*SINET: Ethiop. J. Sci.*, 47(2):147–163, 2024 ISSN: 0379-2897 (PRINT)

eISSN: 2520-7997

Date received: 15 December 2023; Date revised: 28 October 2024; Date accepted: 22 December 2024.
DOI:https://dx.doi.org/10.4314/sinet.v47i2.6

# Some United Boundary-Domain Integral and Integro-Differential Equations to the Dirichlet BVP for a Compressible Stokes System with Variable Viscosity

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**Abstract.** The Dirichlet boundary value problem (BVP) for a compressible Stokes system of partial differential equations (PDEs) with variable viscosity is considered in a bounded three dimensional domain. Using an appropriate parametrix (Levi function), the problem is reduced to the united boundary-domain integro-differential equation (BDIDE) or to a domain integral equation supplemented by the original boundary condition, thus constituting a boundary-domain integro-differential problem (BDIDP). Solvability, solution uniqueness and equivalence of the BDIDE/BDIDP to the original BVP as well as invertibility of the associated operators are analysed in appropriate Sobolev (Bessel potential) spaces.

**Keywords/Phrases: Compressible Stokes system, Equivalence, Invertibility, Parametrix, United BDIEs/BDIDPs.**

**MSC:** 76D07; 35J57; 31A10; 45A05

## Introduction

Boundary integral equations and the hydrodynamic potential theory for the Stokes PDE system with constant viscosity have been extensively studied in numerous publications; cf., e.g., Ladyzhenskaya (1969), Lions and Magenes (1972), Kohr and Wendland (2006), and Hsiao and Wendland (2008). The reduction of different BVPs for the Stokes system to boundary integral equations (BIEs) in the case of constant viscosity was possible because the fundamental solutions for both velocity and pressure are readily available in an explicit form. Such reduction was used not only to analyze the properties of the Stokes system and BVP solutions, but also to solve BVPs by solving numerically the corresponding boundary integral equations.

In this paper, we consider the stationary Stokes system of PDEs with variable viscosity and compressibility, in a bounded three dimensional domain that models the motion of a laminar compressible viscous fluid, e.g., through a variable temperature field that makes both viscosity and compressibility depending on coordinates. Reduction of the BVPs for the Stokes system of PDEs with arbitrarily variable viscosity to explicit boundary integral equations is usually not possible, since the fundamental solution needed for such reduction is generally not available in an analytical form (except for some special dependence of the viscosity on coordinates). Using

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a Parametrix (Levi function) as a substitute for a fundamental solution, it is possible, however, to reduce such BVPs to some systems of boundary-domain integral equations (BDIEs) (cf., e.g., Miranda (1970, Sec. 18); Pomp (1998) where the Dirichlet, Neumann, and Robin problems for some PDEs were reduced to indirect BDIE).

The Dirichlet BVP for the linear stationary diffusion partial differential equation with a variable coefficient for a scalar elliptic differential equation is reduced to a system of BDIEs and analysed in Mikhailov (2005) and Mikhailov and Woldemicheal Zenebe (2019). In Ayele Tsegaye and Dagnaw Mulugeta (2021) the Dirichlet BVP with a compressible Stokes system of PDEs in 2D is transformed to two systems of seregated BDIEs and investigated. Using similar approach as in Mikhailov (2005), Mikhailov and Woldemicheal Zenebe (2019), and Ayele Tsegaye and Dagnaw Mulugeta (2021), we will reduce the Dirichlet BVP for a compressible Stokes system of PDEs in 3D to two different systems of direct united BDIDP(E)s expressed in terms of surface and volume parametrix-based potential type operators for their further analysis. A parametrix for a given PDE system is not unique, and special care will be taken to choose a parametrix that leads to the BDIDP(E)s systems being simply analyzed. The mapping properties of the parametrix- based hydrodynamic surface and volume potentials will be obtained, and the equivalence and invertibility theorems for the operators associated with the BDIDP(E) systems will be proved.

## Formulation of the Dirichlet Problem

Let $\Omega$ be a bounded open three-dimensional region of $\mathbb{R}^3$, and let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$. For simplicity, we assume that the boundary $\partial\Omega$ is a simply connected, infinitely smooth and closed surface of dimension 2.

Let $\boldsymbol{v}$ be the velocity vector field; $p$ the pressure scalar field and $\mu \in C^\infty(\overline{\Omega})$ be the variable kinematic viscosity of the fluid such that $\mu(\boldsymbol{x}) > c > 0$. For an arbitrary couple $(p, \boldsymbol{v})$ the stress tensor operator $\sigma_{ij}$ and the Stokes operator $\mathcal{A}_j(p, \boldsymbol{v})$, for a compressible fluid, are defined as

$$\sigma_{ij}(p,\boldsymbol{v})(\boldsymbol{x}) := -\delta_i^j p(\boldsymbol{x}) \ + \mu(\boldsymbol{x}) \cdot \\ \cdot \left( \frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \frac{2}{3}\delta_i^j \operatorname{div} \boldsymbol{v}(\boldsymbol{x}) \right), \qquad (1)$$

$$\boldsymbol{\mathcal{A}}_j(p,\boldsymbol{v})(\boldsymbol{x}) := \frac{\partial}{\partial x_i}\sigma_{ij}(p,\boldsymbol{v})(\boldsymbol{x}) \qquad (2)$$

for $j, i \in \{1,2,3\}$, where $\delta_i^j$ is a Kronecker symbol.

Here and hereafter we assume the Einstein summation in repeated indices from 1 to 3. We also denote the Stokes operator by $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^3$ and $\mathring{\boldsymbol{\mathcal{A}}} := \boldsymbol{\mathcal{A}}|_{\mu=1}$. We will also use the following notation for derivative operators: $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ with $j = 1,2,3$; $\nabla := (\partial_1, \partial_2, \partial_3)$. In what follows, the set of all infinitely differentiable functions on $\Omega$ with compact support is denoted by $\mathscr{D}(\Omega)$, $H^s(\Omega) = H_2^s(\Omega)$ and $H^s(\partial\Omega)$ are the Bessel potential spaces, where $s$ is a real number; see, e.g., Lions and Magenes (1972) and McLean (2000). We recall that $H^s(\Omega)$ coincide with the Sobolev-Slobodetski spaces $W_2^s(\Omega)$ for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$, $\widetilde{H}^s(\Omega) = \{g : g \in H^s(\mathbb{R}^3), \text{supp g} \subset \overline{\Omega}\}$; similarly, $\widetilde{H}^s(S_1) = \{g : g \in H^s(\partial\Omega), \text{supp g} \subset \overline{S}_1\}$ is the Sobolev space of functions having support in $S_1 \subset \partial\Omega$ and $r_{S_1}$ denotes the restriction operator on $S_1 \subset \partial\Omega$. For $s \geq 1, H_*^{s-1}(\Omega) = \{q \in H^{s-1}(\Omega) : \langle q, 1\rangle_\Omega = 0\}$. We will also use the notations $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^3$, $\mathbf{L}_2(\Omega) = [L_2(\Omega)]^3$, $\boldsymbol{\mathcal{D}}(\Omega) = [\mathscr{D}(\Omega)]^3$ for the 3-dimensional vector space.

We will also make use of the following spaces, see, e.g. Costabel (1988), Chkadua et al. (2009), and Mikhailov and Portillo (2015).

$$\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}) := \{(p,\boldsymbol{v}) \in H^{s-1}(\Omega) \times \mathbf{H}^s(\Omega) : \\ : \boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) \in \mathbf{L}_2(\Omega)\} \qquad (3)$$

endowed with the norm

$$\|(p,\boldsymbol{v})\|^2_{\mathbf{H}^{s,0}(\Omega;\mathcal{A})} := \|v\|^2_{\mathbf{H}^s(\Omega)} + \|p\|^2_{H^{s-1}(\Omega)} \\ + \|\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\|^2_{\mathbf{L}_2(\Omega)}.$$

Let us define a space

$$\mathbf{H}_*^{s,0}(\Omega;\boldsymbol{\mathcal{A}}) := \\ \{(p,\boldsymbol{v}) \in H_*^{s-1}(\Omega) \times \mathbf{H}^s(\Omega) : \boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) \in \mathbf{L}_2(\Omega)\}$$

endowed with the norm

$$\|(p,\boldsymbol{v})\|^2_{\mathbf{H}_*^{s,0}(\Omega;\mathcal{A})} := \|v\|^2_{\mathbf{H}^s(\Omega)} + \|p\|^2_{H_*^{s-1}(\Omega)} \\ + \|\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\|^2_{\mathbf{L}_2(\Omega)}.$$

**Remark 1.** *Note that* $\boldsymbol{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}) = \boldsymbol{H}^{s,0}(\Omega;\mathring{\boldsymbol{\mathcal{A}}})$ *if* $s \geq 1$. *In fact,* $\mathcal{A}_j(p,\boldsymbol{v}) = \mathring{\mathcal{A}}_j(p,\mu\boldsymbol{v}) + B_j(p,\boldsymbol{v})$, *where*

$$B_j(p,\boldsymbol{v}) := -\partial_i\left(v_j\partial_i\mu + v_i\partial_j\mu - \frac{2}{3}\delta_i^j v_l\partial_l\mu\right) \in \boldsymbol{L}_2(\Omega)$$

*if* $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$ *and* $s \geq 1$. *It is also true that* $\boldsymbol{H}_*^{s,0}(\Omega;\boldsymbol{\mathcal{A}}) = \boldsymbol{H}_*^{s,0}(\Omega;\mathring{\boldsymbol{\mathcal{A}}})$ *if* $s \geq 1$.

Similar to Mikhailov (2011, Theorem 3.12) one can prove the following assertion.

**Theorem 1.** *The space* $\mathscr{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ *is dense in* $\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$, $s \in \mathbb{R}$.

The operator $\boldsymbol{\mathcal{A}}$ acting on $(p,\boldsymbol{v})$ is well defined in the weak sense provided $\mu(\boldsymbol{x}) \in L^\infty(\Omega)$ as

$$\langle \boldsymbol{\mathcal{A}}(p,\boldsymbol{v}),\boldsymbol{u}\rangle_\Omega := -\mathcal{E}((p,\boldsymbol{v}),\boldsymbol{u}) \quad \forall \boldsymbol{u} \in \widetilde{\mathbf{H}}^{2-s}(\Omega),$$

$1 \leq s < \dfrac{3}{2}$, where the form $\mathcal{E} : [H^{s-1}(\Omega)\times\mathbf{H}^s(\Omega)] \times \widetilde{\mathbf{H}}^{2-s} \to \mathbb{R}$ is defined as

$$\mathcal{E}((p,\boldsymbol{v}),\boldsymbol{u}) := \int_\Omega E((p,\boldsymbol{v}),\boldsymbol{u})(\boldsymbol{x})\,d\boldsymbol{x}, \tag{4}$$

and the function $E((p,\boldsymbol{v}),\boldsymbol{u})(\boldsymbol{x})$ is defined as

$$E((p,\boldsymbol{v}),u)(\boldsymbol{x}) := \frac{1}{2}\mu(\boldsymbol{x})\cdot \cdot\left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right)\times\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right) - \frac{2}{3}\mu(\boldsymbol{x})\ \mathrm{div}\ \boldsymbol{v}(\boldsymbol{x})\ \mathrm{div}\ \boldsymbol{u}(\boldsymbol{x}) - p(x)\ \mathrm{div}\ \boldsymbol{u}(\boldsymbol{x}).$$

For sufficiently smooth functions $(p,\boldsymbol{v}) \in H^{s-1}(\Omega^\pm) \times \mathbf{H}^s(\Omega^\pm)$ with $s > 3/2$, we can state the classical traction operators $\mathbf{T}^{c\pm} = \{T_j^{c\pm}\}_{j=1}^3$ on the boundary $\partial\Omega$ as

$$T_j^{c\pm}(p,\boldsymbol{v})(\boldsymbol{x}) := [\gamma^\pm\sigma_{ij}(p,\boldsymbol{v})](\boldsymbol{x})n_i(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega, \tag{5}$$

where $n_i(\boldsymbol{x})$ denote components of the unit outward normal vector $\mathbf{n}(\boldsymbol{x})$ to the boundary $\partial\Omega$ of the domain and $\gamma^\pm$ is the trace operator inside and outside $\Omega$. Sometimes we will write $\gamma u$ if $\gamma^+u = \gamma^-u$, and similarly for $\mathbf{T}^c$, etc. The classical traction operators can be continuously extended to the canonical traction operators $\mathbf{T}^\pm : \mathbf{H}^{s,0}(\Omega^\pm;\boldsymbol{\mathcal{A}}) \to \mathbf{H}^{s-3/2}(\partial\Omega)$ for $1 \leq s < \frac{3}{2}$ defined in the weak form (see Mikhailov and Portillo (2015, Sec.34.1)) similar to Costabel (1988, Lemma 3.2); Mikhailov (2011, Definition 3.8) as

$$\left\langle \mathbf{T}^\pm(p,\boldsymbol{v}),\boldsymbol{w}\right\rangle_{\partial\Omega} := \pm\int_{\Omega^\pm}[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\gamma^{-1}\boldsymbol{w} + E((p,\boldsymbol{v}),\gamma^{-1}\boldsymbol{w})]d\boldsymbol{x}, \quad \forall(p,\boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega^\pm;\boldsymbol{\mathcal{A}}), \forall\boldsymbol{w} \in \mathbf{H}^{\frac{3}{2}-s}(\partial\Omega). \tag{6}$$

Here the operator $\gamma^{-1} : \mathbf{H}^{s-1/2}(\partial\Omega) \to \mathbf{H}^s(\mathbb{R}^3)$ denotes a continuous right inverse of the trace operator $\gamma : \mathbf{H}^s(\mathbb{R}^3) \to \mathbf{H}^{s-1/2}(\partial\Omega)$. In addition, for $\mathbf{H}_*^{s,0}(\Omega^\pm;\boldsymbol{\mathcal{A}})$ the traction operator $\mathbf{T}^\pm$ is defined.

Furthermore, if $(p,\boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ and $\boldsymbol{u} \in \mathbf{H}^{2-s}(\Omega)$ for $1 \leq s < \dfrac{3}{2}$ the following first Green identity holds, Mikhailov and Portillo (2015, Eq.(34.2)), cf. also Costabel (1988, Lemma 3.4(i)); Mikhailov (2011, Theorem 3.9)

$$\left\langle \mathbf{T}^+(p,\boldsymbol{v}),\gamma^+\boldsymbol{u}\right\rangle_{\partial\Omega} = \int_\Omega[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\boldsymbol{u} + E((p,\boldsymbol{v})\boldsymbol{u})]\,d\boldsymbol{x}. \tag{7}$$

Equation (7) is also defined for $(p,\boldsymbol{v}) \in \mathbf{H}_*^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ and $\boldsymbol{u} \in \mathbf{H}^{2-s}(\Omega)$. Applying the first Green identity to pairs $(p,\boldsymbol{v}),(q,\boldsymbol{u}) \in \mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting one from the other, we arrive at the second Green identity, cf. Costabel (1988, Lemma 3.4(ii)); Mikhailov (2011, Eq. 4.8),

$$\int_\Omega[\mathcal{A}_j(p,\boldsymbol{v})u_j - \mathcal{A}_j(q,\boldsymbol{u})v_j + q\ \mathrm{div}\ \boldsymbol{v} - p\ \mathrm{div}\ \boldsymbol{u}]\,d\boldsymbol{x} = \left\langle \mathbf{T}^+(p,\boldsymbol{v}),\gamma^+\boldsymbol{u}\right\rangle_{\partial\Omega} - \left\langle \mathbf{T}^+(q,\boldsymbol{u}),\gamma^+\boldsymbol{v}\right\rangle_{\partial\Omega}. \tag{8}$$

Equation (8) is also defined for $(p,\boldsymbol{v}),(q,\boldsymbol{u}) \in \mathbf{H}_*^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$.

We will consider the following Dirichlet BVP for which we aim to derive equivalent BDIDP(E) and investigate the existence and uniqueness of their solutions.

For $\boldsymbol{f} \in \mathbf{L}_2(\Omega), g \in L_2(\Omega), \boldsymbol{\varphi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$, find $(p,\boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ such that:

$$\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{9a}$$

$$\mathrm{div}\ \boldsymbol{v}(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{9b}$$

$$\gamma^+\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\varphi}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{9c}$$

We have the following uniqueness theorem.

**Theorem 2.** *The Dirichlet BVP* (9a)–(9c) *has at most one solution in the space* $\boldsymbol{H}_*^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$.

*Proof.* Let $(p_1, \boldsymbol{v}_1)$ and $(p_2, \boldsymbol{v}_2)$ belonging to the space $\mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ that satisfies the BVP (9a)–(9c). Then the pair $(p, \boldsymbol{v}) = (p_2, \boldsymbol{v}_2) - (p_1, \boldsymbol{v}_1)$ also belongs to the space $\mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and satisfies the following homogeneous Dirichlet BVP

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in \Omega, \tag{10a}$$

$$\operatorname{div} \boldsymbol{v}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega, \tag{10b}$$

$$\gamma^+ \boldsymbol{v}(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in \partial\Omega. \tag{10c}$$

The first Green identity (7) holds for any $(p, \boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$. Therefore, we can choose $\boldsymbol{u} \in \mathbf{H}^1_{\partial\Omega,\ \mathrm{div}}(\Omega) \subset \mathbf{H}^1(\Omega)$ where the space $\mathbf{H}^1_{\partial\Omega,\ \mathrm{div}}(\Omega)$ is defined as:

$$\mathbf{H}^1_{\partial\Omega,\ \mathrm{div}}(\Omega) := \{\boldsymbol{u} \in \mathbf{H}^1(\Omega) : \gamma^+ \boldsymbol{u} = \mathbf{0},\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}$$

Since $(p, \boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, the first Green identity can be applied to $\boldsymbol{u} \in \mathbf{H}^1_{\partial\Omega,\ \mathrm{div}}(\Omega)$ and $(p, \boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$,

$$\int_\Omega \frac{1}{2}\mu(x)\left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j(x)} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i(\boldsymbol{x})}\right) \cdot \cdot \left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j(\boldsymbol{x})} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i(\boldsymbol{x})}\right) d\boldsymbol{x} = 0.$$

In particular, one could choose $\boldsymbol{u} := \boldsymbol{v}$ and taking into account (10a)–(10c), we obtain,

$$\int_\Omega \frac{1}{2}\mu(\boldsymbol{x})\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j(\boldsymbol{x})} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i(\boldsymbol{x})}\right)^2 d\boldsymbol{x} = 0.$$

As $\mu(\boldsymbol{x}) > 0$, the only possibility is that $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, i.e. $\boldsymbol{v}$ is a rigid movement, McLean (2000, Lemma 10.5). Nevertheless, taking into account the Dirichlet condition (10c), we deduce that $\boldsymbol{v} \equiv \mathbf{0}$. Hence $\boldsymbol{v}_1 = \boldsymbol{v}_2$. Now consider $\boldsymbol{v} \equiv \mathbf{0}$ and keep in mind (10a), we have $\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})(\boldsymbol{x}) = \mathbf{0}$ and get $\nabla p = 0$. Since $p \in L_2^*(\Omega)$, we have $p = 0$, which implies that $p_1 = p_2$. Otherwise, if $\nabla p = 0$ and $p \in L_2(\Omega)$, then $p = c$, constant c and thus $p_1 = p_2 + c$ □

**Corollary 1.** *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $g \in L_2(\Omega)$ and $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$. Then, the BVP (9a)–(9c) is uniquely solvable in $\boldsymbol{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and the operator*

$$A^D : \boldsymbol{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \boldsymbol{L}_2(\Omega) \times L_2(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$$

*is continuously invertible.*

*Proof.* (cf. Fresneda-Portillo and Mikhailov (2019, Corollary 7.4)) The operator $A^D$ is evidently continuous and due to the uniqueness theorem, Theorem 2 for BVP it is also injective. And hence follows its invertibility. □

## Parametrix and Remainder

When $\mu = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes the Stokes constant coefficient operator $\mathring{\boldsymbol{\mathcal{A}}}$, for which we know an explicit fundamental solution defined by the pair of distributions $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$, where $\mathring{u}_j^k$ represents components of the fundamental solution of the incompressible velocity and $\mathring{q}^k$ represent the components of the fundamental solution of the pressure; see, e.g., Ladyzhenskaya (1969), Kohr and Wendland (2006), Rjasanow and Steinbach (2007), Hsiao and Wendland (2008), Steinbach (2008), and Mikhailov and Portillo (2015).

$$\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}) = \frac{-(x_k - y_k)}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3} = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\right), \tag{11}$$

$$\mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{8\pi}\left(\frac{\delta_j^k}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^3}\right) \quad j, k \in \{1, 2, 3\}. \tag{12}$$

Therefore, $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$ satisfies

$$\frac{\partial}{\partial x_k}\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_k^2}\left(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\right) = -\delta(\boldsymbol{x} - \boldsymbol{y})$$

and

$$\mathring{A}_j(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y})) = \sum_{i=1}^{3} \frac{\partial^2 \mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y})}{\partial x_i^2} - \frac{\partial \mathring{q}^k(\boldsymbol{x}, \boldsymbol{y})}{\partial x_j} = \delta_j^k \delta(\boldsymbol{x} - \boldsymbol{y}),$$

$$\operatorname{div}_{\boldsymbol{x}} \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y}) = 0.$$

Let us denote $\mathring{\sigma}_{ij}(p, \boldsymbol{v}) := \sigma_{ij}(p, \boldsymbol{v})|_{\mu=1}$, $\mathring{T}_j^c(p, \boldsymbol{v}) := T_j^c(p, \boldsymbol{v})|_{\mu=1}$. Following this, in the particular case, for $\mu = 1$ and the fundamental solution $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)_{k=1,2,3}$ of the operator $\mathring{\boldsymbol{\mathcal{A}}}$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x}, \boldsymbol{y})$ reads as follows:

$$\mathring{\sigma}_{ij}(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y})) = \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^5}.$$

Indeed,

$$\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k) = -\delta_{ij}\mathring{q}^k + \left(\frac{\partial \mathring{u}_i^k}{\partial x_j} + \frac{\partial \mathring{u}_j^k}{\partial x_i}\right) =$$

$$= \frac{(x_k - y_k)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{ij} + \left[\frac{\partial}{\partial x_i}\left(-\frac{1}{8\pi}\left(\frac{\delta_j^k}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right)\right) + \frac{\partial}{\partial x_j}\left(-\frac{1}{8\pi}\left(\frac{\delta_i^k}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_i - y_i)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right)\right)\right] =$$

$$= \frac{x_k - y_k}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\delta_{ij} - \frac{\delta_j^k}{8\pi}\frac{\partial}{\partial x_i}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) - \frac{1}{8\pi}\frac{\partial}{\partial x_i}\left(\frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) - \frac{1}{8\pi}\frac{\partial}{\partial x_j}\cdot \cdot\left(\frac{(x_i - y_i)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) - \frac{\delta_i^k}{8\pi}\frac{\partial}{\partial x_j}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)$$

$$= \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}.$$

The classical boundary traction of the fundamental solution then becomes

$$\mathring{T}_j^c(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x},\boldsymbol{y})) := \mathring{\sigma}_{ij}(\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x},\boldsymbol{y}))n_i(\boldsymbol{x}).$$

Now let's define a pair of functions $(q^k, \boldsymbol{u}^k)_{k=1,2,3}$ as

$$q^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}), \tag{13a}$$

$$u_j^k(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{u}_j^k(\boldsymbol{x},\boldsymbol{y}) \quad j,k \in \{1,2,3\}. \tag{13b}$$

Then

$$\sigma_{ij}(\boldsymbol{x})(q^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y})) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\sigma}_{ij}(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x},\boldsymbol{y})), \tag{14}$$

$$T_j(\boldsymbol{x})(q^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y})) := \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{T}_j(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x},\boldsymbol{y}). \tag{15}$$

By substituting (13a)–(13b) to Stokes system we have

$$\mathcal{A}_j(\boldsymbol{x})(q^k, \boldsymbol{u}^k) = \frac{\partial}{\partial x_i}\left(\sigma_{ij}(q^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y}))\right)$$
$$= \frac{\partial}{\partial x_i}\left(\frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\sigma}_{ij}(\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x},\boldsymbol{y}))\right)$$
$$= \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\frac{\partial}{\partial x_i}\left(\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x},\boldsymbol{y})\right) + \frac{\partial}{\partial x_i}\left(\frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\right)\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x},\boldsymbol{y})$$

$$= \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\mathcal{A}}_j(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x},\boldsymbol{y}) + \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)(\boldsymbol{x},\boldsymbol{y})$$
$$= \frac{\mu(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\delta_j^k}{\mu(\boldsymbol{y})} + \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$$
$$= \frac{\mu(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})\delta_j^k}{\mu(\boldsymbol{y})} + \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$$
$$= \delta_j^k\delta(\boldsymbol{x}-\boldsymbol{y}) + \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\boldsymbol{u}}^k).$$

Thus,

$$\mathcal{A}_j(\boldsymbol{x}; \boldsymbol{u}^k, q^k)(\boldsymbol{x},\boldsymbol{y}) = \delta_j^k\delta(\boldsymbol{x}-\boldsymbol{y}) + R_{kj}(\boldsymbol{x},\boldsymbol{y}), \tag{16}$$

where

$$R_{kj}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\frac{\partial(\mu(\boldsymbol{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)(\boldsymbol{x},\boldsymbol{y})$$
$$= \frac{3}{4\pi\mu(\boldsymbol{y})}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}$$
$$= \mathcal{O}(|\boldsymbol{x}-\boldsymbol{y}|^{-2}) \tag{17}$$

is a weakly singular remainder. This implies that $(q^k, \boldsymbol{u}^k)$ is parametrix of the operator $\boldsymbol{\mathcal{A}}$. Note that the parametrix is generally not unique. The possibility to factor out $\frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}$ in (14)–(15) and $\frac{\nabla\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}$ in (17) is due to the careful choice of the parametrix in the form (13a)–(13b) and this essentially simplifies the analysis of parametrix-based potentials obtained and BDIE systems later.

## Parametrix Based Volume and Surface Potentials

**Definition 1.** *Let $\rho$ and $\boldsymbol{\rho}$ be sufficiently smooth scalar and vector functions on $\overline{\Omega}$, e.g., $\rho \in \mathscr{D}(\overline{\Omega}), \boldsymbol{\rho} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. The parametrix-based Newton type and remainder vector potentials for the velocity* *are defined as,*

$$\boldsymbol{\mathcal{U}}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{U}_{kj}\rho_j(\boldsymbol{y}) := \int_\Omega u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,d\boldsymbol{x}, \quad (18)$$

$$\boldsymbol{\mathcal{R}}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{R}_{kj}\rho_j(\boldsymbol{y}) := \int_\Omega R_{kj}(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,d\boldsymbol{x} \quad (19)$$

*and the scalar Newton-type and remainder potentials for the pressure,*

$$\boldsymbol{\mathcal{Q}}\rho(\boldsymbol{y}) = \mathcal{Q}_j\rho(\boldsymbol{y}) := \int_\Omega q^j(\boldsymbol{y},\boldsymbol{x})\rho(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega q^j(\boldsymbol{x},\boldsymbol{y})\rho(\boldsymbol{x})\,d\boldsymbol{x}, \quad (20)$$

$$\mathcal{Q}\boldsymbol{\rho}(\boldsymbol{y}) = \boldsymbol{\mathcal{Q}}\cdot\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{Q}_j\rho_j(\boldsymbol{y}) := \int_\Omega q^j(\boldsymbol{y},\boldsymbol{x})\rho_j(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega q^j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,d\boldsymbol{x}, \quad (21)$$

$$\mathcal{R}^\bullet\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{R}^\bullet_j\rho_j(\boldsymbol{y}) = -2\,p.v.\int_\Omega \frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}\rho_j(\boldsymbol{x})\,d\boldsymbol{x} - \frac{4}{3}\rho_j\frac{\partial\mu}{\partial y_j}, \quad (22)$$

$$= -2\left\langle\partial_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\right\rangle_\Omega - 2\rho_i(\boldsymbol{y})\partial_i\mu(\boldsymbol{y}), \quad (23)$$

*for $y\in\mathbb{R}^3$, see Mikhailov and Portillo (2015) and Fresneda-Portillo and Mikhailov (2019).*

The integral in (22) is understood as a 3D strongly singular integral in the Cauchy sense (Cauchy principal value sense). The bi-linear form in (23) should be understood in the sense of distribution, and the equalities (22) and (23) hold since

$$\begin{aligned}
&\left\langle\partial_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\right\rangle_\Omega = \\
&= -\left\langle\mathring{q}^j(.,\boldsymbol{y}),\partial_i(\rho_i\partial_j\mu)\right\rangle_\Omega + \left\langle n_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\right\rangle_{\partial\Omega}\\
&= -\lim_{\epsilon\to0}\left\langle\mathring{q}^j(.,\boldsymbol{y}),\partial_i(\rho_i\partial_j\mu)\right\rangle_{\Omega_\epsilon} + \left\langle n_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\right\rangle_{\partial\Omega}\\
&= \lim_{\epsilon\to0}\left\langle\partial_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\right\rangle_{\Omega_\epsilon} - \lim_{\epsilon\to0}\left\langle n_i\mathring{q}^j(.,\boldsymbol{y}),\rho_i\partial_j\mu\right\rangle_{\partial\Omega_\epsilon\setminus\partial\Omega}\\
&= p.v.\int_\Omega\frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\mu_{x_i}\rho_j(\boldsymbol{x})\,d\boldsymbol{x} - \frac{1}{3}\rho_j\frac{\partial\mu}{\partial y_j}
\end{aligned}$$

where $\Omega_\epsilon = \Omega\setminus\overline{B}_\epsilon(\boldsymbol{y})$ and $B_\epsilon(\boldsymbol{y})$ is the ball of radius $\epsilon$ and centered in $\boldsymbol{y}$ which implies that

$$-2\left\langle\partial_i\mathring{q}_j(.,\boldsymbol{y}),\rho_j\partial_j\mu\right\rangle_\Omega - 2\rho_j(\boldsymbol{y})\partial_i\mu(\boldsymbol{y}) = -2v.p\int_\Omega\frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\mu_{x_i}\rho_j(\boldsymbol{x})\,d\boldsymbol{x} - \frac{4}{3}\rho_j\frac{\partial\mu}{\partial y_j} = \mathcal{R}^\bullet\boldsymbol{\rho}(\boldsymbol{y}).$$

Let us now define the parametrix-based velocity single layer potential, double layer potential, and their respective direct values on the boundary as follows (see, e.g., Mikhailov and Portillo (2015) and Fresneda-Portillo and Mikhailov (2019)):

**Definition 2.** *For the velocity, the parametrix-based single-layer and double-layer potentials are defined for $y\notin\partial\Omega$,*

$$\mathbf{V}_k\boldsymbol{\rho}(\boldsymbol{y}) = V_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega}u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_{\boldsymbol{x}},$$

$$\mathbf{W}_k\boldsymbol{\rho}(\boldsymbol{y}) = W_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega}T_j^c(\boldsymbol{x};q^k,u^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_{\boldsymbol{x}}.$$

*The single layer and double layer potentials for pressure in the variable coefficient Stokes system are defined for $y\notin\partial\Omega$,*

$$\Pi^s\boldsymbol{\rho}(\boldsymbol{y}) = \Pi^s_j\rho_j(\boldsymbol{y}) := \int_{\partial\Omega}\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_{\boldsymbol{x}},$$

$$\Pi^d\boldsymbol{\rho}(\boldsymbol{y}) = \Pi^d_j\rho_j(\boldsymbol{y}) := 2\int_{\partial\Omega}\frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{n}(\boldsymbol{x})}\mu(\boldsymbol{x})\rho_j(x)dS_{\boldsymbol{x}}.$$

It is easy to observe that the parametrix-based integral operators, with the variable coefficient $\mu$, can be expressed in terms of the corresponding integral operators for the constant coefficient case $\mu = 1$, marked by $^\circ$

$$\boldsymbol{\mathcal{U}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}, \tag{24}$$

$$\boldsymbol{\mathcal{R}}\boldsymbol{\rho} = -\frac{1}{\mu}\left[\frac{\partial}{\partial y_j}\mathring{\mathcal{U}}_{ki}(\rho_j\partial_i\mu) + \frac{\partial}{\partial y_i}\mathring{\mathcal{U}}_{kj}(\rho_j\partial_i\mu) + \mathring{\mathcal{Q}}_k(\rho_j\partial_j\mu)\right], \tag{25}$$

$$\boldsymbol{\mathcal{Q}}\rho = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu\rho), \tag{26}$$

$$\mathcal{R}^\bullet_j\rho_j = -2\frac{\partial}{\partial y_i}\mathring{\mathcal{Q}}_j(\rho_j\partial_i\mu) - 2\rho_j\frac{\partial\mu}{\partial y_i}, \tag{27}$$

$$\mathbf{V}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{V}}\boldsymbol{\rho}, \quad \mathbf{W}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{W}}(\mu\boldsymbol{\rho}), \tag{28}$$

$$\Pi^s\boldsymbol{\rho} = \mathring{\Pi}^s\boldsymbol{\rho}, \quad \Pi^d\boldsymbol{\rho} = \mathring{\Pi}^d(\mu\boldsymbol{\rho}). \tag{29}$$

To show that the above relations (24)–(29) hold, we simply used their corresponding definitions. Also, note that although the constant coefficient velocity potentials $\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}, \mathring{\boldsymbol{V}}\boldsymbol{\rho}$ and $\mathring{\boldsymbol{W}}\boldsymbol{\rho}$ are divergence-free in $\Omega^\pm$, the corresponding potentials $\boldsymbol{\mathcal{U}}\boldsymbol{\rho}, \boldsymbol{V}\boldsymbol{\rho}$ and $\boldsymbol{W}\boldsymbol{\rho}$ are not divergence-free for the variable coefficient $\mu(\boldsymbol{y})$. Note also that due to (11) and (20),

$$\mathring{\mathcal{Q}}_j\rho = -\partial_j\mathcal{P}_\Delta\rho, \tag{30}$$

where

$$\mathcal{P}_\Delta\rho(\boldsymbol{y}) = -\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho(\boldsymbol{x})\,d\boldsymbol{x}$$

is the harmonic Newtonian potential. Therefore,

$$\text{div } \mathring{\boldsymbol{\mathcal{Q}}}\rho = \partial_j\mathring{\mathcal{Q}}_j\rho = -\Delta\mathcal{P}_\Delta\rho = -\rho. \tag{31}$$

Moreover, for the constant coefficient potentials we have the following well known relations,

$$\mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^s\boldsymbol{\rho}, \mathring{\mathbf{V}}\boldsymbol{\rho}) = \mathbf{0}, \quad \mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^d\boldsymbol{\rho}, \mathring{\mathbf{W}}\boldsymbol{\rho}) = \mathbf{0}, \tag{32}$$

$$\mathring{\boldsymbol{\mathcal{A}}}(\mathring{\mathcal{Q}}\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}) = \boldsymbol{\rho} \text{ in } \Omega^\pm. \tag{33}$$

In addition, by (30) and (31),

$$\mathring{\mathcal{A}}_j(\frac{4}{3}\rho, -\mathring{\boldsymbol{\mathcal{Q}}}\rho) = -\frac{4}{3}\partial_j\rho - \partial_i\left(\partial_i\mathring{\mathcal{Q}}_j\rho + \partial_j\mathring{\mathcal{Q}}_i\rho - \frac{2}{3}\delta_i^j \text{ div } \mathring{\boldsymbol{\mathcal{Q}}}\rho\right)$$
$$= -\left(\Delta\mathring{\mathcal{Q}}_j\rho + \partial_j \text{ div } \mathring{\boldsymbol{\mathcal{Q}}}\rho - \frac{2}{3}\partial_j \text{ div } \mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - \frac{4}{3}\partial_j\rho = 0. \tag{34}$$

The following declarations of this section are well-known for the constant coefficient case, see e.g. Kohr and Wendland (2006) and Hsiao and Wendland (2008). Then, by relations (24)–(29) we obtain their counterparts for the variable coefficient case.

The following theorem is proved in Fresneda-Portillo and Mikhailov (2019, Theorem 4.1)

**Theorem 3.** *The following operators are continuous,*

$$\boldsymbol{\mathcal{U}} : \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{35}$$
$$\boldsymbol{\mathcal{U}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{36}$$
$$\boldsymbol{\mathcal{Q}} : \widetilde{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{37}$$
$$\boldsymbol{\mathcal{Q}} : H^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{38}$$
$$\mathcal{Q} : \widetilde{\boldsymbol{H}}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{39}$$
$$\mathcal{Q} : \boldsymbol{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{40}$$
$$\boldsymbol{\mathcal{R}} : \widetilde{\boldsymbol{H}}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{41}$$
$$\boldsymbol{\mathcal{R}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{42}$$
$$\mathcal{R}^\bullet : \widetilde{\boldsymbol{H}}^s(\Omega) \to H^s(\Omega), \quad s \in \mathbb{R}, \tag{43}$$
$$\mathcal{R}^\bullet : \boldsymbol{H}^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}, \tag{44}$$
$$(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}}) : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+2,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad s \geq 0, \tag{45}$$

$$(\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}}) : H^{s-1}(\Omega) \to \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad s \geq 1, \tag{46}$$
$$(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s+1,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad s \geq 1. \tag{47}$$

The following theorem is proved in Fresneda-Portillo and Mikhailov (2019, Theorem 4.2)

**Proposition 1.** *Let* $s > 1/2$ *The following operators are compact,*

$$\boldsymbol{\mathcal{R}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^s(\Omega),$$
$$\mathcal{R}^\bullet : \boldsymbol{H}^s(\Omega) \to H^{s-1}(\Omega),$$
$$\gamma^+\boldsymbol{\mathcal{R}} : \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega)$$
$$T^\pm(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial\Omega).$$

**Theorem 4.** *The following operators are* *continuous.*

$$\boldsymbol{V}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+3/2}(\Omega), \tag{48}$$

$$\boldsymbol{W}: \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1/2}(\Omega),\ s\in\mathbb{R}, \tag{49}$$

$$\Pi^s: \boldsymbol{H}^{s-3/2}(\partial\Omega) \to H^{s-1}(\Omega), \tag{50}$$

$$\Pi^d: \boldsymbol{H}^{s-1/2}(\partial\Omega) \to H^{s-1}(\Omega), s\in\mathbb{R}, \tag{51}$$

$$(\Pi^s, \boldsymbol{V}): \boldsymbol{H}^{s-3/2}(\partial\Omega) \to \boldsymbol{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}), \tag{52}$$

$$(\Pi^d, \boldsymbol{W}): \boldsymbol{H}^{s-1/2}(\partial\Omega) \to \boldsymbol{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}), \quad 1\le s. \tag{53}$$

*Proof.* We use a similar procedure to that of Fresneda-Portillo and Mikhailov (2019, Theorem 4.3). The continuity of the operators in (49), (51) follows from relations (28), (29) and the continuity of the counterpart operators for the constant coefficient case, see e.g. (Kohr and Wendland (2006) and Hsiao and Wendland (2008)). Let us prove continuity of the operators in (53). We first remark that an arbitrary pair $(p,\boldsymbol{v})$ belongs to $\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ if $(p,\boldsymbol{v})\in H^{s-1}(\Omega)\times\mathbf{H}^s(\Omega)$ and $\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\in\mathbf{L}^2$. By expanding the operator $\mathcal{A}_j(y;p,\boldsymbol{v})$

$$\mathcal{A}_j(\boldsymbol{y};p,\boldsymbol{v}) = \mathring{\mathcal{A}}_j(\boldsymbol{y};p,\mu\boldsymbol{v}) - \partial_i\left[v_j\partial_i\mu + v_i\partial_j\mu - \frac{2}{3}\delta_i^j v_l\partial_l\mu(\boldsymbol{y})\right] \tag{54}$$

we can see that if $\boldsymbol{v}\in\mathbf{H}^s(\Omega)$ then the second term in (54) belongs to $L^2(\Omega)$. Therefore, we only need to check that $\mathring{\mathcal{A}}_j(y;p,\mu\boldsymbol{v})\in L^2(\Omega)$.

Now, let us prove the corresponding mapping property for the pair $(\Pi^s,\mathbf{V})$. Let $\boldsymbol{\Psi}\in\mathbf{H}^{s-3/2}(\partial\Omega)$. Then, $(\Pi^s\boldsymbol{\Psi},\mathbf{V}\boldsymbol{\Psi})\in H^{s-1}(\Omega)\times\mathbf{H}^s(\Omega)$ by virtue of (49),(51). By applying relations (28) and (29), $\mathring{\mathcal{A}}_j(\Pi^s\boldsymbol{\Psi},\mu\mathbf{V}\boldsymbol{\Psi}) = \mathring{\mathcal{A}}_j(\mathring{\Pi}^s\boldsymbol{\Psi},\mu\mathring{\mathbf{V}}\boldsymbol{\Psi}) = 0$ in $\Omega$, which completes the proof for the pair $(\Pi^s,\mathbf{V})$.

Let $\boldsymbol{\Phi}\in\mathbf{H}^{s-1/2}$. By virtue of (49) and (51), $(\Pi^d\boldsymbol{\Phi},\mathbf{W}\boldsymbol{\Phi})\in H^{s-1}(\Omega)\times\mathbf{H}^s(\Omega)$. Furthermore, applying the relations (28) and (29) we deduce $\mathring{\mathcal{A}}_j(\Pi^s\boldsymbol{\Psi},\mu\mathbf{V}\boldsymbol{\Psi}) = \mathring{\mathcal{A}}_j(\mathring{\Pi}^s\boldsymbol{\Psi},\mu\mathring{\mathbf{V}}\boldsymbol{\Psi}) = 0$ in $\Omega$, which completes the proof for the pair $(\Pi^d,\mathbf{W})$. □

Let us now define direct values on the boundary of the parametrix-based velocity single layer and double layer potentials and introduce the notations for the co-normal derivative of the latter,

$$\boldsymbol{\mathcal{V}}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{V}_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega} u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}},$$

$$\boldsymbol{\mathcal{W}}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{W}_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega} T_j^+(\boldsymbol{x};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}},$$

$$\boldsymbol{\mathcal{W}}'_k\rho(\boldsymbol{y}) = \mathcal{W}'_{kj}\rho_j(\boldsymbol{y}) := -\int_{\partial\Omega} T_j^+(\boldsymbol{y};q^k,\boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})dS_{\boldsymbol{x}},$$

$$\boldsymbol{\mathcal{L}}^\pm\boldsymbol{\rho}(\boldsymbol{y}) := \mathbf{T}^\pm(\Pi^d\boldsymbol{\rho},\mathbf{W}\boldsymbol{\rho})(\boldsymbol{y}),$$

where $y\in\partial\Omega$ see, e.g., Mikhailov and Portillo (2015). Here, $\mathbf{T}^\pm$ are the canonical derivative (traction) operators for the compressible fluid that are well defined due to the continuity of the second operator in (53).

Similar to the potentials in the domain, we can also express the boundary operators in terms of their counterparts with the constant coefficient $\mu=1$,

$$\boldsymbol{\mathcal{V}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}, \boldsymbol{\mathcal{W}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{W}}}(\mu\boldsymbol{\rho}), \tag{55}$$

$$\boldsymbol{\mathcal{W}}'_k\boldsymbol{\rho} = \mathring{\boldsymbol{\mathcal{W}}}'_k\boldsymbol{\rho}\left(\frac{\partial_i\mu}{\mu}\mathring{\boldsymbol{\mathcal{V}}}_k\boldsymbol{\rho} + \frac{\partial_k\mu}{\mu}\mathring{\boldsymbol{\mathcal{V}}}_i\boldsymbol{\rho} - \frac{2}{3}\delta_i^k\frac{\partial_j\mu}{\mu}\mathring{\boldsymbol{\mathcal{V}}}_j\boldsymbol{\rho}\right)n_i. \tag{56}$$

The following theorem is proved in Fresneda-Portillo and Mikhailov (2019, Theorem 4.4).

**Theorem 5.** *Let $s\in\mathbb{R}$. Let $S_1$ and $S_2$ be two non-empty manifolds on $\partial\Omega$ with smooth boundaries $\partial S_1$ and $\partial S_2$, respectively. Then the following operators are continuous,*

$$\boldsymbol{\mathcal{V}}: \boldsymbol{H}^s(\partial\Omega)\to\boldsymbol{H}^{s+1}(\partial\Omega),$$
$$\boldsymbol{\mathcal{W}}: \boldsymbol{H}^s(\partial\Omega)\to\boldsymbol{H}^{s+1}(\partial\Omega),$$
$$r_{S_2}\boldsymbol{\mathcal{V}}: \widetilde{\boldsymbol{H}}^s(S_1)\to\boldsymbol{H}^{s+1}(S_2),$$
$$r_{S_2}\boldsymbol{\mathcal{W}}: \widetilde{\boldsymbol{H}}^s(S_1)\to\boldsymbol{H}^{s+1}(S_2),$$
$$\boldsymbol{\mathcal{L}}^\pm: \boldsymbol{H}^s(\partial\Omega)\to\boldsymbol{H}^{s-1}(\partial\Omega),$$
$$\boldsymbol{\mathcal{W}}': \boldsymbol{H}^s(\partial\Omega)\to\boldsymbol{H}^{s+1}(\partial\Omega).$$

*Moreover, the following operators are compact,*

$$r_{S_2}\boldsymbol{\mathcal{V}}: \widetilde{\boldsymbol{H}}^s(S_1)\to\boldsymbol{H}^s(S_2),$$
$$r_{S_2}\boldsymbol{\mathcal{W}}: \widetilde{\boldsymbol{H}}^s(S_1)\to\boldsymbol{H}^s(S_2),$$
$$r_{S_2}\boldsymbol{\mathcal{W}}': \widetilde{\boldsymbol{H}}^s(S_1)\to\boldsymbol{H}^s(S_2),$$

**Theorem 6.** *Let* $\boldsymbol{\tau} \in \boldsymbol{H}^{s-1/2}(\partial\Omega)$ *and* $\boldsymbol{\rho} \in \boldsymbol{H}^{s-3/2}(\partial\Omega), 1 \le s < \frac{3}{2}$. *Then the following jump relations hold on* $\partial\Omega$ :

$$\gamma^{\pm}\boldsymbol{V\rho} = \boldsymbol{\mathcal{V}\rho}, \quad \gamma^{\pm}\boldsymbol{W\tau} = \mp\frac{1}{2}\boldsymbol{\tau} + \boldsymbol{\mathcal{W}\tau},$$
$$\boldsymbol{T}^{\pm}(\Pi^{s}\boldsymbol{\rho}, \boldsymbol{V\rho}) = \pm\frac{1}{2}\boldsymbol{\rho} + \boldsymbol{\mathcal{W}}'\boldsymbol{\rho}.$$

*Proof.* For s = 1, the proof of the theorem directly follows from relations (28), (55)–(56) and the analogous jump properties for the counterparts of the operators for the constant coefficient case of $\mu = 1$, see Hsiao and Wendland, 2008, Lemma 5.6.5, which evidently imply the case $s > 1$. □

Let

$$\mathring{\boldsymbol{\mathcal{L}}}\boldsymbol{\tau}(\boldsymbol{y}) = \mathring{\boldsymbol{\mathcal{L}}}^{\pm}\boldsymbol{\tau}(\boldsymbol{y}) := \mathring{\mathbf{T}}(\mathring{\Pi}^{d}\boldsymbol{\tau}, \mathring{\mathbf{W}}\boldsymbol{\tau})(\boldsymbol{y}),$$
$$\hat{\boldsymbol{\mathcal{L}}}\boldsymbol{\tau}(\boldsymbol{y}) := \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\tau})(\boldsymbol{y}), \quad \boldsymbol{y} \in \partial\Omega,$$

where the first equality is implied by the Lyapnove-Tauber theorem for constant coefficient Stokes potentials. The following assertion can be proved similar to Fresneda-Portillo and Mikhailov (2019, Theorem 4.6)

**Theorem 7.** *Let* $\boldsymbol{\tau} \in \boldsymbol{H}^{s-1/2}(\partial\Omega), 1 \le s < \dfrac{3}{2}$. *Then the following jump relation holds:*

$$(\mathcal{L}_k^{\pm} - \hat{\mathcal{L}}_k)\boldsymbol{\tau} = \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i. \tag{57}$$

*Proof.* By Theorem 4 the operator $(\Pi^d, \mathbf{W}) : \mathbf{H}^{s-1/2}(\partial\Omega) \to \mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ is continuous. By Theorem 1, there exists a unique $(\pi^m, w^m)_{m=1}^{\infty} \subset \mathscr{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ converging to $(\mathring{\Pi}^d(\mu\boldsymbol{\tau}), \mathring{\mathbf{W}}(\mu\boldsymbol{\tau}))$ in $\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$. Then, due to (28)–(29), the sequence $(\pi^m, \frac{1}{\mu}w^m)_{m=1}^{\infty} \subset \mathscr{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ converging to $(\mathring{\Pi}^d(\mu\boldsymbol{\tau}), \frac{1}{\mu}\mathring{\mathbf{W}}(\mu\boldsymbol{\tau})) = (\Pi^d\boldsymbol{\tau}, \mathbf{W}\boldsymbol{\tau})$ in $\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$, and by continuity of the canonical traction operators $\mathbf{T}^{\pm} : \mathbf{H}^{s,0}(\Omega^{\pm};\boldsymbol{\mathcal{A}}) \to \mathbf{H}^{s-1/2}(\partial\Omega)$

$$\mathcal{L}_k^{\pm}\boldsymbol{\tau} := T_k^{\pm}(\Pi^d\boldsymbol{\tau}, \mathbf{W}\boldsymbol{\tau}) = \lim_{m\to\infty} T_k^{\pm}(\pi^m, \frac{1}{\mu}w^m). \tag{58}$$

On the other hand,

$$\begin{aligned} &T_k^{\pm}(\pi^m, \frac{1}{\mu}w^m) \\ &= T_k^{c\pm}(\pi^m, \frac{1}{\mu}w^m) = \gamma^{\pm}\sigma_{ik}(\pi^m, \frac{1}{\mu}w^m)n_i \\ &= \gamma^{\pm}\mathring{\sigma}_{ik}(\pi^m, w^m)n_i + \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)w_k^m + \partial_k\left(\frac{1}{\mu}\right)w_i^m - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)w_j^m\right]\right)n_i \\ &\to \mathring{\mathcal{L}}_k^{\pm}(\mu\boldsymbol{\tau}) + \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i. \end{aligned}$$

Since

$$\gamma^{\pm}\mathring{\sigma}_{ik}(\pi^m, w^m)n_i = \mathring{T}_k^{c\pm}(\pi^m, w^m) \to \mathring{T}_k^{\pm}(\mathring{\Pi}^d(\mu\boldsymbol{\tau}), \mathring{\mathbf{W}}(\mu\boldsymbol{\tau})) = \mathring{\mathcal{L}}_k^{\pm}(\mu\boldsymbol{\tau}) \quad m \to \infty,$$

which implies (57). □

**Remark 2.** *Note that the inverse to the operator* $A^D$ *in Corollary 1 is defined in terms of boundary-domain integral operators genarated by the right-hand side functions of the BVP* (9a)–(9c) *as:*

$$(A^D)^{-1}(\boldsymbol{f}, g, \boldsymbol{\varphi}_0) = \left[((\mathcal{D}^1)^{-1}\mathcal{F}^1)_1, ((\mathcal{D}^1)^{-1}\mathcal{F}^1)_2\right],$$

*where*

$$\mathcal{D}^1 : \boldsymbol{H}_*^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\Omega) \to \boldsymbol{H}_*^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\Omega)$$

$$\mathcal{F}^1 = \left[\mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4}{3}\mu g - \Pi^d\boldsymbol{\varphi}_0, \boldsymbol{\mathcal{U}f} - \boldsymbol{\mathcal{Q}}g - \boldsymbol{W\varphi}_0, \gamma^{+}(\boldsymbol{\mathcal{U}f} - \boldsymbol{\mathcal{Q}}g - \boldsymbol{W\varphi}_0) - \boldsymbol{\varphi}_0\right]$$

*and is continuous since the operator* $\mathcal{D}^1$ *is continuosly invertible Dagnaw Mulugeta and Fresneda-Portillo, 2022, Theorem 5.4 and* $\mathcal{F}^1$ *is a continuous function of* $(\boldsymbol{f}, g, \boldsymbol{\varphi}_0)$ *due to the mapping property of the operators involved.*

**Corollary 2.** *Let* $S_1$ *be non-empty sub-manifold of* $\partial\Omega$ *with smooth boundary,* $0 < s < 1$. *Then, the operators*

$$\widehat{\boldsymbol{\mathcal{L}}} : \widetilde{\boldsymbol{H}}^{s}(S_1) \to \boldsymbol{H}^{s-1}(\partial\Omega),$$
$$(\boldsymbol{\mathcal{L}} - \widehat{\boldsymbol{\mathcal{L}}}) : \widetilde{\boldsymbol{H}}^{s}(S_1) \to \boldsymbol{H}^{s-1}(\partial\Omega), \tag{59}$$

*are continuous and the operator*

$$(\boldsymbol{\mathcal{L}} - \widehat{\boldsymbol{\mathcal{L}}}) : \widetilde{\boldsymbol{H}}^{s}(S_1) \to \boldsymbol{H}^{s-1}(\partial\Omega), \tag{60}$$

*is compact.*

*Proof.* The continuity of operators in (59) follows from Theorems 7 and 5. The compactness of the operators (60) follows from the continuity of the second operators in (59) and the compact embedding $\mathbf{H}^{s}(S_1) \hookrightarrow \mathbf{H}^{s-1}(S_1)$. □

## The Third Green Identities and Integral Relations

**Theorem 8.** *For any $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}), 1 \le s < \frac{3}{2}$, the following third Green identities hold.*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^{s}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \Pi^{d}\gamma^{+}\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) + \frac{4\mu}{3}\ \mathit{div}\ \boldsymbol{v}, \tag{61}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{W}\gamma^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{\mathcal{Q}}\ \mathit{div}\ \boldsymbol{v}, \tag{62}$$

*in* $\Omega$.

*Proof.* For $s = 1$ the proof is provided in Fresneda-Portillo and Mikhailov (2019, Theorem 5.1), which evidently implies the claims of the theorem also for $1 < s < \frac{3}{2}$. □

The analogous third Green identities for a Stokes operator with constant coefficient $\mathring{\boldsymbol{\mathcal{A}}}$ is given by

$$p - \mathring{\Pi}^{s}\mathring{\mathbf{T}}^{+}(p, \boldsymbol{v}) + \mathring{\Pi}^{d}\gamma^{+}\boldsymbol{v} = \mathring{\mathcal{Q}}\mathring{\boldsymbol{\mathcal{A}}}(p, \boldsymbol{v}) + \frac{4}{3}\ \text{div}\ \boldsymbol{v} \quad \text{in } \Omega, \tag{63}$$

$$\boldsymbol{v} - \mathring{\mathbf{V}}\mathring{\mathbf{T}}^{+}(p, \boldsymbol{v}) + \mathring{\mathbf{W}}\gamma^{+}\boldsymbol{v} = \mathring{\boldsymbol{\mathcal{U}}}\mathring{\boldsymbol{\mathcal{A}}}(p, \boldsymbol{v}) - \mathring{\boldsymbol{\mathcal{Q}}}\ \text{div}\ \boldsymbol{v} \quad \text{in } \Omega. \tag{64}$$

If the couple $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDEs (9a)–(9b) with variable viscosity coefficient, then the third Green identities (61) and (62) reduce to

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^{s}\mathbf{T}^{+}(p, \boldsymbol{v}) + \Pi^{d}\gamma^{+}\boldsymbol{v} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \ \text{ in } \Omega, \tag{65}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \mathbf{V}\mathbf{T}^{+}(p, \boldsymbol{v}) + \mathbf{W}\gamma^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \ \text{ in } \Omega. \tag{66}$$

We will also need the trace and traction of the third Green identities for $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ on $\partial\Omega$,

$$\frac{1}{2}\gamma^{+}\boldsymbol{v} + \gamma^{+}\boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{\mathcal{V}}\mathbf{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{\mathcal{W}}\gamma^{+}\boldsymbol{v} = \gamma^{+}\boldsymbol{\mathcal{U}}\boldsymbol{f} - \gamma^{+}\boldsymbol{\mathcal{Q}}g, \tag{67}$$

$$\frac{1}{2}\mathbf{T}^{+}(p, \boldsymbol{v}) + \mathbf{T}^{+}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\boldsymbol{v} - \boldsymbol{\mathcal{W}}'\mathbf{T}^{+}(p, \boldsymbol{v}) + \boldsymbol{\mathcal{L}}^{+}\gamma^{+}\boldsymbol{v} = \mathbf{T}^{+}(\mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g, \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g). \tag{68}$$

Note that the traction operators (6) in (68) are well defined by virtue of the continuity of operators (45)–(47) in Theorem 3 and operators (53) in Theorem 4. Let us now prove the following three assertions, which are instrumental in proving the equivalence of the BDIDP/Es systems to the Dirichlet BVP. The following two assertions are instrumental for proof of the equivalence of the BDIDP/Es and the Dirichlet problem.

**Lemma 1.** *Let $1 \le s < \frac{3}{2}$. Suppose some functions $p \in H^{s-1}(\Omega), \boldsymbol{v} \in \boldsymbol{H}^{s}(\Omega), g \in H^{s-1}(\Omega), \boldsymbol{f} \in \boldsymbol{L}_2(\Omega), \boldsymbol{\Psi} \in \boldsymbol{H}^{s-\frac{3}{2}}(\partial\Omega), \boldsymbol{\Phi} \in \boldsymbol{H}^{s-\frac{1}{2}}(\partial\Omega)$ satisfy the equations*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^{s}\boldsymbol{\Psi} + \Pi^{d}\boldsymbol{\Phi} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \ \text{ in } \Omega, \tag{69}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\Psi} + \boldsymbol{W}\boldsymbol{\Phi} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \ \text{ in } \Omega. \tag{70}$$

*Then $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}})$ on $\partial\Omega$ and solves the equation*

$$\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) = \boldsymbol{f}, \ \mathit{div}\ \boldsymbol{v} = g. \tag{71}$$

*Moreover, the following relations hold true:*

$$\Pi^{s}(\boldsymbol{\Psi} - \boldsymbol{T}^{+}(p, \boldsymbol{v})) - \Pi^{d}(\boldsymbol{\Phi} - \gamma^{+}\boldsymbol{v}) = 0 \ \text{ in } \Omega, \tag{72}$$

$$\boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{T}^{+}(p, \boldsymbol{v})) - \boldsymbol{W}(\boldsymbol{\Phi} - \gamma^{+}\boldsymbol{v}) = \mathbf{0} \ \text{ in } \Omega. \tag{73}$$

*Proof.* The proof follows from equations (44)–(47) in Theorem 3 and operators (53) in Theorem 4, similar to Fresneda-Portillo and Mikhailov (2019, Theorem 5.2). First, the embedding $(p,\boldsymbol{v}) \in \mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})$ is implied by the continuity of operators (44)–(47) in Theorem 3 and operators in (53) in Theorem 4. Hence, the third Green identities (61) and (62) hold. Subtracting from them equations (69) and (1) respectively we obtain

$$\Pi^d\boldsymbol{\Phi}^* - \Pi^s\boldsymbol{\Psi}^* = \mathring{\mathcal{Q}}(\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})-\boldsymbol{f}) + \frac{4\mu}{3}(\text{ div } \boldsymbol{v} - g), \quad (74)$$

$$\mathbf{W}\boldsymbol{\Phi}^* - \mathbf{V}\boldsymbol{\Psi}^* = \boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})-\boldsymbol{f}) - \boldsymbol{\mathcal{Q}}(\text{ div } \boldsymbol{v} - g), \quad (75)$$

where $\boldsymbol{\Psi}^* = \mathbf{T}^+(p,\boldsymbol{v}) - \boldsymbol{\Psi}$, and $\boldsymbol{\Phi}^* = \gamma^+\boldsymbol{v} - \boldsymbol{\Phi}$.

After multiplying (75) by $\mu$ and applying the relations (24) and (28) we will arrive at

$$\mathring{\mathbf{W}}(\mu\boldsymbol{\Phi}^*) - \mathring{\mathbf{V}}(\boldsymbol{\Psi}^*) = \mathring{\boldsymbol{\mathcal{U}}}(\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})-\boldsymbol{f}) - \mathring{\boldsymbol{\mathcal{Q}}}(\mu(\text{ div } \boldsymbol{v} - g)). \quad (76)$$

Applying the divergence operator to both sides of (76) and taking into account that the potentials $\mathring{\boldsymbol{\mathcal{U}}}, \mathring{\mathbf{W}}$, and $\mathring{\mathbf{W}}$ are divergence free, while for $\mathring{\boldsymbol{\mathcal{Q}}}$ we have the equation (31), we obtain the following.

$$0 = -\text{ div } \mathring{\boldsymbol{\mathcal{Q}}}(\mu(\text{ div } \boldsymbol{v} - g)) = \mu(\text{ div } \boldsymbol{v} - g), \quad (77)$$

which implies the second equation in (71). Then equations (74) and (76) reduce to

$$\begin{aligned} \mathring{\Pi}^d(\mu\boldsymbol{\Phi}^*) - \mathring{\Pi}^s\boldsymbol{\Psi}^* &= \mathring{\mathcal{Q}}(\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})-\boldsymbol{f}), \\ \mathring{\mathbf{W}}(\mu\boldsymbol{\Phi}^*) - \mathring{\mathbf{V}}\boldsymbol{\Psi}^* &= \mathring{\boldsymbol{\mathcal{U}}}(\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})-\boldsymbol{f}). \end{aligned}$$

Applying the Stokes operator with $\mu = 1$ to these two equations, by (32) and (33) we obtain $\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) - \boldsymbol{f} = \mathbf{0}$ and hence the first equation in (71). Finally, the relations (73) and (72) follow from the substitution of (71) in (74) and (75). □

**Lemma 2.** *For* $1 \le s$,

(i) *Let* $\boldsymbol{\Psi}^* \in \boldsymbol{H}^{s-3/2}(\partial\Omega)$.
*If*

$$\Pi^s\boldsymbol{\Psi}^* = 0 \quad \text{in} \quad \Omega, \quad (78)$$
$$\boldsymbol{V}\boldsymbol{\Psi}^* = \mathbf{0}, \quad \text{in} \quad \Omega, \quad (79)$$

*then*

$$\boldsymbol{\Psi}^* = \mathbf{0}.$$

(ii) *Let* $\boldsymbol{\Phi}^* \in \boldsymbol{H}^{s-1/2}(\partial\Omega)$.
*If*

$$\Pi^d\boldsymbol{\Phi}^* = 0 \quad \text{in} \quad \Omega, \quad (80)$$
$$\boldsymbol{W}\boldsymbol{\Phi}^* = \mathbf{0}, \quad \text{in} \quad \Omega, \quad (81)$$

*then*

$$\boldsymbol{\Phi}^* = \mathbf{0}.$$

*Proof.* (i) Using the relation in equation (28), we can rewrite equation (79) as,

$$\frac{1}{\mu}\mathring{\mathbf{V}}\boldsymbol{\Psi}^* = \mathbf{0} \ \text{ in } \ \Omega \quad (82)$$

Since $\mu$ in $\Omega$, we can multiply (82) by $\mu$ to have the following relation

$$\mathring{\mathbf{V}}\boldsymbol{\Psi}^* = \mathbf{0} \ \text{ in } \ \Omega. \quad (83)$$

Taking a trace of the relation (83), we have

$$\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\Psi}^* = \mathbf{0} \ \text{ on } \ \partial\Omega \quad (84)$$

A basis of the kernel $\mathring{\mathbf{V}}$ is provided in Reidinger and Steinbach (2003, Proposition 2.2). In our case, since $\Omega$ is a simply connected domain, $\text{Ker}(\mathring{\mathbf{V}})$ has one dimension and is generated by the element

$$\boldsymbol{t}^*(\boldsymbol{x}) = \begin{cases} \boldsymbol{n}(\boldsymbol{x}), & x \in \partial\Omega \\ 0, & \boldsymbol{x} \in \Omega \end{cases}, \quad (85)$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the outer normal vector defined for almost all $\boldsymbol{x} \in \partial\Omega$. The solution of (84) can be written as

$$\boldsymbol{\Psi}^*(\boldsymbol{x}) = c\boldsymbol{t}^*(\boldsymbol{x}), \quad c \in \mathbb{R}, \quad \boldsymbol{x} \in \overline{\Omega} \quad (86)$$

Let us now replace $\boldsymbol{\Psi}^*(\boldsymbol{x})$ in the equation (78) by using the relation (86), we have

$$\begin{aligned} \Pi^s\boldsymbol{\Psi}^* &= \int_{\partial\Omega} \mathring{q}(\boldsymbol{x},\boldsymbol{y})cn_j(\boldsymbol{x})\,dS_{\boldsymbol{x}} \\ &= \int_{\partial\Omega} \frac{x_k - y_k}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3} cn_j dS_{\boldsymbol{x}}, \\ &= \int_{\partial\Omega} \frac{\partial P_\Delta}{\partial x_j}(\boldsymbol{x},\boldsymbol{y})cn_j\,dS_{\boldsymbol{x}} \\ &= \int_{\partial\Omega} \frac{\partial P_\Delta}{\partial x_j}(\boldsymbol{x},\boldsymbol{y})n_j c\,dS_{\boldsymbol{x}}, \\ &= \mathbf{W}_\Delta(c)(\boldsymbol{y}), \quad \boldsymbol{y} \in \Omega, \end{aligned}$$

where $P_\Delta$ and $\mathbf{W}_\Delta$ represent the fundamental solution and the double layer potential of the Laplace equation, defined as

$$P_\Delta(\boldsymbol{x},\boldsymbol{y}) := \frac{-1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|},$$

$$\mathbf{W}_\triangle\rho(\boldsymbol{y}) := \int_{\partial\Omega} \frac{\partial P_\triangle}{\partial x_j}(\boldsymbol{x},\boldsymbol{y}) n_j \rho(\boldsymbol{x})\, dS_{\boldsymbol{x}}, \boldsymbol{y} \in \Omega.$$

From Lions and Magenes, 1972, Sec. 11.2, Remark 8, we have $\mathbf{W}_\triangle(c)(\boldsymbol{y}) = c$. Thus, $\Pi^s\boldsymbol{\Psi}^* = 0$ in $\Omega$ if and only if $c = 0$. Then from (86) it follows that $\boldsymbol{\Psi}^* = \mathbf{0}$ in $\Omega$.

(i) Using the relation in equations (29) and (28) to equations (80) and (81) respectively to have

$$\mathring{\Pi}^d(\mu\boldsymbol{\Phi}^*) = 0 \quad \text{in} \quad \Omega, \tag{87}$$

$$\frac{1}{\mu}\mathring{\mathbf{W}}(\mu\boldsymbol{\Phi}^*) = \mathbf{0}, \quad \text{in} \quad \Omega, \tag{88}$$

Since $\mu > 0$, relation (88) implies that

$$\mathring{\mathbf{W}}(\mu\boldsymbol{\Phi}^*) = \mathbf{0}, \quad \text{in} \quad \Omega \tag{89}$$

Let us now apply the traction operator $\mathring{\mathbf{T}}$ to both sides of the relations (87) and (89) to have

$$\mathring{\mathbf{T}}\left(\mathring{\Pi}^d(\mu\boldsymbol{\Phi}^*), \mathring{\mathbf{W}}(\mu\boldsymbol{\Phi}^*)\right) = \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\Phi}^*) = \mathbf{0} \quad \text{in} \quad \partial\Omega.$$

By virtue of Kohr and Wendland (2006, Theorem 3.8), the solutions of $\mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\Phi}^*) = \mathbf{0}$ can be written in the form

$$\boldsymbol{\Phi}^*(\boldsymbol{y}) = \begin{cases} \dfrac{\boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{y}}{\mu(\boldsymbol{y})}, & \boldsymbol{y} \in \partial\Omega \\ 0, & \boldsymbol{y} \in \Omega \end{cases}, \tag{90}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors in $\mathbb{R}^3$. Let us replace now (90) on the left-hand side of (87) to have

$$\mathring{\Pi}^d(\boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{y}) = 0, \boldsymbol{y} \in \Omega. \tag{91}$$

From Hsiao and Wendland (2008, Lemma 5.6.6), the double layer potential for the pressure is related to the double layer potential of the Laplace equation as follows:

$$\mathring{\Pi}^d\rho_k = -2 \text{ div } \mathbf{W}_\triangle\rho_k. \tag{92}$$

Let us evaluate $\mathbf{W}_\triangle\rho_k$ with $\rho_{\mathrm{k}} := \Phi^*_{\mathrm{k}}$, k = 1, 2, 3, using the corresponding first Green identity for the Laplace operator in $\Omega$, cf. Chkadua et al. (2009, Formula 2.8) with constant coefficient.

$$\left\langle \partial_n u, \gamma^+ v\right\rangle = \langle \Delta u, v\rangle_\Omega + \langle \nabla u, \nabla v\rangle_\Omega \tag{93}$$

Take $u := P_\triangle$, the fundamental solution of the Laplace equation and $v := \Phi^*_k$, for $k = 1, 2, 3$ and substitute them into (93) to have

$$\mathbf{W}_\triangle(\boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{y})_k, \quad \boldsymbol{y} \in \overline{\Omega}, \tag{94}$$

since $\boldsymbol{\Phi}^* \in \mathbf{H}^{s-3/2}$ for $k = 1, 2, 3$. Applying Chkadua et al. (2009, Lemma 4.2.ii), the equation (94) has only one solution, the trivial solution. Thus, $\boldsymbol{\Phi}^* \equiv \mathbf{0}$. □

**Theorem 9.** *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$. A pair of functions $(p, \boldsymbol{v}) \in \boldsymbol{H}^{s,0}_*(\Omega; \boldsymbol{\mathcal{A}}), 1 \le s < \frac{3}{2}$, is a solution of PDE* (9a)-(9b) *in $\Omega$ if and only if it is a solution of* (65)-(66).

*Proof.* If $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}), 1 \le s$, solves PDE (9a)–(9b), then, as follows from (61) and (62), it satisfies (65)–(66). On the other hand, if $(p, \boldsymbol{v}) \in \mathbf{H}^{s,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}), 1 \le s < \frac{3}{2}$, solves (65)–(66), then using Lemma 1 for $\boldsymbol{\Psi} = \mathbf{T}^+(p, \boldsymbol{v}), \boldsymbol{\Phi} = \gamma^+\boldsymbol{v}$ completes the proof. □

**Definition 3.** *(see Munkres (2000, Chapter 2, §15, page 87)) Let $I_p : H^{s-1}(\Omega) \times \boldsymbol{H}^s(\Omega) \to H^{s-1}(\Omega), 1 \le s < \frac{3}{2}$ be a mapping defined by*

$$I_p(p, \boldsymbol{v}) = p;$$

*let $I_{\boldsymbol{v}} : H^{s-1}(\Omega) \times \boldsymbol{H}^s(\Omega) \to \boldsymbol{H}^s(\Omega), 1 \le s < \frac{3}{2}$ be a mapping defined by*

$$I_{\boldsymbol{v}}(p, \boldsymbol{v}) = \boldsymbol{v}.$$

*The maps $I_p$ and $I_{\boldsymbol{v}}$ are called the **projections (projection maps)** of $H^{s-1}(\Omega) \times \boldsymbol{H}^s(\Omega)$ onto its first and second factors, respectively.*

Let us consider reduction of the Dirichlet problem (9a)–(9c) for $(p, \boldsymbol{v}) \in \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$, to a united boundary integro-differential problem or to a united boundary integro-differential equations.

## United Boundary Domain Integro-Differential Problem (GD)

Supplementing BDIDEs (65)–(66) in the domain $\Omega$ with the original Dirichlet condition (9c) on the boundary $\partial\Omega$, we arrive at the following united boundary domain integro-differential problem

BDIDP, for $(p, \boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$,

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} + \Pi^d\gamma^+\boldsymbol{v} - \Pi^s\mathbf{T}^+(p, \boldsymbol{v}) = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \quad \text{in } \Omega,$$
$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} + \mathbf{W}\gamma^+\boldsymbol{v} - \mathbf{V}\mathbf{T}^+(p, \boldsymbol{v}) = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \quad \text{in } \Omega,$$
$$\gamma^+\boldsymbol{v} = \boldsymbol{\varphi}_0 \quad \text{on } \partial\Omega.$$

We can rewrite the system in matrix form with the help of Definition 3:

$$\mathscr{A}^{GD}\mathcal{X} = \mathscr{F}^{GD}, \tag{95}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system; $\mathcal{X} = (p, \boldsymbol{v}) \in L_2^*(\Omega) \times \mathbf{H}^1(\Omega)$,

$$\mathscr{A}^{GD} := \begin{bmatrix} I_p + \mathcal{R}^{\bullet}I_{\boldsymbol{v}} - \Pi^s\mathbf{T}^+ + \Pi^d\gamma^+I_{\boldsymbol{v}} \\ I_{\boldsymbol{v}} + \boldsymbol{\mathcal{R}}I_{\boldsymbol{v}} - \mathbf{V}\mathbf{T}^+ + \mathbf{W}\gamma^+I_{\boldsymbol{v}} \\ r_{\partial\Omega}\gamma^+I_{\boldsymbol{v}} \end{bmatrix} \text{ and}$$

$$\mathscr{F}^{GD} := \begin{bmatrix} \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4}{3}\mu g \\ \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \\ \boldsymbol{\varphi}_0 \end{bmatrix}.$$

The BDIDP system is equivalent to the Dirichlet BVP (9a)–(9c) in $\Omega$, in the following sense.

**Theorem 10.** *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $g \in L_2(\Omega)$ and $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega)$.*

*(i) If a couple $(p, \boldsymbol{v}) \in L_2^*(\Omega) \times \boldsymbol{H}^1(\Omega)$ solves the Dirichlet BVP* (9a)–(9c)*, then it solves the BDIDP system* (95)*.*

*(ii) If a set $(p, \boldsymbol{v}) \in L_2^*(\Omega) \times \boldsymbol{H}^1(\Omega)$ solves the BDIDP system* (95)*, then the pair of functions $(p, \boldsymbol{v})$ belongs to $\boldsymbol{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ and solves the Dirichlet BVP* (9a)–(9c)*.*

*(iii) The BDIDP system* (95) *is uniquely solvable for $(p, \boldsymbol{v}) \in L_2^*(\Omega) \times \boldsymbol{H}^1(\Omega)$.*

*Proof.* (i) A solution of BVP (9a)–(9c) does exist and is unique due to Corollary 1. Let $(p, \boldsymbol{v}) \in L_2^*(\Omega) \times \mathbf{H}^1(\Omega)$ be a solution of BVP (9a)–(9c). Since $\boldsymbol{f} \in \mathbf{L}_2(\Omega)$ then $(p, \boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ and provides a solution to BDIDP (95) due to Theorem 9.

(ii) Let $(p, \boldsymbol{v}) \in L_2^*(\Omega) \times \mathbf{H}^1(\Omega)$ solve the BDIDP system (95). Then the first two equations of the system (95) and Theorems 3 and 4 imply that $(p, \boldsymbol{v}) \in \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$. It satisfies also (9a)–(9b) due to the same Theorem 9 and the boundary condition (9c) due to the analogous construction of the BDIDP.

(iii) The unique solvability of the BDIDP system (95) follows from the unique solvability of the BVP (9a)–(9c) and items (*i*) and (*ii*), see Theorem 2 and Corollary 1.

□

Owing to the mapping properties of operators $\mathbf{V}, \mathbf{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}, \mathcal{R}^{\bullet}, \Pi^s, \Pi^d$ and $\mathring{\mathcal{Q}}$ we have $\mathscr{F}^{GD} \in \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{1/2}(\partial\Omega)$, and the operator $\mathscr{A}^{GD} : L_2^*(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{1/2}(\partial\Omega)$ is continuous. By Theorem 10, it is also injective. Let us now characterise the image of the operator $\mathscr{A}^{GD}$ in the whole space $\mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{1/2}(\partial\Omega)$.

**Definition 4.** *The space $\boldsymbol{Y}_{D_{11}}^1(\Omega; \boldsymbol{\mathcal{A}})$ consists of pairs of functions of the form*

$$(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*) = (\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}})\boldsymbol{f}_* + (\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}})h_* \quad \textit{in } \Omega, \tag{96}$$

*with $\boldsymbol{f}_* \in \boldsymbol{L}_2(\Omega), h_* \in L_2(\Omega)$ and is provided with the norm of space $\boldsymbol{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$,*

$$\|(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*)\|_{\boldsymbol{Y}_{D_{11}}^s(\Omega; \boldsymbol{\mathcal{A}})} := \|(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*)\|_{\boldsymbol{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})}.$$

$\mathbf{Y}_{D_{11}}^1(\Omega; \boldsymbol{\mathcal{A}})$ is a subset of $\mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ (cf. Theorem 3). The completeness of $\mathbf{Y}_{D_{11}}^1(\Omega; \boldsymbol{\mathcal{A}})$ is proved in Lemma 4 below.

**Remark 3.** *A pair of functions $(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*) \in \boldsymbol{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ belongs to $\boldsymbol{Y}_{D_{11}}^1(\Omega; \boldsymbol{\mathcal{A}})$ if and only if*

$$\begin{aligned} &\mathring{\Pi}^d\gamma^+(\mu\boldsymbol{\mathcal{F}}_*) - \mathring{\Pi}^s\mathring{\boldsymbol{T}}^+(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) \\ &\qquad = \frac{4}{3}\ \textit{div}\ (\mu\boldsymbol{\mathcal{F}}_*) \quad \text{in } \Omega, \\ &\mathring{\boldsymbol{W}}\gamma^+(\mu\boldsymbol{\mathcal{F}}_*) - \mathring{\boldsymbol{V}}\mathring{\boldsymbol{T}}^+(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) \\ &\qquad = -\mathring{\boldsymbol{\mathcal{Q}}}\ \textit{div}\ (\mu\boldsymbol{\mathcal{F}}_*) \quad \text{in } \Omega. \end{aligned} \tag{97}$$

*or, the same,*

$$\begin{aligned} &(\mathring{\Pi}^d, \mathring{\boldsymbol{W}})\gamma^+(\mu\boldsymbol{\mathcal{F}}_*) - (\mathring{\Pi}^s, \mathring{\boldsymbol{V}})\mathring{\boldsymbol{T}}^+(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) \\ &\qquad = (\frac{4}{3}I, -\mathring{\boldsymbol{\mathcal{Q}}})\ \textit{div}\ (\mu\boldsymbol{\mathcal{F}}_*) \quad \text{in } \Omega. \end{aligned}$$

*Proof.* Condition (96) can be rewritten as

$$(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) = (\mathring{\mathcal{Q}}, \mathring{\boldsymbol{\mathcal{U}}})\boldsymbol{f}_* + (\frac{4}{3}I, -\mathring{\boldsymbol{\mathcal{Q}}})(\mu h_*) \quad \text{in } \Omega. \tag{98}$$

The third Green identities (63) and (64) by substitute $p = \mathcal{F}_*$ and $\boldsymbol{v} = \mu\boldsymbol{\mathcal{F}}_*$ respectively and for the potentials associated with the operator $\mathring{\boldsymbol{\mathcal{A}}}$

gives

$$\begin{aligned} &\mathcal{F}_* - \mathring{\Pi}^s \mathring{\boldsymbol{T}}^+(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) + \mathring{\Pi}^d \gamma^+(\mu\boldsymbol{\mathcal{F}}_*) \\ &\quad = \mathring{\mathcal{Q}}\mathring{\boldsymbol{\mathcal{A}}}(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) + \frac{4}{3} \text{ div } (\mu\boldsymbol{\mathcal{F}}_*), \\ &\mu\boldsymbol{\mathcal{F}}_* - \mathring{\boldsymbol{V}}\mathring{\boldsymbol{T}}^+(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) + \mathring{\boldsymbol{W}}\gamma^+(\mu\boldsymbol{\mathcal{F}}_*) \\ &\quad = \mathring{\boldsymbol{\mathcal{U}}}\mathring{\boldsymbol{\mathcal{A}}}(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) - \mathring{\boldsymbol{\mathcal{Q}}} \text{ div } (\mu\boldsymbol{\mathcal{F}}_*), \end{aligned} \tag{99}$$

in $\Omega$. Thus, (97) implies (98) with $\boldsymbol{f}_* = \mathring{\boldsymbol{\mathcal{A}}}(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*)$ in $\Omega$.

On the other hand, if (98) is satisfied, then the application of $\mathring{\boldsymbol{\mathcal{A}}}$ to it gives $\boldsymbol{f}_* = \mathring{\boldsymbol{\mathcal{A}}}(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*)$ in $\Omega$, which substitution in (99) comparison with (98) implies (97). □

To realize how narrow is the subspace $\mathbf{Y}^1_{D_{11}}(\Omega : \boldsymbol{\mathcal{A}})$, let us prove the following statement.

**Lemma 3.** *For any pair of functions* $(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*) \in \boldsymbol{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$, *there exists a unique triple* $(\boldsymbol{f}_*, h_*, \boldsymbol{\Phi}_*) = \Theta_{\boldsymbol{\Phi}}(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*) \in \boldsymbol{L}_2(\Omega) \times L_2(\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega)$ *such that,*

$$\begin{aligned} (\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*)(\boldsymbol{y}) &= (\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}})\boldsymbol{f}_*(\boldsymbol{y}) + (\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}})h_* \\ &- (\Pi^d, \boldsymbol{W})\boldsymbol{\Phi}_*(\boldsymbol{y}), \boldsymbol{y} \in \Omega \end{aligned} \tag{100}$$

*and* $\Theta_{\boldsymbol{\Phi}} : \boldsymbol{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \to \boldsymbol{L}_2(\Omega) \times L_2(\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega)$ *is a linear bounded operator.*

*Proof.* Suppose first that there exist some functions $\boldsymbol{f}_*, h_*$ and $\boldsymbol{\Phi}_*$ satisfying (100) and find their expressions in terms of $(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*)(\boldsymbol{y})$. Taking into account definitions for the single layer and volume potentials, (100) can be rewritten as

$$\begin{aligned} (\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) &= (\mathring{\mathcal{Q}}, \mathring{\boldsymbol{\mathcal{U}}})\boldsymbol{f}_* + (\frac{4}{3}I, -\mathring{\boldsymbol{\mathcal{Q}}})(\mu h_*) \\ &- (\mathring{\Pi}^d, \mathring{\boldsymbol{W}})(\mu\boldsymbol{\Phi}_*), \quad \text{in } \Omega. \end{aligned} \tag{101}$$

Applying the $\boldsymbol{\mathcal{A}}_{|\mu=1} = \mathring{\boldsymbol{\mathcal{A}}}$ to (101), we obtain,

$$\boldsymbol{f}_* = \mathring{\boldsymbol{\mathcal{A}}}(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*) \quad \text{in } \Omega. \tag{102}$$

Then (101) can be written as

$$(\mathring{\Pi}^s, \mathring{\boldsymbol{W}})(\mu\boldsymbol{\Psi}_*)(y) = (R_1, R_2)(\boldsymbol{y}), \quad \boldsymbol{y} \in \Omega, \tag{103}$$

where

$$\begin{aligned} (R_1, R_2)(\boldsymbol{y}) &= (\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*)(\boldsymbol{y}) \\ &- \Big((\mathring{\mathcal{Q}}, \mathring{\boldsymbol{\mathcal{U}}})[\mathring{\boldsymbol{\mathcal{A}}}(\mathcal{F}_*, \mu\boldsymbol{\mathcal{F}}_*)](\boldsymbol{y}) \\ &+(\frac{4}{3}I, -\mathring{\boldsymbol{\mathcal{Q}}})(\mu h_*)\Big), \quad \boldsymbol{y} \in \Omega. \end{aligned} \tag{104}$$

We can also rewrite (103) in the following system form;

$$\begin{cases} \mathring{\Pi}^d(\mu\boldsymbol{\Phi}_*)(\boldsymbol{y}) = R_1(\boldsymbol{y}), & \boldsymbol{y} \in \Omega, \\ \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}_*)(\boldsymbol{y}) = R_2(\boldsymbol{y}), & \boldsymbol{y} \in \Omega. \end{cases} \tag{105}$$

The trace of the second equation of (105) on the boundary gives;

$$\left[-\frac{1}{2}I + \mathring{\boldsymbol{\mathcal{W}}}\right](\mu\boldsymbol{\Phi}_*)(\boldsymbol{y}) = \gamma^+ R_2(\boldsymbol{y}), \ \boldsymbol{y} \in \partial\Omega. \tag{106}$$

where $\mathring{\boldsymbol{\mathcal{W}}} = \boldsymbol{\mathcal{W}}_{|\mu=1}$ is the direct value on the boundary of the parametrix-based velocity single layer operator associated with the operator corresponding to the constant coefficient. Since $\left[-\frac{1}{2}I + \mathring{\boldsymbol{\mathcal{W}}}\right] : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega)$, $s \in \mathbb{R}$ is isomorphism (c.f., e.g. Dautray and Lions, 1990, Chapter XI, part B, section 2, Remark 8 and $\mu(\boldsymbol{y}) \neq 0$, we obtain the following expression for $\boldsymbol{\Phi}_*$ :

$$\boldsymbol{\Phi}_*(\boldsymbol{y}) = \frac{1}{\mu}\left[-\frac{1}{2}I + \mathring{\boldsymbol{\mathcal{W}}}\right]^{-1} \gamma^+ R_2(\boldsymbol{y}) \quad \boldsymbol{y} \in \partial\Omega. \tag{107}$$

Consequently, (102), (107) and from Equations (98) implies $h_* = \dfrac{1}{\mu}$ div $(\mu\boldsymbol{\mathcal{F}}_*)$, indicate uniqueness of the triple set $(\boldsymbol{f}_*, h_*, \boldsymbol{\Phi}_*)$. Thus, (102), (107) and (104) give bounded operator

$$\Theta_{\boldsymbol{\Phi}} : \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \to \mathbf{L}_2(\Omega) \times L_2(\Omega) \times \mathbf{H}^{1/2}(\partial\Omega)$$

mapping $(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*)$ to $(\boldsymbol{f}_*, h_*, \boldsymbol{\Phi}_*)$. □

Lemma 3 implies that (96) does not cover the whole space $\mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$. i.e. $\mathbf{Y}^1_{D_{11}}(\Omega; \boldsymbol{\mathcal{A}})$ is more narrow than the space $\mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$. Let us prove $\mathbf{Y}^1_{D_{11}}(\Omega; \boldsymbol{\mathcal{A}})$ a closed subspace of $\mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$.

**Lemma 4.** *The space* $\boldsymbol{Y}^1_{D_{11}}(\Omega; \boldsymbol{\mathcal{A}})$ *is complete.*

*Proof.* Let $\left\{\left(\mathcal{F}^{(n)}_*, \boldsymbol{\mathcal{F}}^{(n)}_*\right)\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbf{Y}^1_{D_{11}}(\Omega; \boldsymbol{\mathcal{A}})$. Then by (96), $\left(\mathcal{F}^{(n)}_*, \boldsymbol{\mathcal{F}}^{(n)}_*\right) = \left(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}}\right)\boldsymbol{f}^{(n)}_* + \left(\dfrac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}}\right)h^{(n)}_*$ in $\Omega$ for some $\left(\boldsymbol{f}^{(n)}_*, h^{(n)}_*\right) \in \mathbf{L}_2(\Omega) \times L_2(\Omega)$. Due to the lemma 3, $\left(\boldsymbol{f}^{(n)}_*, h^{(n)}_*\right) = (\Theta_{\boldsymbol{\Phi}_1}, \Theta_{\boldsymbol{\Phi}_2})\left(\mathcal{F}^{(n)}_*, \boldsymbol{\mathcal{F}}^{(n)}_*\right)$ where $(\Theta_{\boldsymbol{\Phi}_1}, \Theta_{\boldsymbol{\Phi}_2}) : \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \to \mathbf{L}_2(\Omega) \times L_2(\Omega)$ is a linear bounded operator, which implies that $\left\{\left(\boldsymbol{f}^{(n)}_*, h^{(n)}_*\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbf{L}_2(\Omega) \times L_2(\Omega)$. Since $\mathbf{L}_2(\Omega) \times L_2(\Omega)$ is complete, the sequence has a limit (say) $(\boldsymbol{f}_*, h_*) \in \mathbf{L}_2(\Omega) \times L_2(\Omega)$. Due to (45), Theorem 3, the operator $(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}}) : \mathbf{L}_2(\Omega) \to \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ and $\left(\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}}\right) : L_2(\Omega) \to \mathbf{H}^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ (and then their sums) are bounded (continuous) operators, implying $\left(\mathcal{F}^{(n)}_*, \boldsymbol{\mathcal{F}}^{(n)}_*\right)$ converges to $(\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*) = \left(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}}\right) \boldsymbol{f}_* + \left(\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}}\right) h_*$ in $\mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$, which completes the proof. □

Now we are in a position to prove the invertibility theorem.

**Theorem 11.** *The operator*

$$\mathscr{A}^{GD} : \boldsymbol{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \to \boldsymbol{Y}^1_{D_{11}}(\Omega; \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\partial\Omega) \tag{108}$$

*is continuous and continuously invertible.*

*Proof.* The continuity of $\mathscr{A}^{GD}$ is already proved, and we have to prove the existence of a bounded inverse operator $\left(\mathscr{A}^{GD}\right)^{-1}$. For $(p, \boldsymbol{v}) \in \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ the third Green identities (61) and (62) implies that

$$\begin{aligned}\mathscr{A}^{GD}\mathcal{X} &= \Big[\mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) + \frac{4\mu}{3} \operatorname{div} \boldsymbol{v}, \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) - \boldsymbol{\mathcal{Q}} \operatorname{div} \boldsymbol{v}, \\ &\qquad r_{\partial\Omega}\gamma^+\boldsymbol{v}\Big]^T \\ &= \Big[(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) + (\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}}) \operatorname{div} \boldsymbol{v},\ r_{\partial\Omega}\gamma^+\boldsymbol{v}\Big]^T\end{aligned}$$

i.e. operator (108) is continuous. Then, if (any arbitrary) $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \in \mathbf{Y}^1_{M_{11}}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{1/2}(\partial\Omega)$, then $\mathscr{F}_1 = (\mathcal{F}_*, \boldsymbol{\mathcal{F}}_*) = (\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}})\boldsymbol{f}_* + (\frac{4\mu}{3}I, -\boldsymbol{\mathcal{Q}})h_*$ for some $(\boldsymbol{f}_*, h_*)$ in $\mathbf{L}_2(\Omega) \times L_2(\Omega)$. Due to Lemma 3 $(\boldsymbol{f}_*, h_*) = (\Theta_{\boldsymbol{\Phi}_1}, \Theta_{\boldsymbol{\Phi}_2})\mathscr{F}_1$ while $(\Theta_{\boldsymbol{\Phi}_1}, \Theta_{\boldsymbol{\Phi}_2}) : \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \to \mathbf{L}_2(\Omega) \times L_2(\Omega)$ is a linear bounded operator. Then the equivalence theorem, Theorem 10 and invertibility of the BVP operator given by Corollary 1 imply that equation $\mathscr{A}^{GD}\mathcal{X} = \mathscr{F}$ has a unique solution $\mathcal{X} = (p, \boldsymbol{v})$ such that;

$$\begin{aligned}(p, \boldsymbol{v})^T &= (A^D)^{-1}(\boldsymbol{f}_*, g, \mathscr{F}_2)^T \\ &= (A^D)^{-1}\left[(\Theta_{\boldsymbol{\Phi}_1}, \Theta_{\boldsymbol{\Phi}_2})\mathscr{F}_1, \mathscr{F}_2\right]^T \\ &= (A^D)^{-1}\left[\Theta_{\boldsymbol{\Phi}_{12}}\mathscr{F}_1, \mathscr{F}_2\right]^T \\ &= (A^D)^{-1} diag(\Theta_{\boldsymbol{\Phi}_{12}}, I)\mathscr{F}\end{aligned}$$

where

$$\Theta_{\boldsymbol{\Phi}_{12}} := (\Theta_{\boldsymbol{\Phi}_1}, \Theta_{\boldsymbol{\Phi}_2}) : \mathbf{Y}^1_{D_{11}}(\Omega; \boldsymbol{\mathcal{A}}) \to \mathbf{L}_2(\Omega) \times L_2(\Omega)$$

and

$$(A^D)^{-1} : \mathbf{L}_2(\Omega) \times L_2(\Omega) \times \mathbf{H}^{1/2}(\partial\Omega) \to \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$$

is a bounded inverse to the operator $A^D$ of the Dirichlet BVP from Corollary 1. Thus, $(A^D)^{-1}diag(\Theta_{\boldsymbol{\Phi}_{12}}, I)$ is a bounded inverse of the operator (108). □

## United Boundary Domain Integro-Differential Equations ($\tilde{G}_D$)

In this section, we remove the Dirichlet boundary condition to deal with integro-differential equations. Substituting the Dirichlet boundary condition (9c) into (9a)–(9b) leads to the following BDIDEs($\tilde{G}_D$) for $(p, \boldsymbol{v}) \in \mathbf{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$;

$$\begin{aligned}&p + \mathcal{R}^\bullet\boldsymbol{v} - \Pi^s\mathbf{T}^+(p, \boldsymbol{v}) \\ &\quad = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g - \Pi^d\boldsymbol{\varphi}_0 \ \text{ in } \Omega,\end{aligned} \tag{109}$$

$$\begin{aligned}&\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \mathbf{V}\mathbf{T}^+(p, \boldsymbol{v}) \\ &\quad = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g - \mathbf{W}\boldsymbol{\varphi}_0 \ \text{ in } \Omega.\end{aligned} \tag{110}$$

Let us prove the equivalence of the BDIDEs to the Dirichlet BVP

**Theorem 12.** *Let* $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $g \in L_2(\Omega)$ *and* $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega)$.

*(i)* *If a couple* $(p, \boldsymbol{v}) \in L^*_2(\Omega) \times \boldsymbol{H}^1(\Omega)$ *solves the Dirichlet BVP* (9a)–(9c), *then it solves the BDIDP system* (109)–(110).

*(ii)* *If a set* $(p, \boldsymbol{v}) \in L^*_2(\Omega) \times \boldsymbol{H}^1(\Omega)$ *solves the BDIDP system* (109)–(110), *then the pair of functions* $(p, \boldsymbol{v})$ *belongs to* $\boldsymbol{H}^{1,0}_*(\Omega; \mathring{\boldsymbol{\mathcal{A}}})$ *and solves the Dirichlet BVP* (9a)–(9c).

*(iii)* *The BDIDP system* (109)–(110) *is uniquely solvable for* $(p, \boldsymbol{v}) \in L^*_2(\Omega) \times \boldsymbol{H}^s(\Omega)$.

*Proof.* A solution of BVP (9a)–(9c) does exist and is unique due to Corollary 1 and provides a solution to BDIDP (109)–(110) due to Theorem 10. On the other hand, any solution of BDIDP (109)–(110) satisfies also (9a)–(9c) due to the same Theorem 10. The unique solvability of the BDIDP system (109)–(110) follows from the unique solvability of the BVP (9a)–(9c), see Theorem 2, and items (i) and (ii). □

The BDIDP can be written in the form

$$\mathscr{A}^{\tilde{G}_D}\mathcal{X} = \mathscr{F}^{\tilde{G}_D}, \tag{111}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system;

$$\mathcal{X} = (p, \boldsymbol{v}) \in L_2^*(\Omega) \times \mathbf{H}^s(\Omega)$$

and

$$\mathscr{A}^{\tilde{G}_D} := \begin{bmatrix} I_p + \mathcal{R}^{\bullet} I_{\boldsymbol{v}} - \Pi^s \mathbf{T}^+ \\ I_{\boldsymbol{v}} + \boldsymbol{\mathcal{R}} I_{\boldsymbol{v}} - \mathbf{V}\mathbf{T}^+ \end{bmatrix},$$

$$\mathscr{F}^{\tilde{G}_D} := \begin{bmatrix} \mathring{\mathcal{Q}} \boldsymbol{f} + \dfrac{4\mu}{3} g - \Pi^d \boldsymbol{\varphi}_0 \\ \boldsymbol{\mathcal{U}} \boldsymbol{f} - \boldsymbol{\mathcal{Q}} g - \mathbf{W} \boldsymbol{\varphi}_0 \end{bmatrix}.$$

The mapping properties of operators $\mathbf{V}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{U}}, \Pi^s$, and $\mathring{\mathcal{Q}}$ imply the membership $\mathscr{F}^{\tilde{G}_D} \in \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{1/2}(\partial\Omega)$, and the operator $\mathscr{A}^{\tilde{G}_D} : L_2^*(\Omega) \times \mathbf{H}^s(\Omega) \to \mathbf{H}_*^{1,0}(\Omega; \mathring{\boldsymbol{\mathcal{A}}}) \times \mathbf{H}^{1/2}(\partial\Omega)$ is continuous. By Theorem 12, it is also injective.

## Conclusions

In this article, we considered Dirichlet BVP for a compressible Stokes system with variable viscosity and the right-hand side functions from $\mathbf{L}_2(\Omega)$ and $L_2(\Omega)$, respectively and the Dirichlet data from space $\mathbf{H}^{1/2}(\partial\Omega)$. It was shown that BVP can be equivalently reduced to a direct united boundary-domain integro-differential problem, or to a united BDIDEs. This implies unique solvability of the BDIDP/BDIDEs with the right-hand sides generated by the considered BVP. The invertibility of the associated operators in the corresponding Sobolev spaces can also be proved. In similar way one can investigate BDIDP / BDIDE for Dirichlet problems in exterior domains, the incompressible Stokes system, and the BDIEs of the compressible Stokes system formulated and analysed in Mikhailov (2005), Dagnaw Mulugeta and Ayele Tsegaye (2017), Mikhailov and Woldemicheal Zenebe (2019), and Ayele Tsegaye and Dagnaw Mulugeta (2021).

## Acknowledgments

The authors would like to thank Addis Ababa University and the International Science Program (ISP), Sweden. The first author thanks Wolkite University for the opportunities and financial support provided.

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