\bar{A}(\alpha)$ and $I = [0, 1]$.

The arrangement of rest of the paper is given as follows. In section , a brief preliminary of relevant definitions along with some theorems are presented. Section contains a new approach of fuzzy point definitions, their illustrations with examples and some results. Finally, Section 1 draws conclusion.

Preliminaries

Definition 0.1. [19] Let X be a topological space.

(1) A path in X from x_0 to x_1 is a continuous map $f : I \to X$ such that $f(0) = x_0$ to $f(1) = x_1$.
(2) A topological space X is said to be path connected if for every pair of points of X there exist a path in X.

Theorem 0.2. *[13] A subset of Euclidean n-space is compact if and only if it is closed and bounded.*

Definition 0.3. [19] Let $f, g : X \to Y$ be continuous maps. A homotopy between f and g is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all x in X.
Notation: If f is homotopy to g, we write $f \simeq g$.

Definition 0.4. [19] Let X and Y be topological spaces and let $y_0 \in Y$. The constant map at $y_0 \in Y$ is the map $C : X \to Y$ with $C(x) = y_0$ for all x in X.
A continuous map $f : X \to Y$ is nullhomotopic if there is a constant map $C : X \to Y$ such that $f \simeq C$.

Definition 0.5. [19] A topological space X is contractible if id_X is nullhomotopic.

Definition 0.6. [19] Let f be a path in X from x_0 to x_1 and g be a path in X from x_1 to x_2. We define the composition $f * g$ of f and g to be the path given by the equations

$$(f * g)(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}], \\ g(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Definition 0.7. [19] Let X be a topological space and x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop base at x_0. The set of path homotopy classes of loops based at x_0, with the operation * is called the fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.

Theorem 0.8. *[14] Every convex set is connected, but the converse is not true.*

Definition 0.9. [19] A topological space X is said to be simply connected if it is a path connected space and $\pi_1(X, x_0)$ is trivial (one element) group for some $x_0 \in X$ and hence for every $x_0 \in X$.

Theorem 0.10. *[14] A convex set is simply connected but not vice versa.*

Definition 0.11. [19] A subset A of $\mathbb{R}^n$ is star convex if there is $a_0 \in A$ such that the line segment joining a_0 to any point of A lies in A.

Theorem 0.12. *[19] If A is star convex in $\mathbb{R}^n$ then A is contractible.*

Theorem 0.13. *[19] If a topological space is contractible then it is simply connected.*

Theorem 0.14. *[19] If a topological space is star convex then it is simply connected.*

Theorem 0.15. *[19] A topological space X is simply connected if and only if all paths in X with fixed end points are homotopic.*

Definition 0.16. [12] Let X be a non-empty set which should be evaluated with regard to a fuzzy statement. Then the set of order pairs $\bar{A} = \{(x, \mu(x|\bar{A})) : x \in X\}$ is called a fuzzy subset of X.

Definition 0.17. [7] A fuzzy subset $\bar{A}$ of $\mathbb{R}$ is called a real fuzzy number if its membership function has the following properties:

(1) $\mu(x|\bar{A})$ is upper semi-continuous;
(2) $\mu(x|\bar{A}) = 0$ outside some interval $[c, d]$ and
(3) there are real numbers a and b so that $c \le a \le b \le d$ and $\mu(x|\bar{A})$ is increasing on $[c, a]$, $\mu(x|\bar{A})$ is decreasing on $[b, d]$ and $\mu(x|\bar{A}) = 1$ for each x in $[a, b]$.

Remark 1. [3] If $\bar{A}$ be a fuzzy number then $\bar{A}(\alpha)$ is a bounded closed interval for any $\alpha \in [0, 1]$.

Definition 0.18. [12] For a fuzzy subset $\bar{A}$ of $\mathbb{R}^n, n = 1, 2$, its α-cut is defined by:

$$\bar{A}(\alpha) = \begin{cases} \{x : \mu(x|\bar{A}) \ge \alpha\}, & if\ 0 < \alpha \le 1 \\ Closure\{x : \mu(x|\bar{A}) > 0\} & if\ \alpha = 0. \end{cases}$$

Remark 2. [2] If α-cuts of a fuzzy subset are closed sets, then their membership functions are upper semi-continuous.

Definition 0.19. [3] A fuzzy point at (a, b) in $\mathbb{R}^2$, written as $\bar{P}(a, b)$, is defined by its membership function which satisfies the following conditions:

(1) $\mu((x, y)|\bar{P}(a, b))$ is upper semi-continuous;
(2) $\mu((x, y)|\bar{P}(a, b)) = 1$ if and only if $(x, y) = (a, b)$ and
(3) $\bar{P}(a, b)(\alpha)$ is a compact and convex subset of $\mathbb{R}^2$ for each $\alpha \in [0, 1]$.

Definition 0.20. [11] Let $\bar{C}_2$ be a fuzzy circle that passes through three fuzzy points $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$. The center of $\bar{C}_2$, $\tilde{C}$, can be defined by its membership function as $\mu(c|\tilde{C}) = sup\{\alpha$: where c is the center of a circle that passes through the same three points on $\bar{P}_1, \bar{P}_2$ and $\bar{P}_3$ with membership value $\alpha\}$.

Theorem 0.21. *[11] Let C be the center of a fuzzy circle $\bar{C}_2$ that passes through three fuzzy points $\bar{P}_1, \bar{P}_2$ and $\bar{P}_3$. Then for each $\alpha \in [0, 1]$,*

(1) $\tilde{C}(\alpha) = \{c$: where c is the center of a circle that passes through the same three points in $\bar{P}_1(\alpha), \bar{P}_2(\alpha)$ and $\bar{P}_3(\alpha)\}$,

(2) if no pair of the same points in $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are collinear, then $\tilde{C}(\alpha)$ is a compact and connected set.

Remark 3. [11]

i. For a fuzzy circle, its α-cuts must be closed, connected, and arcwise connected. However, α-cuts might not always be convex.
ii. The center of a fuzzy circle that passes through three fuzzy points might not be a fuzzy point. However, the fuzzy center satisfies all of the properties of a fuzzy point, except the convexity for α-cuts.
iii. The α-cuts of the center of a fuzzy circle that passes through three fuzzy points is star convex.

Fuzzy Points

Definition 0.22. A fuzzy point at (a, b) in $\mathbb{R}^2$, written as $\bar{P}(a, b)$, is defined by its membership function which satisfies the following conditions:

(1) $\mu((x, y)|\bar{P}(a, b)) = 1$ if and only if $(x, y) = (a, b)$ and
(2) $\bar{P}(a, b)(\alpha)$ is a compact, connected and simply connected subset of $\mathbb{R}^2$, for each α in $[0, 1]$.

Definition 0.23. Let d denotes the usual Euclidean distance metric on $\mathbb{R}^2$. We now define the fuzzy distance between fuzzy points $\bar{P}_1$ and $\bar{P}_2$. Let $\Omega(\alpha) = \{d(u, v) : u \in \bar{P}_1(\alpha), v \in \bar{P}_2(\alpha)\}$. The fuzzy subset $\bar{D}(\bar{P}_1, \bar{P}_2)$ of $\mathbb{R}$ is defined by $\bar{D}(\bar{P}_1, \bar{P}_2)(r) = \vee\{\alpha : r \in \Omega(\alpha)\}, \forall r \in \mathbb{R}$.

Theorem 0.24. *Let $\bar{P}_1$ and $\bar{P}_2$ be fuzzy points. Then $\bar{D}(\bar{P}_1, \bar{P}_2)(\alpha) = \Omega(\alpha)$ for all α in [0,1].*

Proof. First we show that $\bar{D}(\bar{P}_1, \bar{P}_2)(\alpha) = \Omega(\alpha)$, $0 < \alpha \leq 1$.

i. Let $r \in \Omega(\alpha) = \{r \in R | u \in \bar{P}_1(\alpha), v \in \bar{P}_2(\alpha)$ such that $r = d(u, v)\}$.
From the definition of D, we have $\bar{D}(\bar{P}_1, \bar{P}_2)(r) = \vee\{\theta : r \in \Omega(\theta)\} \geq \alpha$.
Thus $r \in \bar{D}(\bar{P}_1, \bar{P}_2)(\alpha)$. Hence, $\Omega(\alpha)$ is a subset of $\bar{D}(\bar{P}_1, \bar{P}_2)(\alpha)$.
ii. Let $r \in \bar{D}(\bar{P}_1, \bar{P}_2)(\alpha)$. Then $\bar{D}(\bar{P}_1, \bar{P}_2)(r) \geq \alpha$. Put $\bar{D}(\bar{P}_1, \bar{P}_2)(r) = \beta$. Hence we have two cases either $\beta > \alpha$ or $\beta = \alpha$.

Case I: Suppose $\beta > \alpha$.
Then there is $\delta \in [0, 1]$ with $\alpha < \delta \leq \beta$.
Let $t \in \Omega(\delta)$.
Then $t = d(u, v)$ for some $u \in \bar{P}_1(\delta), v \in \bar{P}_2(\delta)$.
It follows that $t = d(u, v)$ for some $u \in \bar{P}_1(\alpha), v \in \bar{P}_2(\alpha)$.
Hence $t \in \Omega(\alpha)$. Therefore $\Omega(\delta)$ is a subset of $\Omega(\alpha)$.
Hence $\bar{D}(\bar{P}_1, \bar{P}_2)(\alpha)$ is a subset of $\Omega(\alpha)$.

Case II: Assume that $\beta = \alpha$
Let $K = \{\delta : r \in \Omega(\delta)\}$. Then $\sup K = \alpha = \beta = D(r)$. Hence there is a sequence in K, say S_n which converges to α. Given $\epsilon > 0$ there is a positive integer N such that $\alpha - \epsilon < S_n$ for all $n \geq N$. Now if $r \in \Omega(S_n)$ for all n then $r \in \Omega(\alpha - \epsilon)$ for $\epsilon > 0$. Thus $r = d(u, v)$ for some $u \in \bar{P}_1(\alpha - \epsilon), v \in \bar{P}_2(\alpha - \epsilon)$.This implies that $\bar{P}_1(u) \geq \alpha - \epsilon$ and $\bar{P}_2(v) \geq \alpha - \epsilon$. Since ϵ was arbitrary, $\bar{P}_1(u) \geq \alpha$ and $\bar{P}_2(v) \geq \alpha$. Thus $r \in \Omega(\alpha)$. Therefore $\bar{D}(\bar{P}_1, \bar{P}_2)(\alpha)$ is a subset of $\Omega(\alpha)$.
Hence $\bar{D}(\bar{P}_1, \bar{P}_2)(\alpha) = \Omega(\alpha)$ for all α in (0,1]. But $\bar{D}(0) = cl(\bigcup_{0<\alpha\leq 1} D(\alpha)) = cl(\bigcup_{0<\alpha\leq 1} \Omega(\alpha)) = \Omega(0)$. So $\bar{D}(0) = \Omega(0)$. Therefore $\bar{D}(\alpha) = \Omega(\alpha)$ for all α in $[0, 1]$.

Corollary 0.25. *$\bar{D}(\bar{P}_1, \bar{P}_2)$ is a fuzzy number.*

Proof. i. Since α-cuts of a fuzzy point are compact subsets of $\mathbb{R}^2$, $\Omega(\alpha)$ is a closed and bounded interval for all α by theorem 0.2. Let $\Omega(\alpha) = [s(\alpha), t(\alpha)]$ for all $\alpha \in [0,1]$. By remark 2 its membership function is upper semi-continuous.

ii. Let $\Omega(0) = [c, d]$. Then $\mu(d/\bar{D}) = 0$ outside $[c, d]$.

iii. Let $\Omega(1) = a$, where $a = d((a_1, b_1), (a_2, b_2))$. We need to show that $\bar{D}$ is increasing from c to a and decreasing from a to d. Since $\bar{D}(\alpha) = \Omega(\alpha)$, $\bar{D}(\alpha)$ is a closed interval. Hence we can write $\bar{D}(\alpha) = \Omega(\alpha) = [l(\alpha), r(\alpha)]$. Let $\alpha_1 \leq \alpha_2$. Then we need to show that $\Omega(\alpha_2) \subseteq \Omega(\alpha_1)$. Let $r \in \Omega(\alpha_2) = \{d(u,v) : u \in \bar{P}_1(\alpha_2), v \in \bar{P}_2(\alpha_2)\}$. So it follows that $r \in \Omega(\alpha_1) = \{d(u,v) : u \in \bar{P}_1(\alpha_1), v \in \bar{P}_2(\alpha_1)\}$, this is because $\bar{P}_1(\alpha_2) \subseteq \bar{P}_2(\alpha_1), i = 1,2$. Hence $\bar{D}(\alpha_2) \subseteq \bar{D}(\alpha_1)$. Now since $\bar{D}(\alpha) = [l(\alpha), r(\alpha)]$ for all α, it follows that $[l(\alpha_2), r(\alpha_2)] \subseteq [l(\alpha_1), r(\alpha_1)]$. This implies $l(\alpha_1) \leq l(\alpha_2)$ and $r(\alpha_2) \leq r(\alpha_1)$. So l is increasing and r is decreasing. But $\Omega(1) = [a, a] = [l(1), r(1)]$. Hence l is increasing in $[c, a]$ and r is decreasing in $[a, d]$. That is $l(\alpha)$ is increasing from c to a and $r(\alpha)$ decreasing from a to d. So we obtain $\mu(d/\bar{D})$ is increasing on $[c, a]$ and decreasing on $[a, d]$ with $\mu(d/\bar{D}) = 1$ at $d = a$. This concludes our argument that $\bar{D}$ is a fuzzy number.

Example 0.26. Let $\bar{P}_1$ be fuzzy point at (1,0) with base $(x-1)^2+y^2 = \frac{1}{4}$ and let $\bar{P}_2$ be a fuzzy point at (3,0) with base $(x-3)^2+y^2 = \frac{1}{9}$. Then D($\bar{P}_1$,$\bar{P}_2$)$(r) = \frac{6r-7}{5}$ for $r \in [\frac{7}{6}, 2]$.

Solution: Let $\bar{P}_1$ be a fuzzy point at (1,0) with base $(x-1)^2 + y^2 = \frac{1}{4}$ and let $\bar{P}_2$ be a fuzzy point at (3,0) with base $(x-3)^2 + y^2 = \frac{1}{9}$. See the figure below.

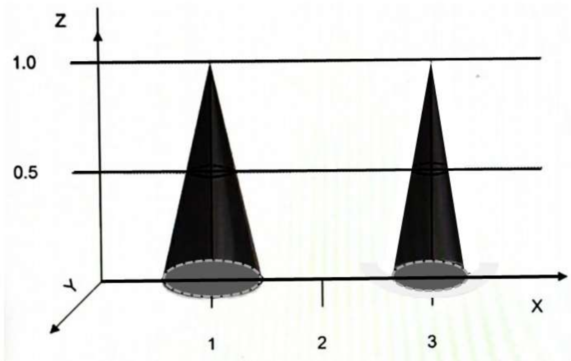

FIGURE 1. Fuzzy Points $\bar{P}_1$ and $\bar{P}_2$

The equation of the right circular cone that defines $\bar{P}_1$ is given by $(x-1)^2 + y^2 = (\frac{1}{2} - \frac{z}{2})^2$ and the equation of the right circular cone that defines $\bar{P}_2$ is given by $(x-3)^2 + y^2 = (\frac{1}{3} - \frac{z}{3})^2$. Consider $\bar{P}_1 : (x-1)^2 + y^2 = (\frac{1}{2} - \frac{z}{2})^2$. This implies $z = 1 - 2\sqrt{(x-1)^2 + y^2}$.
Therefore,

$$\bar{P}_1(x,y) = \begin{cases} 1 - 2\sqrt{(x-1)^2 + y^2}, & if\ (x-1)^2 + y^2 \leq \frac{1}{4} \\ 0, & Otherwise. \end{cases}$$

So $\bar{P}_1(\alpha) = \{(x,y) : \bar{P}_1(x,y) \geq \alpha\}$
$=\{(x,y) : 1 - 2\sqrt{(x-1)^2 + y^2} \geq \alpha\}$
$=\{(x,y) : 1 - \alpha \geq 2\sqrt{(x-1)^2 + y^2}\}$
$= \{(x,y) : (x-1)^2 + y^2 \leq \frac{(1-\alpha)^2}{4}\}$
That is; $\bar{P}_1(\alpha) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq \frac{(1-\alpha)^2}{4}\}$ which is a disc center at(1,0) and radius $r = \frac{1}{2} - \frac{\alpha}{2}$ and similarly
$\bar{P}_2(\alpha) = \{(x,y) \in \mathbb{R}^2 : (x-3)^2 + y^2 \leq \frac{(1-\alpha)^2}{9}\}$ which is a disc center at(3,0) and radius $r = \frac{1}{3} - \frac{\alpha}{3}$. So $D(\bar{P}_1, \bar{P}_2)(r) = \vee\{\alpha : r \in \Omega(\alpha)\}$ and $\Omega(\alpha) = \{d(u,v) : u \in \bar{P}_1(\alpha), v \in \bar{P}_2(\alpha)\}.= [d_{min}, d_{max}]$, where d_{min} is the minimum distance between $\bar{P}_1(\alpha)$ and $\bar{P}_2(\alpha)$ and d_{max} is the maximum distance between $\bar{P}_1(\alpha)$ and $\bar{P}_2(\alpha)$. From geometry it is known that the minimum (maximum) distance between points on two discs lies between points on the line connecting the center of the two discs. Thus $d_{min} = 3 - (\frac{1}{3} - \frac{\alpha}{3}) - (1 + (\frac{1}{2} - \frac{\alpha}{2}) = \frac{7+5\alpha}{6}$ and $d_{max} = 3 + (\frac{1}{3} - \frac{\alpha}{3}) - (1 - (\frac{1}{2} - \frac{\alpha}{2}) = \frac{17-5\alpha}{6}$. Hence

$\Omega(\alpha) = [\frac{7+5\alpha}{6}, \frac{17-5\alpha}{6}]$. Thus, $D(\bar{P}_1, \bar{P}_2)(r) = \vee\{\alpha : r \in [\frac{7+5\alpha}{6}, \frac{17-5\alpha}{6}]\}$. We know that for $\alpha < \beta$, we have $[\frac{7+5\alpha}{6}, \frac{17-5\alpha}{6}] \leq [\frac{7+5\beta}{6}, \frac{17-5\beta}{6}]$. So α is suppremum when $\frac{7+5\alpha}{6} = r$, provided that $r \in [\frac{7}{6}, \frac{12}{6}]$. This implies $\alpha = \frac{6r-7}{5}$. Therefore,

$$D(\bar{P}_1, \bar{P}_2)(r) = \begin{cases} \frac{6r-7}{5}, & if r \in [\frac{7}{6}, \frac{12}{6}] \\ 0, & Otherwise. \end{cases}$$

Definition 0.27. Let $\bar{P}_1$ and $\bar{P}_2$ be fuzzy points in the plane. Define

$$\Omega_3(\alpha) = \Big\{(x, y) : \frac{y - v_1}{x - u_1} = \frac{v_2 - v_1}{u_2 - u_1}, (u_1, v_1) \in \bar{p}_1(\alpha), (u_2, v_2) \in \bar{p}_2(\alpha)\Big\},$$

for $0 \leq \alpha \leq 1$.

Then we let $\bar{L}_3$ denote the fuzzy subset of $\mathbb{R}^2$ defined by for all $(x, y) \in \mathbb{R}^2$.
$\bar{L}_3(x, y) = \vee\{\alpha \in [0, 1] | (x, y) \in \Omega_3(\alpha)\}$.

Theorem 0.28. *$\bar{L}_3(\alpha) = \Omega_3(\alpha)$, $0 \leq \alpha \leq 1$ for $\alpha \in [0, 1]$.*

Proof. The proof is similar to that of theorem 0.24.

Example 0.29. Let $\bar{P}_1$ be a fuzzy point at $(1, 0)$ with base $(x-1)^2+y^2 = \frac{1}{9}$ and let $\bar{P}_2$ be a fuzzy point at $(3, 0)$ with base $(x-3)^2+y^2 = \frac{1}{9}$. Find $\bar{L}_3(0.5)$.

Solution: Let $\bar{P}_1$ be a fuzzy point at $(1, 0)$ with base $(x-1)^2+y^2 = \frac{1}{9}$ and let $\bar{P}_2$ be a fuzzy point at $(3, 0)$ with base $(x-3)^2+y^2 = \frac{1}{9}$. Then the equation of the right circular cone that define $\bar{P}_1$ and $\bar{P}_2$ respectively are :

$$\bar{P}_1(x, y) = \begin{cases} 1 - 3\sqrt{(x-1)^2 + y^2}, & if (x-1)^2 + y^2 \leq (\frac{1}{3})^2 \\ 0, & Otherwise \end{cases}$$

$$\bar{P}_2(x, y) = \begin{cases} 1 - 3\sqrt{(x-3)^2 + y^2}, & if (x-3)^2 + y^2 \leq (\frac{1}{3})^2 \\ 0, & Otherwise. \end{cases}$$

Hence,
$\bar{P}_1(\alpha) = \{(x, y) : 1 - 3\sqrt{(x-1)^2 + y^2} \geq \alpha\}$ and $= \{(x, y) : (x-1)^2 + y^2 \leq (\frac{1}{3} - \frac{\alpha}{3})^2\}$ which is a disc centered at $(1, 0)$ and radius $(\frac{1}{3} - \frac{\alpha}{3})$ and
$\bar{P}_2(\alpha) = \{(x, y) : 1 - 3\sqrt{(x-3)^2 + y^2} \geq \alpha\}$.
$= \{(x, y) : (x-3)^2 + y^2 \leq (\frac{1}{3} - \frac{\alpha}{3})^2\}$ which is a disc centered at $(3, 0)$ and radius $(\frac{1}{3} - \frac{\alpha}{3})$.

We know that

$$\bar{L}_3(\alpha) = \Omega_3(\alpha) = \Big\{(x, y) : \frac{y - v_1}{x - u_1} = \frac{v_2 - v_1}{u_2 - u_1}, (u_1, v_1) \in \bar{p}_1(\alpha), (u_2, v_2) \in \bar{p}_2(\alpha)\Big\}$$

$$\Rightarrow \bar{L}_3(0.5) = \Big\{(x, y) : \frac{y - v_1}{x - u_1} = \frac{v_2 - v_1}{u_2 - u_1}, (u_1, v_1) \in p_1(0.5), (u_2, v_2) \in p_2(0.5)\Big\}$$

But $\bar{P}_1(0.5) = \{(x, y) : (x-1)^2+y^2 \leq (\frac{1}{3} - \frac{0.5}{3})^2\} = B_1$
and $\bar{P}_2(0.5) = \{(x, y) : (x-3)^2+y^2 \leq (\frac{1}{3} - \frac{0.5}{3})^2\} = B_2$
Thus

$$\bar{L}_3(0.5) = \Big\{(x, y) : \frac{y - v_1}{x - u_1} = \frac{v_2 - v_1}{u_2 - u_1}, (u_1, v_1) \in B_1, (u_2, v_2) \in B_2\Big\}$$

which is the set of all lines through any point $(u_1, v_1) \in B_1, (u_2, v_2) \in B_2$. It is thin between B_1 and B_2 but gets wider and wider as we move along $\bar{L}_3(1)$ in the x-axis.

Corollary 0.30. *For $\alpha \in [0, 1]$, the α-cuts of L_3 are closed and connected.*

Proof.

i. We need to show $L_3(\alpha)$ closed. Let $\alpha \in [0, 1]$. We know that by theorem 0.28 $\bar{L}_3(\alpha) = \Omega_3(\alpha)$. So

$$\Omega_3(\alpha) = \Big\{(x, y) : \frac{y - v_1}{x - u_1} = \frac{v_2 - v_1}{u_2 - u_1}, (u_1, v_1) \in \bar{p}_1(\alpha), (u_2, v_2) \in \bar{p}_2(\alpha)\Big\},$$

for $0 \leq \alpha \leq 1$ Since $\bar{p}_1(\alpha)$ and $\bar{p}_2(\alpha)$ are fuzzy points, then the α-cuts are compact. By theorem 0.2, the α-cuts are closed and bounded. So it follows that $\Omega_3(\alpha)$ is bounded and closed subset of $\mathbb{R}^2$ for all α. Hence $L_3(\alpha)$ is closed.

ii. We need to show that $L_3(\alpha)$ is connected. Suppose not. Let $L_3(\alpha)$ is not connected. This implies there exist a pair of disjoint non empty open sets of $L_3(\alpha)$ whose union is $L_3(\alpha)$ .Since U and V are non empty open set and union of open sets is open , $L_3(\alpha) = U \cup V$ is open. But this contradicts the fact that $L_3(\alpha)$ is closed. Hence $L_3(\alpha)$ is connected.

Theorem 0.31. *Let C be the center of C_2 that passes through three fuzzy points $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$.*

If no pair of the same points in $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ are collinear then C is a fuzzy point.

Proof. We need to show X=$C(\alpha)$ is simply connected.
Let $f, g : I \to Y$ such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$.
Consider the map $H : I \times I \to X$ defined as $H(s,t) = (1-t)f(s) + tg(s)\ \forall s,t \in I$.
So $H(s,0) = f(s)$ and $H(s,1) = g(s)$ and $H(0,t) = x_0$, $H(1,t) = x_1$. Therefore H is a homotopy between f and g. Hence by theorem 0.15, $C(\alpha)$ is simply connected. Thus C is a fuzzy point.

Theorem 0.32. *The α-cut of the center of the fuzzy circle that passes through three fuzzy points is contractible.*

Proof. The proof directly follows from theorem 0.12 and remark 3.

Theorem 0.33. *The α-cut of the center of the fuzzy circle that passes though three fuzzy points is simply connected.*

Proof. The proof directly follows from theorem 0.14 and remark 3.

1. Conclusion

In this paper, we proposed a new definition of fuzzy points. We applied this definition to study fuzzy distance, fuzzy lines, fuzzy circles and their properties. In the preliminaries from theorem 0.8 and theorem 0.10 it follows that every convex set is connected and simply connected so the statement using convex sets is stronger. Moreover, from remark 2 and Theorem 0.2 if the α-cuts are compact, we have noticed that the mapping is upper semi-continuous. Therefore, our definition of fuzzy point reduces the condition (1) of Buckley and Islami definition of fuzzy point. Finally, in Ghosh and Chakraborty paper the center of a fuzzy circle is not in general fuzzy point but we have also shown that the center of a fuzzy circle is fuzzy point. Similarly, one can ask the same question for other conic sections.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] Bogomolny, A. On the perimeter and area of fuzzy sets. *Fuzzy Sets and Systems 23*, 2 (1987), 257–269.

[2] Buckley, J., and Qu, Y. Solving linear and quadratic fuzzy equations. *Fuzzy sets and systems 38*, 1 (1990), 43–59.

[3] Buckley, J. J., and Eslami, E. Fuzzy plane geometry I: Points and lines. *Fuzzy sets and systems 86*, 2 (1997), 179–187.

[4] Buckley, J. J., and Eslami, E. Fuzzy plane geometry II: Circles and polygons. *Fuzzy Sets and systems 87*, 1 (1997), 79–85.

[5] Chakraborty, D., and Das, S. Fuzzy geometry: Perpendicular to fuzzy line segment. *Information Sciences 468* (2018), 213–225.

[6] Chakraborty, D., and Ghosh, D. Analytical fuzzy plane geometry II. *Fuzzy Sets and Systems 243* (2014), 84–109.

[7] Clark, T. D., Larson, J. M., Mordeson, J. N., Potter, J. D., and Wierman, M. J. *Applying fuzzy mathematics to formal models in comparative politics.* Springer, 2008.

[8] Das, S., and Chakraborty, D. Conceptualizing fuzzy line as a collection of fuzzy points. *Information Sciences 598* (2022), 86–100.

[9] Ghosh, D., and Chakraborty, D. Analytical fuzzy plane geometry I. *Fuzzy Sets and Systems 209* (2012), 66–83.

[10] Ghosh, D., and Chakraborty, D. On general form of fuzzy lines and its application in fuzzy line fitting. *Journal of Intelligent & Fuzzy Systems 29*, 2 (2015), 659–671.

[11] Ghosh, D., and Chakraborty, D. Analytical fuzzy plane geometry III. *Fuzzy Sets and Systems 283* (2016), 83–107.

[12] Ghosh, D., and Chakraborty, D. *An Introduction to Analytical Fuzzy Plane Geometry.* Springer, 2019.

[13] Kelley, J. L. *General topology.* Courier Dover Publications, 2017.

[14] Malla, K. R. A study of the relationship between convex sets and connected sets. *Journal of Advanced Academic Research 6*, 1 (2019), 18–28.

[15] Özekinci, S., Aycan, C., et al. Constructing the fuzzy hyperbola and its applications in analytical fuzzy plane geometry. *Journal of Mathematics 2022* (2022).

[16] Özekinci, S., and YORMAZ, C. Constructing the ellipse and its applications in analytical fuzzy plane geometry. *Ikonion Journal of Mathematics 5*, 1 (2023), 20–38.

[17] Rosenfeld, A. The diameter of a fuzzy set. *Fuzzy sets and Systems 13*, 3 (1984), 241–246.

[18] Rosenfeld, A., and Haber, S. The perimeter of a fuzzy set. *Pattern recognition 18*, 2 (1985),

125–130.

[19] ROTMAN, J. *An introduction to algebraic topology*, vol. 119. Springer Science, 2013.

[20] YUAN, X., AND SHEN, Z. Notes on "fuzzy plane geometry I, II". *Fuzzy sets and systems 121*, 3 (2001), 545–547.

[21] ZADEH, L. A., KLIR, G. J., AND YUAN, B. *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, vol. 6. World Scientific, 1996.