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# ALMOST ANALYTIC EXTENSIONS AND ULTRADIFFERENTIABLE FUNCTIONS

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Abstract. The notion of almost analytic extension has found numerous applications in various fields. This paper provides a comprehensive characterization of ultradifferentiable functions by examining the existence of almost analytic extensions.

**Key words/phrases:** Ultradifferentiable classes, almost analytic extensions

## 1. Introduction

The construction of almost analytic extensions has been explored by varios authors. In chapter 8 of Dimassi and Sjöstrand (1999), two different almost analytic constructions were presented for $f \in C_0^\infty$. The first one is following Hörmander's approach, based on Borel's construction.

$$\tilde{f}(x+iy) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!}(iy)^k \chi(\lambda_k y),$$

where $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 near 0 and $\lambda_k$ tending to $\infty$ sufficiently fast. The other is the construction introduced by Mather, Jensen and Nakamura based on Fourier inversion formula: If $\chi(x) \in C_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of the support of $f.\chi$ as above.

$$\tilde{f}(x+iy) = \int_{\mathbb{R}} e^{(x+iy)\xi}\chi(y\xi)\hat{f}(\xi)d\xi,$$

where $\hat{f}$ is the Fourier transform of $f$.
These constructions have garnered significant attention due to their potential applications in diverse fields. By extending analytic functions to almost analytic ones, these constructions offer a broader range of mathematical tools and techniques for analysis and problem-solving. In recent years, researchers have delved deeper into the intricacies of these constructions, seeking to refine and expand upon the existing methodologies. By investigating the properties and limitations of almost analytic extensions, they aim to uncover new insights and possibilities for their utilization in various domains.
The comprehensive almost analytic description of ultradifferentiable classes can be traced back to Petzsche and Vogt (2009). In Berhanu and Hailu (2017), the authors discussed the local and microlocal characterization of Gevrey functions as boundary values of almost analytic functions. A natural extension of the Gevrey classes can be achieved by considering a sequence of real numbers $M = (M_j)_{j\in\mathbb{N}}$ that satisfy certain properties. The objective of this study is to characterize ultradifferentiable functions based on the existence of almost analytic extentions. To accomplish this, we utilize the higher dimensional version of the inhomogeneous Cauchy integral formula to construct almost analytic extension of ultradifferentiable functions.
The structure of this paper is as follows: Section 1 deals with introduction. Section 2 presents the definition of ultradifferentiable functions and explores some of their properties. In Section 3, we establish the characterization of ultradifferentiable functions in terms of the existence of almost analytic extensions.

## 2. Ultradifferentiable functions and some properties

We will start by recalling a sequence of numbers that possess certain properties as mentioned in Jemal and Tadesse (2020). Consider a sequence $M = (M_j)$ consisting of positive real numbers that satisfy the following conditions. (1) Initial Conditions:

$$M_0 = M_1 = 1. \tag{2.1}$$

(2) Non-quasianalyticity:

$$\sum_{j=1}^{\infty} \frac{M_j}{M_{j+1}} < \infty. \tag{2.2}$$

(3) Stability under differential operators: There exists a constant $A, H > 0$, independent of $j, k$, such that for all $j, k \in \mathbb{N}$

$$M_{j+k} \leq AH^{j+k} M_j M_k. \tag{2.3}$$

(4) Invariance under composition: For all $j, k \in \mathbb{N}$ with $0 \leq j \leq k$, we have

$$\binom{k}{j} M_{k-j} M_j \leq M_k. \tag{2.4}$$

(5) Logarithmic convexity: For all $j \in \mathbb{N}$

$$M_j^2 \leq M_{j-1} M_{j+1}, \tag{2.5}$$

and this in turn implies that for all $j, k \in \mathbb{N}$

$$M_j M_k \leq M_{j+k}. \tag{2.6}$$

(6) Invariance under division: The sequence $Q_0 = 1$ and $Q_j = \left(\frac{M_j}{j!}\right)^{\frac{1}{j}}$ for $j \geq 1$ is increasing, that is, for all $j < k$

$$Q_j \leq Q_k. \tag{2.7}$$

For the reader who are interested in delving deeper into to this sequence and its properties, as well as the subsequent function spaces, we highly recommend referring to the papers Adwan and Hoepfner, (2010), and Adwan and Hoepfner, (2015), Komatsu, (1973) and references herein.
Here we remark that:

*Remark* 2.1. i) From (**??**), (**??**), and (**??**), it follows that
(7) Faá di Bruno: For all $j, k \in \mathbb{N}$, if $n = jk$, there is a constant $C > 1$, independent of $n$, so that

$$M_j^k \leq C^n M_{n-k}. \tag{2.8}$$

This condition is utilized when there is a requirement to apply the multi-variable Faá di Bruno formula for calculating the derivatives of composition of function.
ii) If $M$ satisfies (**??**)and (**??**), then it satisfies the following: For all $j = 1, 2, \ldots$

$$M_j \geq j!. \tag{2.9}$$

**Example 2.2.** Let $s > 1$ be a real number and choose

$$M_j = (j!)^s.$$

Then $M = (M_j)$ satisfies (**??**) to (**??**). If $M_j = j!$, then $M = (M_j)$ satisfies all conditions except (**??**).

**Definition 2.3.** Let $\Omega \subset \mathbb{R}^m$ be an open set and $(M_j)_{j\in\mathbb{N}}$ be a sequence of positive real numbers that is increasing and satisfies certain properties mentioned above. The ultradifferentiable (Denjoy-Carleman) spaces, denoted as $\mathcal{E}^M(\Omega)$, is defined as the set of all functions $f$ in $C^\infty(\Omega)$ that satisfies the following property: for each $K \subset\subset \Omega$ there exist a constant $C > 0$, depending on $K$ and $f$, such that

$$|\partial^\alpha f(x)| \leq C^{|\alpha|+1} M_{|\alpha|}, \ \ \forall \alpha \in \mathbb{N}_0^m, \ \forall x \in K. \tag{2.10}$$

**Example 2.4.** Let $s > 1$ be a real number and

$$M_j = (j!)^s.$$

Then $\mathcal{E}^M(\Omega) = G^s(\Omega)$ denotes the $s-$ Gevrey space. If $M_j = j!$, then $\mathcal{E}^M(\Omega) = C^\omega(\Omega)$ (the space of real analytic functions).

**Example 2.5.** (For more examples, see Rainer, (2009)) a) Let $q > 1$. Put $M_j = q^{j^2}$, $j \in \mathbb{N}$. The corresponding $\mathcal{E}^M$ functions are called $q-$Gevrey regular. Then $M = (M_j)$ is non-quasianalytic.
b) Let $\delta > 0$ and $M_j = \Big(\log(j + e)\Big)^{\delta j}$ for $j \in \mathbb{N}$. Then $M = (M_j)$ is quasianalytic for $0 < \delta \leq 1$ and non-quasianalytic for $\delta > 1$.

Note that if $M = (M_j)$ and $N = (N_j)$ satisfy $M_j \leq C^j N_j$, $\forall j$ and a constant $C$, then $\mathcal{E}^M(\Omega) \subset \mathcal{E}^N(\Omega)$. The converse is true as well by the logarithimic convexity assumption. In particular, if $f \in G^s(\Omega)$ and $s \leq t$, then $f \in G^t(\Omega)$. Thus $G^1 \subset G^s$, $\forall s \geq 1$.

## 3. Ultradifferentiable functions and almost analytic extensions

We begin the section by defining the $M-$ almost analytic extensions.

**Definition 3.1.** Let $\Omega \subset \mathbb{R}^m$ be an open set and $f = f(x) \in \mathcal{E}^M(\Omega)$. A function $F = F(x, y) \in \mathcal{E}^M(\Omega \times (-1, 1)^m)$ is said to be an $M-$ almost analytic extension of $f$ if the following holds:
i) $F(x, 0) = f(x)$ for all $x \in \Omega$; and
ii) for all $(x, y) \in \Omega \times (-1, 1)^m$ and for all $N = 1, 2, \ldots$ there exists a constant $C > 0$, independent of $N$, such that, for all $j = 1, \ldots, m$ it holds

$$\left|\frac{\partial F}{\partial \bar{z}_j}(z)\right| \leq \frac{C^{N+1}}{N!} M_N |y|^N$$

where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$.

We will use the following Faá di Bruno generalized formula to prove a lemma in this section and in the subsequent section.

**Theorem 3.2** ([**?**], Corollary 2.10)**.** *Let* $\alpha \in \mathbb{N}_0^m$ *and* $h(x_1, \ldots, x_d) = f(g(x_1, \ldots, x_d))$ *with* $g \in C^{\alpha}(U_{x_0})$ *and* $f \in C^{|\alpha|}(V_{y_0})$, *where* $y_0 = g(x_0)$, *and* $U_{x_0} \subset \mathbb{R}^d$ *and* $V_{y_0} \subset \mathbb{R}$ *open neighborhoods of* $x_0$ *and* $y_0$, *respectively. Then*

$$\partial^{\alpha} h = \sum_{r=1}^{|\alpha|} \partial^r f \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{|\alpha|} \frac{(\partial^{\alpha_j} g)^{k_j}}{k_j!(\alpha_j!)^{k_j}},$$

*where*

$$\begin{aligned} &p(\alpha, r) = \{(k_1, \ldots, k_{|\alpha|}; \alpha_1, \ldots, \alpha_{|\alpha|}) \\ &\textit{for some } 1 \leq s \leq |\alpha|, k_i = 0 \textit{ and } \alpha_i = 0 \\ &\textit{for } 1 \leq i \leq |\alpha| - s; k_i > 0 \textit{ for } |\alpha| - s + 1 \leq i \leq |\alpha|; \\ &\textit{and } 0 < \alpha_{|\alpha|-s+1} < \ldots < \alpha_{|\alpha|} \textit{ are such that} \\ &\qquad \sum_{i=1}^{|\alpha|} k_i = r, \ \sum_{i=1}^{|\alpha|} k_i \alpha_i = \alpha\}. \end{aligned}$$

In particular, we have (see [**?**], page 515) that there exist $C > 0$ such that

$$r! \sum_{p(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} = \binom{|\alpha| - 1}{r - 1} \leq C^{|\alpha|}. \qquad (3.1)$$

**Lemma 3.3.** *Let* $\alpha \in \mathbb{N}_0^m$, $x \in \mathbb{R}^m \setminus \{0\}$ *and* $h(x) = |x|^{-2m}$. *Then there exist* $C > 0$ *such that*

$$|\partial^{\alpha} h(x)| \leq C^{|\alpha|+1} (m + |\alpha| - 1)! |x|^{-2m-|\alpha|}.$$

*Proof:*

Let

$$f(t) = t^{-m}, \ g(x) = \sum_{j=1}^{m} x_j^2$$

so that $h(x) = f(g(x))$. We will use Faa di Bruno formula to compute this derivative, that is,

$$\partial^{\alpha} h(x) = \sum_{r=1}^{|\alpha|} f^{(r)}(g(x)) \sum_{P(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial^{\alpha_j} g)^{k_j}}{k_j!(\alpha_j!)^{k_j}},$$

where

$$\begin{aligned} &p(\alpha, r) = \{(k_1, \ldots, k_{|\alpha|}; \alpha_1, \ldots, \alpha_{|\alpha|}) \\ &\quad \text{for some } 1 \leq s \leq |\alpha|, k_j = 0 \text{ and } \alpha_j = 0 \\ &\quad \text{for } 1 \leq j \leq |\alpha| - s; k_j > 0 \\ &\quad \text{for } |\alpha| - s - 1 \leq j \leq |\alpha|; \text{ and} \\ &\quad 0 < \alpha_{|\alpha|-s+1} < \ldots < \alpha_{|\alpha|} \text{ are such that} \\ &\quad \sum_{i=1}^{|\alpha|} k_j = r, \ \sum_{j=1}^{|\alpha|} k_j \alpha_j = \alpha\}. \end{aligned}$$

Now $f'(t) = -mt^{-m-1}, f''(t) = -m(-m-1)t^{-m-2}$ and in general

$$\begin{aligned} f^{(r)}(t) &= (-1)^r r! \binom{m + r - 1}{r} t^{-m-r} \\ &= (-1)^r (m + r - 1)(m + r - 2) \ldots m t^{-m-r}. \end{aligned}$$

Let $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^m)$, $j = 1, 2, \ldots, |\alpha|$. Then $\partial^{\alpha_j} g(x) = \frac{\partial^{|\alpha_j|}}{\partial x_1^{\alpha_j^1} \ldots \partial x_m^{\alpha_j^m}} g(x) = 0$ except when $\alpha_j = e_j = (0, \ldots 1, 0 \ldots, 0)$ in that case it is $2x_j$ and when $\alpha_j = 2e_j$ the derivative is 2.
Thus when $\alpha_j = e_j$ from $\sum_{j=1}^{|\alpha|} k_j = r$ and $\sum_{j=1}^{|\alpha|} k_j \alpha_j = \alpha$ we have $|\alpha| = r, \alpha = (k_1, \ldots, k_{|\alpha|})$. When $\alpha_j = 2e_j$, we have $|\alpha| = 2r$ and $\alpha = 2(k_1, \ldots, k_{|\alpha|})$.

Thus, there are nonzero terms only when $r = |\alpha|$ and $2r = |\alpha|$. Hence, using the fact that $(\frac{|\alpha|}{2})!\binom{m+\frac{|\alpha|}{2}-1}{\frac{|\alpha|}{2}} \leq |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}$, we have

$$\begin{aligned}
|\partial^\alpha h(x)| &\leq |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-2|\alpha|}\left|\sum_{k_1+\ldots+k_{|\alpha|}=|\alpha|} k_1!\ldots k_{|\alpha|}!\prod_{j=1}^{|\alpha|}\frac{(2x_j)^{k_j}}{k_j!}\right| \\
&+ |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\sum_{2k_1+\ldots+2k_{|\alpha|}=|\alpha|}(2k_1)!\ldots(2k_{|\alpha|})!\prod_{j=1}^{|\alpha|}\frac{(2)^{k_j}}{k_j!(2!)^{k_j}} \\
&= |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-2|\alpha|}2^{|\alpha|}\left|x^{(k_1,\ldots,k_{|\alpha|})}\right|\sum_{k_1+\ldots+k_{|\alpha|}=|\alpha|}1 \\
&+ |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\sum_{2k_1+\ldots+2k_{|\alpha|}=|\alpha|}(2k_1)!\ldots(2k_{|\alpha|})!\prod_{j=1}^{|\alpha|}\frac{1}{k_j!} \\
&\leq |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\left(2^{|\alpha|}\sum_{k_1+\ldots+k_{|\alpha|}=|\alpha|}1+|\alpha|!\sum_{2k_1+\ldots+2k_{|\alpha|}=|\alpha|}\prod_{j=1}^{|\alpha|}\frac{1}{k_j!}\right) \\
&\leq |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\left(2^{|\alpha|}|\alpha|+C^{|\alpha|}\right) \text{ from (??)} \\
&\leq |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\left(2^{2|\alpha|}+C^{|\alpha|}\right) \\
&\leq C^{|\alpha|+1}|\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|} \\
&\leq C^{|\alpha|+1}(m+|\alpha|+1)!|x|^{-2m-|\alpha|}.
\end{aligned}$$

**Theorem 3.4.** *Let $\Omega \subset \mathbb{R}^m$ be an open set. $f \in \mathcal{E}^M(\Omega)$ if and only if there exist $F(x,y) \in \mathcal{E}^M(\Omega\times(-1,1)^m)$ such that*
*(1) $F(x,0)=f(x)$ on $\Omega$ and*
*(2) $\left|\frac{\partial F}{\partial \bar{z}_j}(z)\right| \leq \frac{C^{N+1}}{N!}M_N|y|^N$, $\forall j=1,2,\ldots,m$ on $\Omega\times(-1,1)^m$ for some constant $C>0$, where $z_j = x_j+iy_j$.*

*Proof:*
Suppose there exist $F=F(x,y)\in\mathcal{E}^M(\Omega\times(-1,1)^m))$ such that
(1) $F(x,0)=f(x)$ on $\Omega$ and
(2) $\left|\frac{\partial F}{\partial \bar{z}_j}(z)\right| \leq \frac{C^{N+1}}{N!}M_N|y|^N$, $\forall j=1,2,\ldots,m$ on $\Omega\times(-1,1)^m$ for some constant $C>0$. (Of course it suffice to assume that $F\in C^1(\Omega\times(-1,1)^m)$). We will show that $f\in\mathcal{E}^M(\Omega)$. It suffice to show that $f\in\mathcal{E}^M(B_r)$ for each sufficiently small ball in $\Omega$. Let $B_{2r}=\{x\in\Omega : |x|<2r\}$ such that $\overline{B_{2r}}\subset\Omega$. Let $F(x,y)$ be as given above on a neighborhood of the closure of $\Omega_r = B_{2r}\times B_r$. We may assume that $F(x,y)=0$ for $y\in B_r\setminus(-1,1)^m$.
Set $\omega(z)=dz_1\wedge\ldots\wedge dz_m$. We will identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$. For $k=1,\ldots,m$, let

$$\omega_k(\bar{z}) = (-1)^{k-1}d\bar{z}_1\wedge\ldots\wedge d\bar{z}_{k-1}\wedge d\hat{\bar{z}}_k\wedge d\bar{z}_{k+1}\wedge\ldots\wedge d\bar{z}_m,$$

where $d\bar{z}_k$ is removed. Let $\sigma_n$ denotes the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

Then for each $x\in B_r$, from the higher dimensional version of the inhomogeneous Cauchy Integral Formula, we have

$$\begin{aligned}
f(x) = F(x,0) = &\\
&\frac{2(2i)^{-m}}{\sigma_{2m}}\int_{\partial\Omega_r}F(w)\sum_{k=1}^m(\overline{w}_k-x_k)|w-x|^{-2m}\omega_k(\overline{w})\wedge\omega(w) \\
&-\frac{2(2i)^{-m}}{\sigma_{2m}}\int_{\Omega_r}\sum_{k=1}^m\frac{\partial F}{\partial\overline{w}_k}(w)(\overline{w}_k-x_k)|w-x|^{-2m}\omega(\overline{w})\wedge\omega(w) \\
&= f_1(x)+f_2(x).
\end{aligned}$$

Since $f_1(x)$ is real analytic on $B_r$, $f_1 \in \mathcal{E}^M(B_r)$. It remains to show that $f_2 \in \mathcal{E}^M(B_r)$. Let $\alpha = (\alpha_1, \dots, \alpha_m)$. Then

$$\partial^\alpha f_2(x) = -\frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^m \frac{\partial F}{\partial \overline{w}_k}(w) \partial_x^\alpha \left[ (\overline{w}_k - x_k)|w-x|^{-2m} \right] \omega(\overline{w}) \wedge \omega(w).$$

For $x \neq w$,

$$\begin{aligned}\partial^\alpha \left[ (\overline{w}_k - x_k)|w-x|^{-2m} \right] &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial_x^\beta (\overline{w}_k - x_k) \partial_x^{\alpha-\beta} [|w-x|^{-2m}] \\ &= (\overline{w}_k - x_k)\partial_x^\alpha [|w-x|^{-2m}] - \frac{\alpha!}{(\alpha - e_k)!} \partial_x^{\alpha - e_k} [|w-x|^{-2m}] \\ &= (\overline{w}_k - x_k)\partial_x^\alpha [|w-x|^{-2m}] - \alpha_k \partial_x^{\alpha - e_k} [|w-x|^{-2m}].\end{aligned}$$

Using Lemma **??**, we have

$$\begin{aligned}&|\partial^\alpha \left[ (\overline{w}_k - x_k)|w-x|^{-2m} \right]| \leq \\ &|w-x||\partial_x^\alpha [|w-x|^{-2m}]| + \alpha_k |\partial_x^{\alpha - e_k}[|w-x|^{-2m}]| \\ &\leq (m+|\alpha|-1)! \Big( C^{|\alpha|+1}|w-x||w-x|^{-2m-|\alpha|} + \alpha_k C^{|\alpha|-1}|w-x|^{-2m-|\alpha|+1} \Big) \\ &\leq C^{|\alpha|+1}(m+|\alpha|-1)!|w-x|^{-2m-|\alpha|+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\partial^\alpha f_2(x)| &= -\frac{2^{1-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^m \left| \frac{\partial F}{\partial \overline{w}_k}(w) \right| |\partial_x^\alpha \left[ (\overline{w}_k - x_k)|w-x|^{-2m} \right]| |\omega(\overline{w}) \wedge \omega(w)| \\ &\leq \frac{22^{-m}}{\sigma_{2m}} c^{|\alpha|+1}(m+|\alpha|-1)! C^{N+1} \frac{M_N}{N!} \int_{\Omega_r} \sum_{k=1}^m \frac{|\Im w|^N}{|w-x|^{2m+|\alpha|-1}} \omega(\overline{w}) \wedge \omega(w)| \\ &\leq \frac{22^{-m}}{\sigma_{2m}} c^{|\alpha|+1}(m+|\alpha|-1)! C^{N+1} \frac{M_N}{N!} \int_{\Omega_r} |\Im w|^{N-2m-|\alpha|+1} dv \\ &\leq \frac{22^{-m}}{\sigma_{2m}} c^{|\alpha|+1}(m+|\alpha|-1)! C^{|\alpha|+2m} \frac{M_{2m+|\alpha|-1}}{(2m+|\alpha|-1)!} \int_{\Omega_r} dv \text{ (let } N = 2m+|\alpha|-1) \\ &\leq \frac{22^{-m}}{\sigma_{2m}} c^{|\alpha|+1} C^{|\alpha|+2m} M_{2m+|\alpha|-1} \\ &\leq \frac{22^{-m}}{\sigma_{2m}} c^{|\alpha|+1} C^{|\alpha|+2m} A H^{2m+|\alpha|-1} M_{2m-1} M_{|\alpha|} \\ &\leq C'^{|\alpha|+1} M_{|\alpha|}.\end{aligned}$$

Therefore, $f_2 \in \mathcal{E}^M(B_r)$ and hence $f \in \mathcal{E}^M(B_r)$.

Conversely, Suppose $f \in \mathcal{E}^M(\Omega)$. Let $h \in D^M(\mathbb{R}^m)$ such that $h \equiv 1$ for $|y| \leq \frac{1}{2}$ and $h \equiv 0$ for $|y| \geq 1$. Let $\{\mu_k\}_{k=0}^\infty$ be increasing sequence of positive numbers to be chosen appropriately such that $\mu_k \to \infty$.
Define

$$F(x,y) = \sum_\gamma \frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|} y) \tag{3.2}$$

Clearly, $F(x,0) = f(x)$. Fix $y \neq 0$. Then since $\lim_{k\to\infty} \mu_k |y| = \infty$, there is $k_0 \geq 1$ such that

$$\mu_k |y| \geq 1, \ \forall k \geq k_0.$$

Then $h(\mu_{|\gamma|} y) = 0 \ \forall |\gamma| \geq k_0$. Hence,

$$F(x,y) = \sum_{|\gamma| \leq k_0} \frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|} y),$$

which is a finite sum. Therefore, $F$ is well-defined. We will show that $F$ is in $\mathcal{E}^M$ for $y$ in a neighborhood of 0.
For this, let $K \subset\subset \Omega$ and fix $\alpha, \beta \in \mathbb{N}_0^m$. Then for $x \in K$

$$\begin{aligned}
&\left|\partial_y^\beta \partial_x^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|}y)\right)\right| \\
&= \left|\frac{\partial_y^\alpha \partial_x^\gamma f(x)}{\gamma!}\sum_{\delta\le\beta}\binom{\beta}{\delta}\partial_y^\delta(y^\gamma)\partial_y^{\beta-\delta}(h(\mu_{|\gamma|}y))\right| \\
&= \left|\frac{\partial_x^{\gamma+\alpha} f(x)}{\gamma!}\sum_{\delta\le\gamma,\delta\le\beta}\binom{\beta}{\delta}\frac{\gamma!}{(\gamma-\delta)!}y^{\gamma-\delta}\mu_{|\gamma|}^{|\beta|-|\delta|}(\partial_y^{\beta-\delta}h)(\mu_{|\gamma|}y)\right| \\
&(\text{since } \partial_y^\delta(y^\gamma)=0 \text{ if } \delta_j>\gamma_j \text{ for some } j) \\
&\le |\partial_x^{\gamma+\alpha}f(x)|\sum_{\delta\le\gamma,\delta\le\beta}\binom{\beta}{\delta}\frac{1}{(\gamma-\delta)!}|y|^{|\gamma|-|\delta|}\mu_{|\gamma|}^{|\beta|-|\delta|}|(\partial_y^{\beta-\delta}h)(\mu_{|\gamma|}y)| \\
&\le |\partial_x^{\gamma+\alpha}f(x)|\sum_{\delta\le\gamma,\delta\le\beta}\binom{\beta}{\delta}|y|^{|\gamma|-|\delta|}\mu_{|\gamma|}^{|\beta|-|\delta|}|(\partial_y^{\beta-\delta}h)(\mu_{|\gamma|}y)| \\
&\le |\partial_x^{\gamma+\alpha}f(x)|\sum_{\delta\le\gamma,\delta\le\beta}\binom{\beta}{\delta}\frac{1}{\mu_{|\gamma|}^{|\gamma|-|\delta|}}\mu_{|\gamma|}^{|\beta|-|\delta|}|(\partial_y^{\beta-\delta}h)(\mu_{|\gamma|}y)| \\
&(\text{since } h(\mu_{|\gamma|}y)=0 \text{ for } |y|\ge\frac{1}{\mu_{|\gamma|}}) \\
&\le M_{|\alpha|+|\gamma|}C_1^{|\alpha|+|\gamma|+1}\sum_{\delta\le\beta}\binom{\beta}{\delta}\frac{1}{\mu_{|\gamma|}^{|\gamma|-|\beta|}}|(\partial_y^{\beta-\delta}h)(\mu_{|\gamma|}y)|
\end{aligned}$$

Set $C_\beta = \sup\limits_{\beta'\le\beta}\left\{\left(\partial_y^{\beta'}h\right)(y) : y\in\mathbb{R}^m\right\}$. Then $C_\beta \le C_2^{|\beta|+1}M_{|\beta|}$.
Thus

$$\begin{aligned}
&\left|\partial_y^\beta \partial_x^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|}y)\right)\right| \\
&\le M_{|\alpha|+|\gamma|}C_1^{|\alpha|+|\gamma|+1}C_\beta\sum_{\delta\le\beta}\binom{\beta}{\delta}\frac{1}{(\mu_{|\gamma|})^{|\gamma|-|\beta|}} \\
&\le M_{|\alpha|+|\gamma|}C_1^{|\alpha|+|\gamma|+1}C_\beta\sum_{\delta\le\beta}\binom{\beta}{\delta}\frac{1}{(\mu_{|\gamma|})^{|\gamma|-|\beta|}} \\
&= M_{|\alpha|+|\gamma|}C_1^{|\alpha|+|\gamma|+1}C_\beta 2^{|\beta|}\frac{1}{(\mu_{|\gamma|})^{|\gamma|-|\beta|}} \\
&\le C_1^{|\alpha|+|\gamma|+1}M_{|\alpha|+|\gamma|}C_2^{|\beta|+1}M_{|\beta|}2^{|\beta|}\frac{1}{(\mu_{|\gamma|})^{|\gamma|-|\beta|}} \\
&\le C^{|\alpha|+|\beta|+|\gamma|+1}M_{|\alpha|}M_{|\gamma|}M_{|\beta|}\frac{1}{(\mu_{|\gamma|})^{|\gamma|-|\beta|}} \text{ (by \textbf{??})}
\end{aligned}$$

Choose $\{\mu_k\}_{k=0}^\infty$ such that $\mu_0=\mu_1=\ldots=\mu_{|\beta|-1}$ and for $k\ge|\beta|$

$$\mu_k = \sup\left\{\left(\mu_0 M_{|\gamma'|}|\gamma'|!\right)^{\frac{1}{|\gamma'|-|\beta'|}} : |\gamma'|\le k, |\beta'|\le|\gamma'|\right\}+k,$$

where $\mu_0$ is chosen that $\mu_0|y|\le\frac{1}{2}$ for $|y|<N,\ N>0$. Thus $\{\mu_k\}_{k=0}^\infty$ increases to $\infty$ and for $|\gamma|\ge|\beta|$

$$\mu_{|\gamma|} \ge \left(\mu_0 M_{|\gamma|}|\gamma|!\right)^{\frac{1}{|\gamma|-|\beta|}}.$$

Thus for $|\gamma| \geq |\beta|$, we get

$$\left|\partial_y^\beta \partial_x^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|}y)\right)\right| \leq C^{|\alpha|+|\beta|+1} M_{|\alpha|} M_{|\beta|} \frac{C^{|\gamma|}}{|\gamma|!}$$
$$\leq C^{|\alpha|+|\beta|+1} M_{|\alpha|+|\beta|} \frac{C^{|\gamma|}}{|\gamma|!} \text{ (by \textbf{??})}$$

Also, for $x \in \mathbb{R}^m$ and for $|\gamma| < |\beta|$ since $h(\mu_{|\gamma|}y) \equiv 1$,

$$\left|\partial_y^\beta \partial_x^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|}y)\right)\right| = 0,$$

we have

$$\sum_\gamma \left|\partial_y^\beta \partial_x^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|y})\right)\right|$$
$$= \sum_{|\gamma|\geq|\beta|} \left|\partial_y^\beta \partial_x^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|y})\right)\right|$$
$$\leq C^{|\alpha|+|\beta|+1} M_{|\alpha|+|\beta|} \sum_\gamma \frac{C^{|\gamma|}}{|\gamma|!}$$
$$\leq C^{|\alpha|+|\beta|+1} M_{|\alpha|+|\beta|} \sum_\gamma \frac{C^{|\gamma|}}{\gamma!}$$
$$\leq C^{|\alpha|+|\beta|+1} M_{|\alpha|+|\beta|} e^{Cm}$$

Therefore, letting

$$g_\gamma(x,y) = \frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|}y),$$

we have shown that the series $\sum_\gamma g_\gamma(x,y)$ and any series of the derivatives

$$\sum_\gamma \partial_y^\beta \partial_x^\alpha g_\gamma(x,y)$$

converges uniformly on $K \times \mathbb{R}^m$.
For each $k \geq 1$, let

$$h_k(x,y) = \sum_{|\gamma|\leq k} g_\gamma(x,y).$$

Then $h_k(x,y) \to F(x,y)$ and

$$\partial_y^\beta \partial_x^\alpha h_k(x,y) = \sum_{|\gamma|\leq k} \partial_y^\beta \partial_x^\alpha g_\gamma(x,y) \to \sum_\gamma \partial_y^\beta \partial_x^\alpha g_\gamma(x,y)$$

uniformly on $K \times \mathbb{R}^m$.
Therefore, $F(x,y) \in \mathcal{E}^M(\Omega \times \mathbb{R}^m)$ and

$$\partial_y^\beta \partial_x^\alpha F(x,y) = \sum_\gamma \partial_y^\beta \partial_x^\alpha g_\gamma(x,y).$$

We are left to show that

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \leq C^{N+1} \frac{M_N}{N!} |y|^N.$$

Now for all $|\alpha| \geq 0$,

$$\partial_y^\alpha F(x,y)|_{y=0} = \sum_\gamma \partial_y^\alpha \left(\frac{\partial_x^\gamma f(x)}{\gamma!}(iy)^\gamma h(\mu_{|\gamma|}y)\right)|_{y=0} = (i)^{|\alpha|} \partial_x^\alpha f(x).$$

Therefore, for all $\alpha$,

$$\begin{aligned}
\partial_y^\alpha\left(\frac{\partial F}{\partial \bar{z}_j}(x,y)\right)|_{y=0} &= \frac{1}{2}\partial_y^\alpha\left(\frac{\partial F}{\partial x_j}(x,y)+i\frac{\partial F}{\partial y_j}(x,y)\right)|_{y=0} \\
&= \frac{1}{2}[\partial_{x_j}\partial_y^\alpha F(x,y)+i\partial_{y_j}\partial_y^\alpha F(x,y)]|_{y=0} \\
&= \frac{1}{2}[\partial_{x_j}\partial_y^\alpha F(x,y)|_{y=0}+i(i)^{|\alpha+e_j|}\partial_x^{\alpha+e_j}f(x)] \\
&= \frac{1}{2}[\partial_{x_j}\partial_y^\alpha F(x,y)|_{y=0}+(i)^{|\alpha|+2}\partial_x^{\alpha+e_j}f(x)] \\
&= \frac{1}{2}[\partial_{x_j}\partial_y^\alpha F(x,y)|_{y=0}-(i)^{|\alpha|}\partial_x^{\alpha+e_j}f(x)] \\
&= \frac{1}{2}\left[\sum_\gamma \frac{(i)^{|\gamma|}}{\gamma!}\partial_y^\alpha\left(y^\gamma h(\mu_{|\gamma|}y)\partial_x^{\alpha+e_j}f(x)\right)|_{y=0}-(i)^{|\alpha|}\partial_x^{\alpha+e_j}f(x)\right] \\
&= 0,\ j=1,2,\ldots,m.
\end{aligned}$$

Then by Taylor's theorem for $x\in\Omega'$ and $|y|<M$, $(M>0)$ there is a point $y_0=y_0(x,y,\alpha)$ between 0 and $y$ such that

$$\begin{aligned}
\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| &= \left|\sum_{|\alpha|=N}\frac{1}{\alpha!}\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,y_0)y^\alpha\right| \\
&\le \sum_{|\alpha|=N}\frac{1}{\alpha!}\left|\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,y_0)\right||y|^N \\
&\le C^{N+1}M_N|y|^N\sum_{|\alpha|=N}\frac{1}{\alpha!}\ \text{since}\ \frac{\partial F}{\partial \bar{z}_j}\in\mathcal{E}^M \\
&\le C^{N+1}\frac{M_N}{N!}|y|^N
\end{aligned}$$


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