***SINET*: *Ethiop. J. Sci.*,47(2),164-169, 2024** **ISSN: 0379-2897 (PRINT)**

eISSN: 2520-7997

Date received: November 22,2023; Date revised: Semptember 15,2024; Date accepted:December 22,2024
DOI:https://dx.doi.org/10.4314/sinet.v47i2.7

PERFECT ROLE OF ATOM IN WEAK IDEMPOTENT RINGS

Tamiru Abera[1]**,*Yibeltal Yitayew***[2*] **and *Kolluru Venkateswarlu***[3]

Department of Mathematics, Addis Ababa University, P.O.Box 1176, Addis Ababa, Ethiopia.
Department of Mathematics, Addis Ababa Science and Technology University, Artificial Intelligences and Robotics Center of Excellence, Addis Ababa, Ethiopia.
Department of Computer Science and System Engineering, College of Engineering, Andhra University, Visakhapatnam, AP, India.
E-mail: Email: drkvenkateswarlu@gmail.com

ABSTRACT: In a commutative weak idempotent ring R with unity, we prove that $bR = \{0, b\}$ or $\{0, b, n, b+n\}$ for idempotent and nilpotent atoms b and n of R respectively provided that any two nilpotent atoms have an upper bound in R. Further, we prove that the subgroup generated by $\{n_i\}_{i\in I}$ in R is a lattice, where $\{n_i\}_{i\in I}$ is the collection of nilpotent atoms of R corresponding to the idempotent atoms $\{b_i\}_{i\in I}$ of R. We also prove that R is atomic if and only if R_B is atomic provided that the set of all nilpotent elements of R is nil-free and any two nilpotent atoms have an upper bound in R. Finally, we state and prove the direct product decomposition theorem of R.

Keywords/phrases: commutative weak idempotent ring, atom, nil potent, and idempotent.

INTRODUCTION

Foster(1946) defined a Boolean-like ring (BLR, for short) as a commutative ring with unity R in which $ab(1-a)(1-b) = 0$ and $a + a = 0$ for all $a, b \in R$. A weak idempotent ring (WIR, for short) is a ring $(R, +, \cdot)$ of characteristic 2 such that $a^4 = a^2$ for every element a in R. It is clear that a BLR is a WIR but not conversely. For an element a in R: $a = a^2 + (a + a^2)$ and if a is nilpotent, then $a^2 = 0$. Observe that the product of any two nilpotent elements of R need not be equal to zero (See Dereje Wasihun et al.(2022)). In (Tamiru Abera et al. 2024; Lemma 2.1), an order relation is defined as $y < x$ if and only if there exists $b \in R_B$ such that $bx = y$, a non zero element m is called an atom if for every x in the ring , $x < m$ implies either $x = m$ or $x = 0$, and $b < a$ implies that $bc < ac$ for any a, b, c in R.
In Section 2, we prove that the subgroup generated by $\{n_i\}_{i\in I}$ in a commutative WIR with unity is a lattice.
In the last section, we prove that R_B is a symmetric Boolean ring, N is nil free and the set of all nilpotent atoms is complete if and only if R is isomorphic to the direct product of WIR each of which is either a copy of 2 element field or a four element BLR H_4 (see Foster (1946)).

Lattice on the Nil Radical

In this section, we use the concepts of atom and partial order in commutative weak idempotent ring with unity. Throughout this paper, R denotes a commutative WIR (cWIR, for short) with unity, R_B and N denote the set of all idempotent and nilpotent elements of R respectively.

Lemma 0.1. *If n_1 and n_2 are nilpotent elements of R such that $n_1 < n_2$, then $n_1 n_2 = 0$.*

Proof. For $n_1 = n_2$, clearly $n_1 n_2 = 0$. Suppose that n_1 and n_2 are distinct nilpotent elements of R such that $n_1 < n_2$. Then, there exists $b \in R_B$ such that $bn_2 = n_1$. Hence, $n_1 n_2 = 0$.

Lemma 0.2. *If n_1 and n_2 are distinct atoms of N, then $n_1 + n_2$ is the least upper bound of n_1 and n_2. Further $n_1 + n_2$ is not an atom.*

Proof. Suppose that n_1 and n_2 are distinct atoms of N. Then, $n_i < n_1 + n_2$ for $i = 1, 2$. For: if $n_1 + n_2 < n_1$, then $n_1 = n_2$ or $n_2 = 0$ which contradicts the hypothesis. Therefore, $n_i < n_1 + n_2$ for $i = 1, 2$. Let $n_1 < n$ and $n_2 < n$ for an

[*]Author to whom correspondence should be addressed.

arbitrary element $n \in R$. Then, there exist $b_1, b_2 \in R_B$ such that $b_1 n = n_1$ and $b_2 n = n_2$. This implies that $(b_1+b_2)n = n_1+n_2$ and hence $n_1+n_2 < n$. Therefore, n_1+n_2 is the least upper bound of n_1 and n_2. Suppose, if possible, n_1+n_2 is an atom. This and $n_1 < n_1+n_2$ imply that $n_1 = 0$ or $n_2 = 0$ which is a contradiction. Hence, n_1+n_2 is not an atom.

Lemma 0.3. *For $0 \neq b \in R_B$, the following are equivalent.*

i. b is an atom of R.
ii. b is an atom of R_B.
iii. For every $x \in R_B$, either $b < x$ or $bx = 0$.

Proof. $(i) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (iii)$: Suppose that b is an atom of R_B. Let $x \in R_B$. Assume that $bx \neq 0$. As both b and x are idempotent elements, $bx = y$ for some $y \in R_B$. Thus, $y < b$. Since b is an atom of R_B, $y = 0$ or $y = b$. Hence, $bx = b$. Therefore, $b < x$.
$(iii) \Rightarrow (i)$: Suppose that for every $x \in R_B$, either $b < x$ or $bx = 0$. Let $r \in R$ and $r < b$. Then, $bz = r$ for some $z \in R_B$. From our assumption, $bz = b$ or $bz = 0$ as b is an atom. Thus, $r = 0$ or $r = b$. Hence, b is an atom of R.

Lemma 0.4. *If x is an atom of R, then x_B and x_N are atoms of R provided that $x_B \neq 0$, $x_N \neq 0$, where x_B and x_N represent the idempotent and nilpotent part of x.*

Proof. Let x be an atom of R. If $x_B \neq 0$ and $x_N = 0$, then $x_B = x$ and is an atom of R as x is so. If $x_B = 0$ and $x_N \neq 0$, then $x_N = x$ and is an atom of R.

The converse of the above statement is not in general true. See the following.

Example 0.1. *Consider the ring $H_4 = \{0, 1, p, 1+p\}$, where $+$ and $\cdot$ are defined by the following tables (See Tamiru Abera et al.(2024), Example 2.1).*
Let $B = \{0, a, b, a+b\}$ be a Boolean group of 4 elements. Define a unitary H_4^2-module structure on B^2 by the multiplication generated from the following: $pa = a$ and $pb = 0$. Consider the ring $(H_4^2 \times B^2, +, .)$ where the operations are defined as:
$((a_1, a_2), (b_1, b_2)) + ((c_1, c_2), (d_1, d_2)) = ((a_1 + c_1, a_2 + c_2), (b_1 + d_1, b_2 + d_2))$
and $((a_1, a_2), (b_1, b_2)).((c_1, c_2), (d_1, d_2))$
$= ((a_1c_1, a_2c_2), (a_1, a_2)(d_1, d_2) + (c_1, c_2)(b_1, b_2))$, *where the operation between $(H_4)^2$ and B^2 is component-wise. Then, $(H_4^2 \times B^2, +, .)$ is a cWIR with unity.*

Let $x = ((1, 0), (0, a))$. Then, $x_B = ((1, 0), (0, 0))$ is an atom of R and $x_N = ((0, 0), (0, a))$ is an atom of R. But x is not an atom of R as $((0, 1), (0, 0)).((1, 0), (0, a)) = ((0, 0), (0, a)) \neq x$ or 0.

Lemma 0.5. *Let a be an atom of R and $r \in R$. Then, ra is an atom of R provided that $ra \neq 0$.*

Proof. Let a be an atom of R such that $ra \neq 0$ for $r \in R$. Then $\forall b \in R_B, ba < a$ as $ba = ba$ and $b \in R_B$. This implies that either $ba = a$ or $ba = 0$. So, $bra = ra$ or $bra = 0$. Hence ra is an atom of R.

Lemma 0.6. *Let b be an atom of R_B and any two nilpotent atoms have an upper bound. Then, either $bN = \{0\}$ or $bN = \{0, n\}$, where n is an atom of N. Furthermore, $bN = \{0, n\}$, $bn_1 = 0$ for all atoms $n_1 \neq n$.*

Proof. Let bn_1 and bn_2 be non-zero elements of bN. Then, bn_1 and bn_2 are atoms of R by Lemma 0.5 and hence atoms of N by Lemma 0.3 as $bn_1, bn_2 \in N$. Thus, there exists $x \in R$ (particularly x in N) such that $bn_1 < x$ and $bn_2 < x$ as any two nilpotent atoms have an upper bound by the hypothesis. Then, $yx = bn_1$ for some $y \in R_B$. For: if $x \in R_B$, then $0 = (bn_1)^2 = (yx)^2 = yx = bn_1$ as $x, y \in R_B$ which is a contradiction. Thus, $x \in N$ and so that $ybx = bn_1$ as $b \in R_B$. Hence, $bn_1 < bx$. Similarly, $bn_2 < bx$. But $0 < a$ for every $a \in R$. Thus, $bx \neq 0$ and it is an atom of N by Lemma 0.5. Hence, $bn_1 = bn_2 = bx$ as $bn_1 \neq 0, bn_2 \neq 0$. Therefore, $bN = \{0\}$ or $bN = \{0, n\}$. Suppose that $bN \neq \{0\}$. Then, there exists $n_1 \in N$ such that $bn_1 \neq 0$. Let $bn_1 = n$. Then $bn_1 = bn = n$ and hence $bN = \{0, n\}$. By Lemma 0.5, $bn = n$ is an atom of N. Let n_2 be an atom of N and $n \neq n_2$. Assume that $bn_2 \neq 0$. Then, $bn_2 = n$. But $bn_2 = n_2$ as n_2 is an atom of N and $bn_2 \neq 0$ from our assumption. Thus, $n = n_2$ which is a contradiction. Therefore, $bn_2 = 0$, $\forall n_2 \neq n$.

Lemma 0.7. *Let any two nilpotent atoms have an upper bound in R. Then, $b_1N \cap b_2N = \{0\}$ for every two distinct idempotent atoms b_1 and b_2.*

Proof. Suppose that b_1 and b_2 are two distinct idempotent atoms with $b_1N = \{0, n_1\}$ and $b_2N = \{0, n_2\}$. Assume that $b_1N \cap b_2N \neq \{0\}$.Then, $n_1 = n_2$ by Lemma 0.6 which implies $b_2n_1 = b_2n_2 = n_2$. Thus, $b_1b_2n_1 = n_1 = n_2$. But

$b_1b_2 < b_i$ for $i = 1, 2$ which implies $b_1b_2 = 0$ as b_1 and b_2 are distinct idempotent atoms. Hence, $n_1 = n_2 = 0$ which is a contradiction. Therefore, $b_1N \cap b_2N = \{0\}$ for every two distinct idempotent atoms b_1 and b_2.

Lemma 0.8. *Let any two nilpotent atoms have an upper bound in R. For an idempotent atom b, $bR = \{0, b\}$ or $bR = \{0, b, n, b + n\}$, where n is a nilpotent atom.*

Proof. In a cWIR R with unity, $R = \{r_B + r_N : r \in R\}$. Thus, $bR = \{br_B + br_N : r \in R\}$. By Lemma 0.6, $br_N = 0$ or $br_N = n$ for all r_N. As $br_B < b$ and b is an idempotent atom, either $br_B = b$ or $br_B = 0$. Hence, $bR = \{0, b\}$ or $bR = \{0, b, n, b + n\}$.

Definition 0.1. *Let any two nilpotent atoms have an upper bound and $\{b_i\}_{i\in I}$ be the set of all idempotent atoms such that $b_iN \neq \{0\}$. If $b_iN = \{0, n_i\}$, then $\{n_i\}_{i\in I}$ is called the set of nilpotent atoms corresponding to the idempotent atom b_i of $\{b_i\}_{i\in I}$.*

Note: Here and after, we use the notations $\{b_i\}_{i\in I}$ and $\{n_i\}_{i\in I}$ in the context of definition 0.1.

Lemma 0.9. *If nilpotent atoms n_1 and n_2 have an upper bound in R, then $n_1n_2 = 0$.*

Proof. Suppose that n_1 and n_2 have an upper bound in R. By Lemma 0.2(i), n_1+n_2 is the least upper bound of n_1 and n_2. So, $n_1 < n_1 + n_2$ and $n_1(n_1 + n_2) = 0$ by Lemma 0.1. Hence, $n_1n_2 = 0$.

Note.

i. From now onwards, we use $\{n_i\}_{i\in I}$ to denote the set of nil potent atoms corresponding to each idempotent atom b_i for all $i \in I$, $\{b_i\}_{i\in I}$ be the set of all idempotent atoms as in definition 0.1.

ii. Let $r_1, r_2 \in R$. If there exists an upper bound of r_1 and r_2, then the least upper bound of r_1 and r_2 is $r_1 \vee r_2$.

iii. Let $r_1, r_2 \in R$. If there exists a lower bound of r_1 and r_2, then the greatest lower bound of r_1 and r_2 is $r_1 \wedge r_2$.

Lemma 0.10. *Let $n_1, n_2 \in \{n_i\}_{i\in I}$ be in R. Then, $n_1 \vee n_2$ and $n_1 \wedge n_2$ exist and are equal to $n_1 + n_2$ and 0 respectively.*

Proof. Let $n_1, n_2 \in \{n_i\}_{i\in I}$. By Lemma 0.6, $b_1n_1 = n_1$ and $b_1n_2 = 0$. Consider $b_1(n_1 + n_2) = b_1n_1 + b_1n_2 = n_1$. By Lemma 0.2(ii), $n_1 + n_2$ is not an atom. Hence, $n_1 + n_2$ is the least upper bound of n_1 and n_2 by Lemma 0.2. Let $x < n_1$ and $x < n_2$. If $x \neq 0$, then $x = n_1 = n_2$ which is a contradiction as n_1 and n_2 are distinct atoms. Hence $x = 0$. Therefore, 0 is the greatest lower bound of n_1 and n_2.

Theorem 0.1. *The subgroup generated by $\{n_i\}_{i\in I}$ in N is a lattice.*

Proof. Let A be a subgroup generated by $\{n_i\}_{i\in I}$ and $a, b \in A$. Then, $a = \sum_{j\in F_a} n_j$ and $b = \sum_{k\in F_b} n_k$, where F_a and F_b are finite subsets of I. We point out that $a + b = \sum_{t\in F_a\cup F_b} n_t$. Let $F_a = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ and $F_b = \{\beta_1, \beta_2, ..., \beta_s\}$, where $F_a \cap F_b = \{\gamma_1, \gamma_2, ..., \gamma_p\}$ and $p \leq r$ and $p \leq s$. Then $(b_{\alpha_1} + b_{\alpha_2} + ... + b_{\alpha_r})(a+b) = a$ and $(b_{\beta_1} + b_{\beta_2} + ... + b_{\beta_s})(a+b) = b$. Hence, $a < a+b$ and $b < a + b$. That is, $a + b$ is an upper bound of a and b. Let x be an upper bound of a and b. Then, there exist $c, d \in R_B$ such that $cx = a$ and $dx = b$. Thus, $(c + d)x = a + b$ and hence $a + b < x$. Therefore, $a + b$ is the least upper bound of a and b. Let $e = n_{\gamma_1} + n_{\gamma_2} + ... + n_{\gamma_p}$. Then, $(b_{\gamma_1} + b_{\gamma_2} + ... + b_{\gamma_p})a = e$ and $(b_{\gamma_1} + b_{\gamma_2} + ... + b_{\gamma_p})b = e$. Thus, e is the lower bound of a and b. Let x be the lower bound of a and b. Then, there exist $c, d \in R_B$ such that $ca = x$ and $db = x$. If $x \neq 0$, then $c(n_{\alpha_1} + n_{\alpha_2} + ... + n_{\alpha_r}) = x$ and $d(n_{\beta_1} + n_{\beta_2} + ... + n_{\beta_s}) = x$ implies $cn_{\alpha_j} \neq 0$ for some $\alpha_j \in F_a$ and $dn_{\beta_k} \neq 0$ for some $\beta_k \in F_b$. Let $c(n_{\alpha_{j_1}} + n_{\alpha_{j_2}} + ... + n_{\alpha_{j_q}}) = x$ and $d(n_{\beta_{k_1}} + n_{\beta_{k_2}} + ... + n_{\beta_{k_v}}) = x$. Thus, $n_{\alpha_{j_1}} + n_{\alpha_{j_2}} + ... + n_{\alpha_{j_q}} = x$ and $n_{\beta_{k_1}} + n_{\beta_{k_2}} + ... + n_{\beta_{k_v}} = x$ since $b_{\alpha_j}n_{\alpha_j} = n_{\alpha_j}$ and $b_{\beta_k}n_{\beta_k} = n_{\beta_k}$. If $b_{\alpha_{j_i}} \notin \{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\}$ for some $i = 1, 2, ..., q$, then $b_{\alpha_{j_i}}x = b_{\alpha_{j_i}}(n_{\beta_{k_1}} + n_{\beta_{k_2}} + ... + n_{\beta_{k_v}}) = 0$. Thus, $n_{\alpha_{j_i}} = 0$ which is a contradiction. Hence, $\{b_{\alpha_{j_1}}, b_{\alpha_{j_2}}, ..., b_{\alpha_{j_q}}\} \subseteq \{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\}$. Similarly,
$\{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\} \subseteq \{b_{\alpha_{j_1}}, b_{\alpha_{j_2}}, ..., b_{\alpha_{j_q}}\}$.
This gives us $\{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\}$
$= \{b_{\alpha_{j_1}}, b_{\alpha_{j_2}}, ..., b_{\alpha_{j_q}}\} \subseteq \{b_{\gamma_1}, b_{\gamma_2}, ..., b_{\gamma_p}\}$. Thus, $(b_{\beta_{k_1}} + b_{\beta_{k_2}} + ... + b_{\beta_{k_v}})e = x$. Therefore, e is the greatest lower bound of a and b.

Nil free Nil radical

Recall that an element x in R is said to be nil if $bx = 0$ for each idempotent atom b. The nil radical N of R is called nil free if the only nil nilpotent element is zero.

Lemma 0.11. *Suppose that R_B has atoms and any two nilpotent atoms have an upper bound. Then, the following are equivalent.*

i. N is nil free.

ii. For any $0 \neq n \in N$, there exists an atom b of R_B such that $bn \neq 0$.

iii. N is atomic and the set of all atoms of N is precisely $\{n_i\}_{i\in I}$.

Proof. (i) and (ii) are clearly equivalent by definition. Suppose that for any $0 \neq n \in N$, there exists an atom b of R_B such that $bn \neq 0$. Then, $bN \neq \{0\}$ for every idempotent atom b. Thus, $\{n_i\}_{i\in I} \neq \emptyset$. Suppose that there exists a nilpotent element $n \notin \{n_i\}_{i\in I}$. Thus, $bn = 0$ for all atom $b \in R_B$ which is a contradiction. Hence, $\{n_i\}_{i\in I}$ is the set of all nilpotent atoms. Let $0 \neq n \in N$ and there exists an idempotent atom b such that $bn \neq 0$. So, $bn < n$. Therefore, N is atomic since bn is an atom. Assume that N is atomic and $\{n_i\}_{i\in I}$ is the set of all nilpotent atoms. Let $0 \neq n \in N$. Then, $n_i < n$ for some $i \in I$. Hence, $bn = n_i \neq 0$ for some $b \in R_B$.

Remark 0.1. *If R_B has atoms and N is nil free, then N has atoms.*

Proof. Suppose that R_B has an atom say b and N is nil free. Then b is an atom of R and $bn \neq 0$ for some $0 \neq n \in N$ as N is nil free. Then bn is an atom of R as $bn \neq 0$ and b is an atom of R. But $bn \in N$. This implies bn is an atom of N.

Lemma 0.12. *Let N be nil free and any two nilpotent atoms have an upper bound. R is atomic if and only if R_B is atomic.*

Proof. Obviously, R_B is atomic if R is atomic. Suppose that R_B is atomic. Since N is nil free, it is atomic by Lemma 0.11. Let $r \in R$ be neither nilpotent nor idempotent. So, $r = r_B + r_N$ and either $r_B \neq 0$ or $r_N \neq 0$. Thus, there exists an atom $b \in R_B$ such that $b < r_B$ which implies $br_B = b$. Hence, $br = br_B + br_N = b + br_N \neq 0$ as b is non zero. By Lemma 0.5, br is an atom of R and also $br < r$. Therefore, R is atomic.

Lemma 0.13. *Let R_B be atomic, N be nil-free and any two nilpotent atoms have an upper bound. For any $0 \neq n \in N$,*

i. Every atom of R_Bn is of the form b_jn for some atom $b_j \in \{b_i\}_{i\in I}$.

ii. R_Bn is atomic.

Proof. i. Let bn be an atom of R_Bn. Then, bn is an atom of N. By Lemma 0.11, $bn \in \{n_i\}_{i\in I}$. Thus, $bn = n_j$ for some $n_j \in \{n_i\}_{i\in I}$. By definition, there exists an atom b_j of R_B such that $b_jn_j = n_j$. If $bb_j = 0$, then $bn = bn_j = bb_jn_j = 0$ which is a contradiction. Thus, $bb_j \neq 0$ and hence $bb_j = b_j$. Therefore, $bn = bn_j = bb_jn_j = b_jn_j = b_jbn = b_jn$.

ii. Let $bn \in R_Bn$. Then, there exists $n_k \in \{n_i\}_{i\in I}$ such that $n_k < bn$ by Lemma 0.11. Thus, $b_kn_k = n_k < b_kbn$. If $b_kb = 0$, then $n_k = 0$ which is a contradiction. Therefore, $b_kb = b_k$ and hence $n_k < b_kn$. So, there exists $c \in R_B$ such that $cb_kn = n_k$. If $cb_k = 0$, then $n_k = 0$ which is a contradiction. Thus, $b_kn = n_k$ and hence $b_kn < bn$. Therefore, R_Bn is atomic since b_kn is an atom in R_Bn.

Lemma 0.14. *Let R_B be atomic and any two nilpotent atoms have an upper bound. If N is nil free, then for all $n(\neq 0) \in N$, $n = \bigvee_{j\in J} n_j$ where J is some subset of I.*

Proof. Let $n \in N$. Then, by Lemma 0.13 R_Bn is atomic and every atom of R_Bn is of the form b_jn for some atom b_j of $\{b_i\}_{i\in I}$. In **??**, $\bigvee_{k\in K} b_k = 1$, where $\{b_k\}_{k\in K}$ is the set of all atoms of R_B. Now, $n = (\bigvee_{k\in K} b_k)n = (\bigvee_{i\in I} b_i)n = (\bigvee_{j\in J} b_j)n$, where $J \subseteq I$ and $b_jn \neq 0$. Therefore, $n = \bigvee_{j\in J} b_jn$. $b_jn \neq 0$ implies that b_jn is an atom of R_Bn and $b_jn = n_j$. Hence, $n = \bigvee_{j\in J} n_j$.

Lemma 0.15. *Let R_B be atomic and $\{b_k\}_{k\in K}$ be the set of all atoms of R_B. If $\{b_k\}_{k\in K}$ is join-complete, then*
$b\wedge(\bigvee_{j\in J\subset K} b_j) = \bigvee_{j\in J\subset K}(b\wedge b_j)$ for any $b \in R_B$.

Proof. Suppose that $\{b_k\}_{k\in K}$ is join-complete and $b \in R_B$. Take $\{b_j\}_{j\in J}$ for $J \subseteq K$. Then, there exists $x \in R_B$ such that $x = \bigvee_{j\in J} b_j$. Thus, $b_j < x$ and hence $bb_j < bx$ for all $j \in J$. So, bx is an upper bound of $\{bb_j\}_{j\in J}$. Let u be an upper bound of $\{bb_j\}_{j\in J}$. Then, $bb_j < u$ for all $j \in J$. By definition of partial order, $bb_ju = bb_j$ as both $bb_j, u \in R_B$. So, $b_j = bb_ju + bb_j + b_j = bb_ju + bb_jx + b_jx$ as $bb_j < bx \Rightarrow bb_jx = bb_j$ by definition of partial order for $b, bb_j, x \in R_B$. Thus, $b_j(bu+bx+x) = b_j$ which gives us $b_j < bu+bx+x$ for all $j \in J$. So,we obtain that $x < bu + bx + x$ as x is lub$\{b_j\}_{j\in J}$. This gives us $bxu = bx$ and hence $bx < u$. Therefore, $bx = \bigvee_{j\in J} bb_j$. That is, $b \wedge (\bigvee_{j\in J\subset K} b_j) = \bigvee_{j\in J\subset K}(b \wedge b_j)$ for any $b \in R_B$ as R_B is a Boolean lattice.

Lemma 0.16. *Let R_B be atomic, N be nil free and any two nilpotent atoms have an upper bound. Let $\{n_i\}_{i\in I}$ be join-complete and b be an idempotent atom of R. Then:*

i. $b\ (\bigvee_{j\in J\subset I} n_j) = 0$ *if* $b \notin \{b_i\}_{i\in I}$.

ii. $b_i\ (\bigvee_{j\in J} n_j) = n_i$ *if and only if* $n_i \in \{n_j\}_{j\in J}$.

Proof. Suppose that $\{n_i\}_{i\in I}$ is join-complete, b is an idempotent atom of R and $J \subset I$. Then, $n = \bigvee_{j\in J} n_j$ exists in N by the hypothesis.

i. If $b \notin \{b_i\}_{i\in I}$, then $bN = \{0\}$ by definition 0.1 and Lemma 0.6. Therefore, $bn = 0$ and hence $b\ (\bigvee_{j\in J} n_j) = 0$.

ii. Suppose that $n_i \in \{n_j\}_{j\in J}$. Then, $n_i < n$ and hence $0 \neq b_i n_i < b_i n$. By Lemma 0.6, $b_i N = \{0, n_i\}$ and hence $n_i = b_i n_i = b_i n$. So, we obtain that $b_i(\bigvee_{j\in J} n_j) = n_i$ and also $n_i < n$. Hence, $n_i < n$ for all atoms n_i. Suppose that $n_i \notin \{n_j\}_{j\in J}$ by Definition 0.1. Then, $n_i = b_i n = (b_i n) \wedge (\bigvee_{j\in J} b_j n)$ since $n_j = b_j n$ and $R_B n$ is a Boolean lattice. Therefore, $n_i = \bigvee_{j\in J}[(b_i n) \wedge (b_j n)]$ from (i). That is, $n_i = \bigvee_{j\in J}(b_i \wedge b_j)n = 0$ since $b_i \neq b_j$ for any $j \in J$ which is a contradiction. Hence, $n_i \in \{n_j\}_{j\in J}$.

Theorem 0.2. *The set of all nilpotent atoms is complete, R_B is a symmetric Boolean ring and N is nil free if and only if R is isomorphic to the direct product of WI-rings each of which is a copy of a two element field or a four elements BLR H_4.*

Proof. Let the set of all nilpotent atoms be complete, R_B be a symmetric Boolean ring and N be nil free. Assume that $\{b_k\}_{k\in K}$ is the set of all idempotent atoms, $\{b_i\}_{i\in I\subset K}$ is the set of all idempotent atoms for which $b_i N \neq \{0\}$ and $\{n_i\}_{i\in I}$ is the set of all nilpotent atoms corresponding to $\{b_i\}_{i\in I}$. Then, Rb_r is either $\{0, b_r\}$ or $\{0, b_r, n_r, b_r + n_r\}$ and $Rb_{r_1} \cap Rb_{r_2} = \{0\}$ for all $r_1, r_2 \in K$, $r_1 \neq r_2$ by Lemma 0.8 and Lemma 0.7. Hence, Rb_r is either a copy of the 2-element field or a copy of the four-element BLR H_4. Consider the direct product $\bigoplus_{r\in K} Rb_r$. Define $\psi : R \to \bigoplus_{r\in K} Rb_r$ by $\psi(x) = (xb_r)_{r\in K}$.
Then, $\psi(x + y) = ((x + y)b_r)_{r\in K} = (xb_r)_{r\in K} + (yb_r)_{r\in K} = \psi(x) + \psi(y)$.
$\psi(xy) = (xyb_r)_{r\in K} = (xb_r)_{r\in K} \cdot (yb_r)_{r\in K} = \psi(x)\psi(y)$ as $b_r \in R_B$ and R is commutative.
$\psi(1) = (b_r)_{r\in K} = \bigvee_{r\in K} b_r = 1$ as R_B is a symmetric Boolean ring.Therefore, ψ is a homomorphism. Let $\psi(x) = \psi(y)$. So, we get $(xb_r)_{r\in K} = (yb_r)_{r\in K}$. Then, $xb_r = yb_r$ for all $r \in K$. This implies $x_B b_r = y_B b_r$ and $x_N b_r = y_N b_r$ for all $r \in K$.
Now, $x_B = x_B(\bigvee_{r\in K} b_r) = \bigvee_{r\in K} x_B b_r = \bigvee_{r\in K} y_B b_r = y_B(\bigvee_{r\in K} b_r) = y_B$.
Again $x_N = \bigvee_{i\in I} b_i x_N$ by Lemma 0.14 and hence $x_N = \bigvee_{r\in K} b_r x_N = \bigvee_{r\in K} b_r y_N$
$= (\bigvee_{r\in K} b_r)y_N = y_N$. Therefore, $x = y$ and so , ψ is one to one. Let $(x_r b_r)_{r\in K}$ be any element of $\bigoplus_{r\in K} Rb_r$. Then,
$x_r b_r = s_r b_r + t_r n_r$ where $s_r, t_r \in \{0, 1\}$. Consider $\bigvee_{r\in K} s_r b_r + \bigvee_{r\in K} t_r n_r$ in R.
$\psi(\bigvee_{r\in K} s_r b_r + \bigvee_{r\in K} t_r n_r) = ([\bigvee_{r\in K} s_r b_r + \bigvee_{r\in K} t_r n_r]b_r)_{r\in K}$
$= (\bigvee_{r\in K}[s_r b_r + t_r n_r]b_r)_{r\in K} = \bigvee_{r\in K}[(x_r b_r)_{r\in K}]$
$= (x_r b_r)_{r\in K}$ by Lemma 0.15 and Lemma 0.16. Hence, ψ is onto and so that R is isomorphic to $\bigoplus_{r\in K} Rb_r$.
Let R be isomorphic to $\bigoplus_{i\in I} R_i$ where $R_i = \{0, b_i\}$ or $\{0, b_i, n_i, b_i + n_i\}$. Let $x \in \bigoplus_{i\in I} R_i$. Then, $x = (x_i)_{i\in I}$ where $x_i \in R_i$. x is an idempotent if and only if $(x_i^2)_{i\in I} = (x_i)_{i\in I}$ and $x_i^2 = x_i, \forall i \in I$ which implies either $x_i = 0$ or b_i. x is nilpotent if and only if $x^2 = 0$ which implies either $x_i = 0$ or n_i. Hence, the set of all idempotent elements of $\bigoplus_{i\in I} R_i$ is $R_B = \{(x_i)_{i\in I} : x_i = 0 \text{ or } b_i\}$ and the set of nilpotent elements of $\bigoplus_{i\in I} R_i$ is $N = \{(x_i)_{i\in I} : x_i = 0 \text{ or } n_i\}$. Let $0 \neq x = (x_i)_{i\in I} \in R_B$. x is an atom if and only if $x_i = b_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ and $j \in I$. For: if x is an atom, then $x_i \neq 0$ for at least one $i \in I$. If $x_i \neq 0$ and $x_j \neq 0$ with $i \neq j$ and $i, j \in I$, then we have $x_i = b_i$ and $x_j = b_j$ from the hypothesis. So, $x = (x_i)_{j\neq i\in I} + x_j = (b_i)_{j\neq i\in I} + b_j$. Let $y = (y_i)_{i\in I}$ with $y_i = b_i, y_j = 0$ for all $j \neq i$ and $i, j \in I$. Then, y is neither equal to zero nor x.
Considering $xy = (x_i)_{i\in I}(b_i)_{i\in I} + x_j(b_i)_{j\neq i\in I} = (x_i b_i)_{i\in I} + (x_j b_i)_{j\neq i\in I} = (b_i)_{i\in I} = y$ which contradicts the fact that x is an atom. Hence, $x_i \neq 0$ holds for at most one $i \in I$. Conversely, let $0 \neq x \in R_B$ such that $x_i = b_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ with $i, j \in I$. Let $y < x$. Then, $y \in R_B$ and $y_i = b_i$ for some $i \in I$ and $y_j = 0$ for all $j \neq i$ with $i, j \in I$ by the hypothesis. Thus, $xy = (x_i)_{i\in I}(y_i)_{i\in I} = (b_i)_{i\in I} = x$. Hence, x is an atom. Let $x = (x_i)_{i\in I} \in R_B$. Then, there exists at least one $i \in I$ such that $x_i \neq 0$. Let $0 \neq y = (y_i)_{i\in I} \in R_B$ with $y_i = b_i$ and $y_j = 0$ for all $j \neq i, j \in I$. Then, y is an

atom and $xy = y$. This implies $y < x$ and hence R_B is atomic. Let $\{y_k\}_{k\in K}$ be any set of atoms of R_B such that $y_i = b_i$ if $y_{k_i} = b_i$ for some $k \in K$ and $y_i = 0$ if $y_{k_i} = 0$ for all $k \in K$. Then, $y = (y_k)_{k\in K} = (y_{k_i})_{i\in I} = (y_i)_{i\in I}$. Hence, y is the upper bound of $\{y_k\}_{k\in K}$. Let x be an upper bound of $\{y_k\}_{k\in K}$. Then, there exists an element $z_k \in R_B$ such that $y_k = z_k x$ by definition. So, we have $y = (y_k)_{k\in K} = (z_k x)_{z_k\in R_B} = x(z_k)_{z_k\in R_B}$ by Lemma 0.15 which gives us $y < x$. Thus, y is the supremum of $\{y_k\}_{k\in K}$. Therefore, R_B is complete. Let $0 \neq x \in N$ and $x = (x_i)_{i\in I}$. Then, there exists an $i \in I$ such that $x_i \neq 0$. Therefore, $x_i = n_i$. Consider $y = (y_i)_{i\in I}$ with $y_i = b_i$ and $y_j = 0$ for all $j \neq i, j \in I$. Then, y is an atom of R_B and $yx \neq 0$. Hence, N is nil free. Also, atoms of N are of the form $(x_i)_{i\in I}$ with $x_i = n_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ and $i, j \in I$. If $\{z_k\}_{k\in K}$ is any set of atoms of N, then $z = (z_i)_{i\in I}$ is the supremum of $\{z_k\}_{k\in K}$, where, $z_i = n_i$ if $z_{k_i} = n_i$ for some $k \in K$ and $z_i = 0$ if $z_{k_i} = 0$ for all $k \in K$. And 0 is the infimum of $\{z_k\}_{k\in K}$. Hence, the set of all nilpotent atoms of $\bigoplus_{i\in I} R_i$ is complete.

Conclusions

In this paper we have studied the lattice of the sub group generated by the collections of nilpotent atoms corresponding to idempotent atoms of a commutative weak idempotent ring with unity. We also studied the isomorphic properties of a cWIR with unity to the direct product of WIRs each of which is a copy of two elements field and a BLR H_4. This may motivate to study further isomorphic properties and lattice structure of a cWIR with unity.

ACKNOWLEDGEMENTS

The authors thank the anonymous reviewers for their valuable suggestions and comments that helped to improve the manuscript. The first author would like to thank the authorities of Assosa and Addis Ababa Universities for supporting in doing a PhD dissertation thesis research. And also gratefully acknowledge ISP (International Science Program, Uppsala University, Sweden) for supporting in fundraising for the completion of this thesis research.

References

1 Dereje Wasihun,Tilahun Abebaw, Yibeltal Yitayew and Venkateswarlu Kolluru (2022). *Weak idempotent rings.* SINET: Ethiop. J. Sci., 45(1): 97-104.

2 Foster, A.L.(1946). *The theory of Boolean like rings.* Trans.Amer.Math.Soc. Vol.59.

3 ivant, S. and Halmos, P. (2009). *Introduction to Boolean Algebras.* Springer, New York.

4 Kolluru, Venkateswarlu, Dereje Wasihun. and Tilahun Abebaw(2020). *Structure of weak idempotent Rings.* Bull.Int. Math. Virtual Inst., Vol. 10(3), 535-545.

5 Tamiru Abera, Dereje Wasihun, Yibeltal Yitayew and Kolluru, Venkateswarlu (2024). *Ordered Weak Idempotent Rings.* Palestine Journal of Mathematics, Vol. 13(3),215-222.