SINET: Ethiop. J. Sci., 46(3): 345-355, 2023 ISSN: 0379–2897 (PRINT)

eISSN: 2520–7997

Date received: 20 May 2023; Date revised: 10 March 2024; Date accepted: 20 March 2024
DOI: https://dx.doi.org/10.4314/sinet.v46i3.11

Congruences and Filters of BH-Lattices

K. Venkateswarlu [*1], Mekonnen Mamo Elema [2], Yibeltal Yitayew Tessema [2]

[1] Department of computer science and systems engineering, college of engineering, Andhra University, Visakhapatnam, India. E -mail: drkvenkateswarlu@gmail.com
[2] Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia. E-mail: mekonnenmamo@hu.edu.et; yibeltal.yitayew@aau.edu.et

ABSTRACT: In this paper, we introduce the concept of filters in BH-lattices and furnish certain examples. We obtain certain basic properties of BH-lattices. Further, we characterize the filter generated by a given sub set of a BH-lattice. Also we prove that, the set of all filters with set inclusion forms a Heyting algebra. We obtain a one to one correspondence between the set of congruences and filter of BH-lattices.

AMS Mathematics Subject Classification (2020): 06B10, 06F05, 06D25, 18B35.

Keywords/phrases: Boolean algebra, Heyting algebra, l-group, Brouwer-Heyting (BH) monoid, BH-lattice, filter.

INTRODUCTION

K.L.N.Swamy [13] initiated the study of BH-monoids by generalizing the notion of DRl-semigroups. He obtained certain properties concerning this class. He also obtained decomposition theorems for both BH-monoids and BH-lattices. i.e; every BH-monoid L is a direct product of po-group and a BH-monoid with greatest element if and only if $e \to x$ is invertible and $x \to x = e, \forall x \in L$. Moreover, every BH-lattice L is a direct product of a commutative l-group and a BH-lattice with greatest element.

In this article we introduce the notion of filter in BH-lattices, furnish examples and prove certain properties of filters. We obtain the basic properties concerning congruences and as well as for filters. Also we prove one to one correspondence between the set of congruences and filters of BH-lattices which gives more insight for constructing quotient BH-lattices. Finally, we obtain a one to one correspondence between the set of filters and the set of all congruences in a Heyting algebra as a special case. We prove that if L be a BH-lattice with the special property $e \leq (x \to y) \vee (y \to x)$ and subdirectly irreducible, then L is totally ordered.

PRELIMINARIES

In this section, we recall certain definitions and results concerning Heyting algebra, l-group and Brouwer-Heyting lattices which will be used in the sequel.

Definition 2.1. A bounded lattice $(L, \vee, \wedge)$ is called a Heyting algebra if for any given elements a and b in L, there is a greatest x such that $x \wedge a \leq b$.

Remark 2.1. The greatest element x is denoted by $a \to b$. Clearly $a \to b$is unique.

Lemma 2.1. Every Boolean algebra is a Heyting algebra, with $a \to b$given by $a' \vee b$.

Theorem 2.1. Let L be a Heyting algebra and $x, y, z \in L$. Then the following hold

1. $y \leq x \to y$
2. $1 = x \to 1$
3. $x \leq y \Leftrightarrow x \to y = 1$.
4. $x \wedge (x \to y) = x \wedge y$
5. $y \wedge (x \to y) = y$ and $((x \wedge y) \to x) \wedge z = z$
6. $y = 1 \to y$
7. If $x \leq y$, then $(z \to x) \leq (z \to y)$

Theorem 2.2. Let L be a Heyting algebra and $x, y, z \in L$. Then the following hold

1. If $x \leq y$, then $(y \to z) \leq (x \to z)$
2. $x \to (y \to z) = (x \wedge y) \to z = (x \to y) \to (x \to z)$
3. $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$
4. $(x \to y) \wedge z = ((z \wedge x) \to (z \wedge y)) \wedge z$

*Author to whom correspondence should be addressed.

5. L is a bounded distributive lattice.

Note: An equivalent definition of 2.1 is

Definition 2.2. [12] A non- empty set L with three binary operations $\wedge$, $\vee$ and $\rightarrow$ and two distinguished elements 0 and 1 is a Heyting algebra if the following conditions hold:

(H_1) $(L, \vee, \wedge, 0, 1)$ is a lattice with 0, 1

(H_2) $x \wedge (x \rightarrow y) = x \wedge y$

(H_3) $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$

(H_4) $(x \wedge y) \rightarrow x = 1$.

Definition 2.3. [2, 8] A partially ordered group (po-group) is a non-empty set G with binary operation . and binary relation $\leq$ such that (G, .) is a group, $(G, \leq)$ is a poset and the following axioms are satisfied.

1. $x \leq y \Rightarrow xz \leq yz$,
2. $x \leq y \Rightarrow zx \leq zy, \forall x, y, z \in G$.

Definition2.4. [2, 8] An l-group is a po-group where $(G, \leq)$ is a lattice.

Lemma2.2. [2, 8] Let G be a po-group. Then the following correlations are true for all elements $x, y, z, t \in G$

1. $x \leq y \Rightarrow z^{-1}xz \leq z^{-1}yz$;
2. $x \leq y \Leftrightarrow y^{-1} \leq x^{-1}$;
3. $x \leq y, z \leq t \Rightarrow xz \leq yt$.

Theorem2.3. [8] In any l-group G

1. $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$ and $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$
2. $x(x \wedge y)^{-1}y = x \vee y$ and $x(x \vee y)^{-1}y = x \wedge y$.

Theorem2.4. [8] In any l-group G, the lattice $(G, \vee, \wedge)$ is distributive.

Definition 2.5. [13] A system $(L, \circ, e, \leq, \rightarrow)$ is a Brouwer-Heyting (for short BH) monoid if

1. $(L. \circ, e)$ is a commutative semigroup with identity "e"
2. $(L, \leq)$ is a partially ordered set and $\rightarrow$ is a binary operation on L such that for all x, a, b in L, $(x \circ b) \leq a \Leftrightarrow x \leq (a \rightarrow b)$.

Example 2.1. Let $(G, \circ, e, \leq)$ is a commutative po-group. Define $a \rightarrow b = a \circ b^{-1}$

Example 2.2. $(B, \vee, \wedge, 0, 1)$ is a Boolean algebra. Let $\circ = \wedge$, $\leq$ defined by $a \leq b$ if $a \wedge b = a$, $e = 1$, $a \rightarrow b = a \vee b'$.

Example 2.3. Let $(G, \vee, \wedge, 1)$ be a Heyting algebra. Let $o = \wedge$, $e = 1$, $\leq$ is the lattice order, $a \rightarrow b$ is the largest x such that $b \wedge x \leq a$. That is, the new arrow operation is defined in terms of the arrow operation in the Heyting algebra by $a \rightarrow_N b = b \rightarrow a$.

Theorem2.5. [13] In BH monoid $(L, o, e, \leq, \rightarrow)$ the following hold for x, y, $z \in L$

1. $y \leq z \Rightarrow x \circ y \leq x \circ z$
2. $x \leq (x \circ y) \rightarrow y$
3. $(x \rightarrow y) \circ y \leq x$
4. $z \rightarrow (x \circ y) = (z \rightarrow y) \rightarrow x = (z \rightarrow x) \rightarrow y$
5. $e \rightarrow e = e$
6. $x \rightarrow e = x$
7. $e \leq x \rightarrow x$
8. $x \leq y \Rightarrow x \rightarrow z \leq y \rightarrow z$
9. $x \leq y \Rightarrow z \rightarrow y \leq z \rightarrow x$
10. $y \leq x \Leftrightarrow e \leq x \rightarrow y$
11. $(x \rightarrow y) \circ (y \rightarrow z) \leq x \rightarrow z$
12. If $x \vee y$ exists, then $(z \circ x) \vee (z \circ y)$ exists for any z and $z \circ (x \vee y) = (z \circ x) \vee (z \circ y)$
13. If $x \vee y$ exists, then $(z \rightarrow x) \wedge (z \rightarrow y)$ exist and $z \rightarrow (x \vee y) = (z \rightarrow x) \wedge (z \rightarrow y)$
14. If $x \wedge y$ exists, then $(x \rightarrow z) \wedge (y \rightarrow z)$ exists and $(x \wedge y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

Definition2.6. [13] A BH monoid $(L, \circ, e, \leq, \rightarrow)$ is a BH-lattice if

1. $(L, \leq)$ is a lattice with glb and lub denoted by $\wedge$ and $\vee$ respectively
2. $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z), \forall x, y, z \in L$
3. $((y \rightarrow x) \wedge e) \circ x = x \wedge y, \forall x, y \in L$.

Theorem 2.6. [13] A lattice $(L, \vee, \wedge, o, e, \rightarrow)$, where $(L, \circ, e)$ is a commutative monoid and $\rightarrow$ is a binary operation on L, is a BH-lattice if and only if

1. $(y \rightarrow x) \circ x \leq y$
2. $(x \wedge z) \rightarrow y \leq x \rightarrow y$
3. $x \leq (x \circ y) \rightarrow y, \forall x, y \in L$
4. $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z), \forall x, y, z \in L$
5. $((y \rightarrow x) \wedge e) \circ x = x \wedge y, \forall x, y \in L$

Since $\wedge$ is commutative, it follows that $((y \rightarrow x) \wedge e) \circ x = ((x \rightarrow y) \wedge e) \circ y, \forall x, y \in L$.

Definition 2.7. [2] A class of algebras is said to be equational if it can be defined by a set of identities.

Hence we have the following remark.

Remark 2.2. [13] Theorem 2.6 shows that BH-lattices are equational.

Example 2.4. [9] Heyting algebra and l-groups are BH-lattices. Heyting algebra is an example of bounded BH-lattice while the unbounded l-groups are examples of unbounded BH-lattices.

Theorem 2.7. [13], [9] Let $(L, \vee, \wedge, o, e, \rightarrow)$ be a BH-lattice and a, b, $x \in L$, then the following hold.

1. $x \rightarrow x = e$.
2. L is distributive
3. $(x\vee y)\circ(x\wedge y) = x\circ y$
4. $x = x\circ e = (x\vee e)\circ(x\wedge e)$
5. If $e \leq x$, then x is invertible
6. $e \vee x$ is invertible
7. If x is invertible, then $e \rightarrow x$ is inverse of x.
8. If y is invertible, then $x \rightarrow y = x\circ(e\rightarrow y)$
9. If x and y are invertible, then [a] xoy is invertible and $(e\rightarrow x)o(e\rightarrow y)$ is the inverse of xoy [b] $x\rightarrow y$ is invertible and $y \rightarrow x$ is the inverse of $x\rightarrow y$.
10. $e \rightarrow x$ is invertible.
11. If G is the set of all invertible elements, then G is an l-group.

Further Properties of BH-lattics

Theorem 3.1. Let L be a BH-lattice and a, b, x, $y \in L$ such that $a \leq b$ and $x \leq y$, then $aox \leq boy$

Proof. Suppose $a \leq b$ and $x \leq y$. By 1 in Theorem 2.5 $a \leq b$ implies $aox \leq box$ and $x \leq y$ implies $box \leq boy$. Hence $aox \leq boy$.

Notation: For a BH-lattice L and x, $y \in L$, $x * y = (x\rightarrow y)\wedge(y\rightarrow x)$.

Theorem 3.2. Let L be any BH-lattice. For any x, y, $z \in L$, the following properties hold.

1. $x *y \leq e$ with equality iff $x = y$.
2. $(x\vee y) *(x\wedge y) = x *y$
3. $(x*y)o(y*z) \leq x*z$
4. $(x\rightarrow y)*e \geq x*y$
5. $x \leq e \Rightarrow x*e= x$.

Proof. By 13 and 14 of Theorem 2.5,
$x\wedge y\rightarrow x\vee y = [(x\wedge y)\rightarrow x] \wedge [(x\wedge y)\rightarrow y]$
$= [(x \rightarrow x) \wedge (y\rightarrow x)]\wedge[(x\rightarrow y) \wedge (y\rightarrow y)]$
$= e\wedge(y\rightarrow x) \wedge (x\rightarrow y)$ (i)
And also by 1, 3 and 12 of Theorem 2.5, $(x\vee y)o\{(x\rightarrow y)\wedge(y\rightarrow x)\} = [xo\{(x\rightarrow y)\wedge(y\rightarrow x)\}]\vee[yo\{(x\rightarrow y)\wedge(y\rightarrow x)\}] \leq \{xo(y\rightarrow x)\} \vee \{yo(x\rightarrow y)\} \leq x \vee y$. That is $(x \vee y)o\{(x \rightarrow y) \wedge (y \rightarrow x)\} \leq (x \vee y)$
$\Rightarrow (x\rightarrow y)\wedge(y\rightarrow x)\leq (x\vee y)\rightarrow(x\vee y)= e$
$\Rightarrow (x\rightarrow y)\wedge(y\rightarrow x)\wedge e= (x\rightarrow y)\wedge(y\rightarrow x)$ (ii)
Hence from (i) and (ii) we have $x\wedge y\rightarrow x\vee y = (x\rightarrow y)\wedge(y\rightarrow x)$. Since $x\wedge y \leq x\vee y$, by 8 of Theorem 2.5, $x*y = (x\rightarrow y)\wedge(y\rightarrow x) = (x\wedge y)\rightarrow(x\vee y)\leq e$. If $x = y$, then by 1 of Theorem 2.7, $x*y = e$. Conversely if $x*y = e$, then $x*y = (x\wedge y)\rightarrow(x\vee y)= e \Rightarrow e \leq (x\wedge y)\rightarrow(x\vee y) \Rightarrow (x \vee y) \leq (x \wedge y) \Rightarrow x \vee y = x \wedge y$ and consequently it follows that $x = y$. Hence (1) holds. Since $(x \vee y) * (x \wedge y) = [(x \vee y) \wedge (x \wedge y)] \rightarrow [(x \vee y) \vee (x \wedge y)] = (x \wedge y)\rightarrow (x \vee y) = x * y$. Hence (2) holds. Again $(x\rightarrow y) \wedge (y\rightarrow x)\leq x\rightarrow y, y\rightarrow x$ and $(y\rightarrow z) \wedge (z\rightarrow y)\leq y\rightarrow z, z\rightarrow y$.$(x * y)o(y* z) = [(x\rightarrow y)\wedge(y\rightarrow x)]o[(y\rightarrow z)\wedge(z\rightarrow y)]\leq (x\rightarrow y)o(y\rightarrow z)$ (by Theorem 3.1) $\leq x\rightarrow z$ (by 11 of Theorem 2.5). That is $(x*y)o(y*z) \leq x\rightarrow z$. By a similar argument $(x*y)o(y*z) \leq z\rightarrow x$. So it follows that $(x * y)o(y* z) \leq (x\rightarrow z) \wedge (z\rightarrow x) = x * z$. Hence (3) holds. Further, by 4 and 9 of Theorem 2.5, $(x\rightarrow y)oy \leq x$. This implies that $y\rightarrow x \leq y\rightarrow(x\rightarrow y)oy = e\rightarrow(x\rightarrow y)$. Hence $x * y = (x\rightarrow y) \wedge (y\rightarrow x) \leq (x\rightarrow y) \wedge (e\rightarrow(x\rightarrow y)) = (x\rightarrow y) * e$. So $x * y \leq (x\rightarrow y) * e$. Finally, let $x\leq e \Rightarrow e\rightarrow e = e \leq e\rightarrow x$ (by 9 of Theorem 2.5). $\Rightarrow x = e\wedge x \leq x\wedge(e\rightarrow x) = x *e \leq x$. Hence $x *e = x$.

Definition 3.1. Let L is a BH-lattice. For $n \in N$ and $x \in L$, define $x^n = xoxo\ldots ox$ (n factors).

Theorem 3.3. In a BH-lattice L, for any $n \in N$, $t^n \leq e \Leftrightarrow t \leq e$.

Proof. Let $t^n \leq e$. Repeatedly applying the associative and distributive properties of o over the operation $\vee$, we have $(t \vee e)^n = t^n \vee t^{n-1} \vee \ldots \vee t \vee e = (t^n \vee e) \vee (t^{n-1} \vee \ldots \vee t \vee e) = t^{n-1} \vee \ldots \vee t \vee e = (t \vee e)^{n-1}$. That is $(t\vee e)o(t\vee e)^{n-1} = (t\vee e)^{n-1}$ $\Rightarrow (t\vee e) \leq [(t\vee e)o(t\vee e)^{n-1}] \rightarrow (t\vee e)^{n-1} = (t\vee e)^{n-1} \rightarrow (t\vee e)^{n-1} = e \Rightarrow t\vee e = e \Rightarrow t \leq e$.
Hence $t \leq e$ (by 2 of Theorem 2.5).
The converse follows from Theorem 3.1 by induction.

Filters (i)

In this section we introduce the notion of filters in BH-lattices, furnish examples and prove certain properties of filters.

Definition 4.1. A non empty subset F of a BH-lattice L is called a filter of L iff

1. x, $y \in F \Rightarrow xoy \in F$
2. $x*e \leq y*e$ and $x \in F \Rightarrow y \in F$.

Theorem 4.1. Let F be a filter of BH-lattice L. Then $e \in F$.

Proof. Since F is nonempty subset of L,

there exists an element $x \in F$. By 13 and 14 of Theorem 2.5, $x \wedge e \to x \vee e = [(x \wedge e) \to x] \wedge [(x \wedge e) \to e]$
$= [(x \to x) \wedge (e \to x)] \wedge [(x \to e) \wedge (e \to e)]$
$= e \wedge (e \to x) \wedge x$
And also by 1, 3 and 12 of Theorem 2.5, $(x \vee e) o \{x \wedge (e \to x)\} = [x o \{x \wedge (e \to x)\}] \vee [e o \{x \wedge (e \to x)\}] \leq \{x o (e \to x)\} \vee (e o x) \leq x \vee e$. That is $(x \vee e) o \{x \wedge (e \to x)\} \leq (x \vee e) \Rightarrow x \wedge (e \to x) \leq (x \vee e) \to (x \vee e) = e \Rightarrow x \wedge (e \to x) \wedge e = x \wedge (e \to x)$

Hence by 8 of Theorem 2.5, $x * e = x \wedge (e \to x) = (x \wedge e) \to (x \vee e) \leq e$. So, by the definition of a filter $e \in F$.

Theorem 4.2. Let F be a filter of BH-lattice L. Then the following properties hold.

1. If $x, y \in F$ and $x \wedge y \leq a \leq x \vee y$, then $a \in F$.
2. F is closed under all the operations in L.

Proof. Let $x \in F$. By 1 of Theorem 3.2 and 9 of Theorem 2.5, $x * e \leq e \Rightarrow e \to e = e \leq e \to (x * e)$. This implies that $e \wedge (x * e) = x * e \leq (x * e) \wedge (e \to (x * e)) = (x * e) * e$. Hence $x * e \in F$.

Let $x, y, a \in L$ such that $x \wedge y \leq a \leq x \vee y$. Then by 1, 9 and 13 of Theorem 2.5 and 1 and 5 of Theorem 3.2, $a * e = a \wedge (e \to a) \geq a \wedge (e \to (x \vee y)) \geq (x \wedge y) \wedge (e \to (x \vee y)) = (x \wedge y) \wedge (e \to x) \wedge (e \to y) = (x * e) \wedge (y * e) \geq (x * e) o (y * e) = [(x * e) o (y * e)] * e$. Further, for $x, y \in F$, $x * e, y * e \in F$, and hence $(x * e) o (y * e) \in F$. Hence $a \in F$. By Theorem 3.1 and 1, 3 and 5 of Theorem 3.2, we have $[(x * e) o (y * e)] * e = (x * e) o (y * e) \leq x * y = (x * y) * e$. Thus it follows that $x * y \in F$. Furthermore, by 4 and 9 of Theorem 2.5, $(x \to y) o y \leq x$. This implies that, $y \to x \leq y \to (x \to y) o y = e \to (x \to y)$. So that $(x \to y) * e = (x \to y) \wedge (e \to (x \to y)) \geq (x \to y) \wedge (y \to x) = x * y = (x * y) * e$. Hence by definition it follows that $x \to y \in F$. Thus (2) holds.

Corollary 4.1. A filter F of a BH-lattice L is a sub BH-lattice of L.

A sub BH-lattice H of a BH-lattice may not be a filter of L. For this take L the addative group of real numbers with the usual order and H the set of integers.

Corollary 4.2. If x is an element of a filter F of a BH-lattice L, then the interval $[x \wedge e, x \vee e] \subseteq F$.

Definition 4.2. Every BH-lattice L is a filter of itself, called the improper filter; all other filters will be called proper.

Theorem 4.3. For any BH-lattice L, $E = \{e\}$ is a filter of L.

Proof. Clearly $e * e = e$. Let $x \in L$ such that $e * e \leq x * e \Rightarrow e \leq x * e \leq e$ (by Theorem 3.2) $\Rightarrow x \wedge (e \to x) = e \Rightarrow e \leq x$ and $e \leq e \to x$ (by the definition of $\wedge$) $\Rightarrow e \leq x$ and $x \leq e$ (by defining condition of $\to$). Hence $x = e \in E$. Thus E is a filter.

Definition 4.3. The filter $E = \{e\}$ is called the trivial filter. (ii)

Theorem 4.4. Z is the only non trivial filter in the BH-lattice $(Z, +, 0, \leq, -)$, where $\leq$ is the usual ordering.

Proof. Let F be a filter of BH-lattice $(Z, +, 0, \leq, -)$ and $0 \neq x_0 \in Z$. If there exists an element $y \in F$ such that $y * 0 \leq x_0 * 0$, then $x_0 \in F$ and $Z \subseteq F$ and the proof is over. Suppose not. This implies, for all $y \in F$, $x_0 * 0 \leq y * 0$. $\Rightarrow x_0 \wedge (-x_0) \leq y \wedge (-y)$ (because in l-group $x \to y = x o y^{-1}$). $\Rightarrow -|x_0| \leq -|y| \Rightarrow |y| \leq |x_0|$.

Since F is a filter and $y \in F$, $|y| \in F$ and further by Theorem 4.2, for all $n \in N$, $n|y| \in F$. Hence $n|y| \leq |x_0|$, $\forall n \in N$ which is impossible. Hence there exists an element $y \in F$ such that $y * 0 \leq x_0 * 0$. Thus $x_0 \in F$.

Example 4.1. Let $L = \{0, x, y, z, 1\}$ be the lattice given by the Hasse diagram in fig.1. For $o = \wedge$ and $\to$ defined by the table 1.

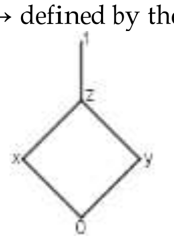

Figure 1: Hasse diagram of L

$\to$	0	x	y	z	1
0	1	y	x	0	0
x	1	1	x	x	x
y	1	y	1	y	y
z	1	1	1	1	z
1	1	1	1	1	1

Table1: definition of $\to$

Then, L is a BH-lattice, and $F_1 = \{x, z, 1\}$, $F_2 = \{y, z, 1\}$ and $F_3 = \{1\}$ are all filters of L.

Example 4.2. Let $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$, positive factors of 30. It is a Boolean algebra under divisibility. That is, for $x, y \in B$, $x \leq y$ means x divides y,

$x \vee y = \mathrm{lcm}\{x, y\}$, $x \wedge y = \gcd\{x, y\}$ and $1' = 30$, $2' = 15$, $3' = 10$, $5' = 6$. Then clearly $F_1 = \{2, 6, 10, 30\}$, $F_2 = \{3, 6, 15, 30\}$ and $F_3 = \{30\}$ are filters of B.

Theorem 4.5. The set of filters of a BH-lattice are closed under arbitrary intersection.

Proof. Follows from the definition of filter.

Consider the filters F_1 and F_2 in example 4.2 above. Clearly $F_1 \cup F_2$ is not a filter. Hence we have the following.

Theorem 4.6. Let L be a BH-lattice and for all $x \in L$, $e \to (e \to x) = x$. Let F and G be filters of L. Then $F \cup G$ is a filter of L if and only if $G \subseteq F$ or $F \subseteq G$.

Proof. Let $F \cup G$ be a filter of L. Suppose that $F \not\subseteq G$ and $G \not\subseteq F$. Thus there exists elements $x \in F - G$ and $y \in G - F$. Hence $xoy \in F \cup G$. And consequently $xoy \in F$ or $xoy \in G$. If $xoy \in F$, then $x \to xoy = e \to y \in F$. Thus $e \to (e \to y) = y \in F$, which is a contradiction. If $xoy \in G$, then $y \to xoy = e \to x \in G$. Thus $e \to (e \to x) = x \in G$, which is also a contradiction. In both cases there is a contradiction. Hence $G \subseteq F$ or $F \subseteq G$. The converse is trivial.

Corollary 4.3. If a BH-lattice L is an l-group and F and G are filters of L, then $F \cup G$ is a filter of L if and only if $G \subseteq F$ or $F \subseteq G$.

Theorem 4.7. Let F be a non empty subset of BH-lattice L such that

1. $x \in F$ and $x*e \leq y*e \Rightarrow y \in F$
2. If $x \in F$ and $y \to x \in F$, then $y \in F$.

Then F is a filter of L. Moreover, if L is bounded above and F is a filter of L, then F satisfies the conditions in (1) and (2).

Proof. Let a non empty subset F of the BH-lattice L satisfies the conditions (1) and (2). Since F is non-empty, there exists an element, say $x \in F$. Then $x*e \leq e = e*e$. So that $e \in F$. Let $x, y \in F$. By 4 of Theorem 2.5, $z \to (xoy) = (z \to x) \to y$. For $z = xoy$, by condition (2) in the hypothesis, we have $e = (xoy) \to (xoy) = ((xoy) \to x) \to y \in F$. Hence $(xoy) \to x \in F$. Thus $(xoy) \in F$. Therefore F is a filter of L.

Now let L is bounded above and F is a filter of L. If there exists an element $x \in L$ such that $e < x$, then using (1) of Theorem 2.5 the elements of the sequence $\{x^{2^n}\}$ are all distinct and the sequence is strictly increasing. And this contradict the given hypothesis that L is bounded above. Hence e is the upper bound of L and by Theorem 4.1, $e \in F$. Let $x, y \to x \in F$. Then $x \wedge y = [(y \to x) \wedge e]ox = xo(y \to x) \in F$. Moreover, by Theorem 2.6 and Theorem 3.2, $[xo(y \to x)]*e = xo(y \to x) \leq y = y*e$. Hence $y \in F$.

Theorem 4.8. Let L be a BH-lattice and F be a filter of L. Then, for x, y, $z \in L$ the following hold.

1. $x*y \in F \Leftrightarrow x \to y, y \to x \in F$
2. $x*y, y*z \in F \Rightarrow x*z \in F$
3. $x, y \to x \in F \Rightarrow x \wedge y \in F$.

Proof. Follow from the definition of filter, Theorem 3.2, Theorem 4.1 and Theorem 4.2.

Theorem 4.9. Let L be a BH-lattice, F a filter of L, $x \in L$. Then

1. $x \in F \Leftrightarrow (x \vee e), (x \wedge e) \in F$
2. $x \in F \Leftrightarrow (x*e) \in F$.

Proof. Follow from the definition of filter and Theorem3.2.

Theorem 4.10. Let L be a BH-lattice, $x, y \in L$ and F be a filter of L. Then

1. $(x \vee y)*e \leq (x*e) \vee (y*e)$
2. If $x \vee y \in F$, then $(x*e) \vee (y*e) \in F$.

Proof. By 13 of Theorem 2.5 and distributive property, $(x \vee y)*e = (x \vee y) \wedge (e \to (x \vee y)) = (x \vee y) \wedge \{(e \to x) \wedge (e \to y)\} = \{x \wedge ((e \to x) \wedge (e \to y))\} \vee \{y \wedge ((e \to x) \wedge (e \to y))\} = \{(x*e) \wedge (e \to y)\} \vee \{(y*e) \wedge (e \to x)\} \leq (x*e) \vee (y*e)$. Hence (1) holds. Thus by Theorem 3.1 and Theorem 3.2 in (1), we obtain that $(x \vee y)*e \leq (x*e \vee y*e)*e$. Hence $x \vee y \in F$ implies that $(x*e) \vee (y*e) \in F$.

Theorem 4.11. Let L be a BH-lattice bounded above and F be a filter of L. Then for x, y, $z \in L$, the following hold.

1. $x \to y, y \to z \in F \Rightarrow x \to z \in F$
2. $x \to y, yoz \in F \Rightarrow xoz \in F$
3. $x, z \in F, x \leq y \to z \Rightarrow y \in F$
4. $x \leq y \Rightarrow y \to x \in F$.

Proof. Let L be a BH-lattice bounded above and F be a filter of L. By 11 of Theorem 2.5 and definition of filter, $x \to y, y \to z \in F$ implies that $(x \to y)o(y \to z) \leq x \to z$ and $(x \to y)o(y \to z) \in F$. Hence $x \to z \in F$. Suppose that, $x \to y$, $yoz \in F$. Then by 1 and 3 of Theorem 2.5, $(x \to y)o(yoz) = ((x \to y)oy)oz \in F$ and $((x \to y)oy)oz \leq xoz$. Hence $xoz \in F$. Further, by Theorem 4.7 and the definition, $x, z \in F, x \leq y \to z \Rightarrow y \to z, z \in F$. Hence $y \in F$. Finally to prove (4), let $x \leq y \Rightarrow e \leq y \to x$. Hence $y \to x \in F$.

Theorem 4.12. Let L be a BH-lattice which is bounded above. Let F and G be filters of L. Then $F \cup G$ is a filter of L if and only if $G \subseteq F$ or $F \subseteq G$.

Proof. Let $F \cup G$ is a filter of L. Suppose that $F \not\subseteq G$ and $G \not\subseteq F$. This implies there exist elements $x \in F - G$ and $y \in G - F$. Hence $x*y = (x \to y) \wedge (y \to x) \in F \cup G$. It follows that $(x \to y) \wedge (y \to x) \in F$ or $(x \to y) \wedge (y \to x) \in G$. If $(x \to y) \wedge (y \to x) \in F$, then $y \to x \in F$. Thus by Theorem 4.7, $y \in F$, which is a contradiction. If $x*y \in G$, then $x \to y \in G$. Thus by the same argument as above, $x \in G$, which is also a contradiction. In both cases there is a contradiction. Hence $G \subseteq F$ or $F \subseteq G$. The converse is trivial.

Theorem 4.13. Let L be a BH-lattice which is bounded above. Then $L^e = \{x \in L : e \to x = e\}$ is a filter of L.

Proof. Let $x, y \in L^e$. Then $e \to xoy = (e \to x) \to y = e \to y = e$. Hence $xoy \in L^e$. Let $x \in L^e$ and $x \leq y$. Then $x \leq y \leq e$. Thus $e = e \to e \leq e \to y \leq e \to x = e$. Hence $y \in L^e$.

Theorem 4.14. Let L be a BH-lattice with unity 1 and bounded above. Then $L_1 = \{x \in L : x \to 1 = e\}$ is a filter of L.

Proof. Since L_1 is a sub BH-lattice of L [[9], Theorem 2.25], it follows from 8 of Theorem 2.5.

Definition4.4. Let L be a BH-lattice. A non empty set $H \subseteq L$ is called

1. multiplicatively closed if $xoy \in H$, whenever x and y be long to H.
2. implicatively closed if $x \to y$ H, whenever x and y belong to H.

Example 4.3. For any BH-lattice L, $L^e = \{x \in L : e \to x = e\}$ is both multiplicatively and implicatively closed set.

Definition 4.5. Given a BH-lattice L and a non empty set $X \subseteq L$. $F(X)$ or $[X)$ denote the least filter containing X, i.e., the intersection of all filters containing X, called the filter generated by X. The elements of X are called generators of the filter $[X)$.

If $X = \{x_1, \ldots, x_n\}$, then the filter $[X)$ is simply denoted by $[x_1, x_2, \ldots, x_n)$ and said to be finitely generated. We can write $F(a)$ instead of writing $F(\{a\})$, when $X = \{a\}$ and we call it the principal filter generated by a.

Theorem 4.15. Let L be a BH-lattice and $\emptyset = X \subseteq L$. Then the filter generated by X is given by, $F(X) = \{x \in L : (a_1*e)o(a_2*e)o\ldots o(a_k*e) \leq x*e,\ a_i \in X,\ k \in N\}$, a_i's are not necessarily distinct.

Proof. Since for any $a \in X$, $a*e \leq a*e$, $X \subseteq F(X)$. Let $x, y \in F(X)$, then there exist elements $a_1, \ldots, a_k, b_1, \ldots, b_t \in X$ such that $(a_1*e)o(a_2*e)o\ldots o(a_k*e) \leq x*e$ and $(b_1*e)o(b_2*e)o\ldots o(b_t*e) \leq y*e$. Hence by Theorem 3.1 and distributive property of o over $\wedge$ (use twice), we have $\{(a_1*e)\ o\ (a_2*e)\ o \ldots o\ (a_k*e)\}\ o\ \{(b_1*e)\ o\ (b_2*e)\ o \ldots o\ (b_t*e)\} \leq (x*e)o(y*e) = \{x \wedge (e \to x)\}\ o\ \{y \wedge (e \to y)\} = (xoy) \wedge (yo(e \to x)) \wedge (xo(e \to y)) \wedge \{(e \to x)o(e \to y)\} \leq xoy \wedge [(e \to x)o(e \to y)]$. But by 4 and 11 of Theorem 2.5, $(e \to x)o(e \to y) \leq (e \to x)o(x \to yox) \leq e \to xoy$. Hence

$(x*e)o(y*e) \leq (xoy) \wedge (e \to (xoy)) = (xoy)*e$. Hence $xoy \in F(X)$. Let $x \in F(X)$ and $x*e \leq y*e$. Then clearly, $y \in F(X)$. Moreover, let F be any filter of L containing X. Then by Theorem 4.2 and Theorem 4.9, $F(X) \subseteq F$. Therefore it is the filter generated by X.

Corollary 4.4. Let L be a BH-lattice and $\emptyset = X \subseteq L$ such that X is implicatively closed and $e \in X$. Then $F(X) = \{x \in L : (a_1 \wedge a_2)o(a_3 \wedge a_4)o\ldots o(a_{k-1} \wedge a_k) \leq x*e, a_i \in X\}$, where a_i's are not necessarily distinct.

Proof. Directly follows from Theorem 4.15 and the definition of $*$.

Remark 4.1. For BH-lattice L, the filter generated by empty set is the intersection of all filters of L.

Corollary 4.5. For a BH-lattice L and $a \in L$, the filter generated by a is given by $F(a) = \{x \in L : (a*e)^n \leq x*e,\ n \in N\}$

Proof. Directly follows from Theorem 4.15.

Corollary 4.6. Let L be a BH-lattice and $\emptyset = X \subseteq L$ such that X is bounded above by e. Then $F(X) = \{x \in L : a_1 o a_2 o \ldots o a_k \leq x*e,\ a_i \in X\}$.

Proof. Follows from 9 of Theorem 2.5 and Theorem 4.15.

From the above results we observe that

1. if $a \in L$, $a \leq e$, then $F(a) = \{x \in L : a^n \leq x*e,\ n \in N\}$.
2. if L is a BH-lattice bounded above, then for any $\emptyset = X \subseteq L$, $F(X) = \{x \in L : x_1 o x_2 o \ldots o x_n \leq x,\ x_i \in X,$

$\forall i = 1, 2, 3, \ldots, n\}$ and $F(a) = \{x \in L : a^n \leq x,\ n \in N\}$. Moreover, if X is multiplicatively closed subset of L, then $F(X)$ reduces to the form $F(X) = \{x \in L : a \leq x,\ a \in X\}$.

Corollary 4.7. For a Heyting algebra H and $a \in H$, $\{x : a^n \leq x$, for some positive integer $n\}$ is the filter generated by a.

Corollary 4.8. In a commutative l-group G and $a \in G$ and $a \leq e$, $\{x : a^n \leq x \wedge x^{-1},\ n \in N\}$ is the filter generated by a.

Recall that a subdirectly irreducible algebra is an algebra that cannot be factored as a subdirect product of "simpler" algebras [3].

Theorem 4.16. Let L be a BH-lattice with the special property $e \leq (x \rightarrow y) \vee (y \rightarrow x)$, $\forall x, y \in L$. If L is subdirectly irreducible, then L is totally ordered.

Proof. Suppose L is subdirectly irreducible and let L is not totally ordered. Hence there exist a pair of incomparable elements $x, y \in L$. By 10 of Theorem 2.5, this implies that $(x \rightarrow y) \wedge e = e$ and $(y \rightarrow x) \wedge e = e$. Let $F_1 = F((x \rightarrow y) \wedge e)$ and $F_2 = F((y \rightarrow x) \wedge e)$. Let $a \in F_1 \cap F_2$. Then there exist positive integers m and n such that $\{(x \rightarrow y) \wedge e\}^m \leq a * e$ and $\{(y \rightarrow x) \wedge e\}^n \leq a * e$. Hence $[\{(x \rightarrow y) \wedge e\}^m] \vee [\{(y \rightarrow x) \wedge e\}^n] \leq a * e$. Since L is a distributive lattice and by the hypothesis given $[\{(x \rightarrow y) \wedge e\}] \vee [\{(y \rightarrow x) \wedge e\}] = \{(x \rightarrow y) \vee (y \rightarrow x)\} \wedge e = e$. For any $t, s \in L$ such that $t \vee s = e$, by 1 of Theorem 2.5 and Theorem 3.1, $s^2 \leq e$, $(t \vee s) o t \leq t$. Thus $t \vee s^2 \leq e$ and $(t \vee s) o t \vee s o s = t o t \vee s o t \vee s o t \vee s o s = (t \vee s) o (t \vee s) = e$. And $e = t o (t \vee s) \vee (s o s) \leq t \vee s^2$. Therefore $t \vee s^2 = e$. Therefore by induction we have $t \vee s^n = e$. By a similar argument by interchanging the role of t and s in this result, we obtain $t^m \vee s^n = e$. Now, taking $t = \{(x \rightarrow y) \wedge e\}$ and $s = \{(y \rightarrow x) \wedge e\}$, gives, $e = [\{(x \rightarrow y) \wedge e\}^m] \vee [\{(y \rightarrow x) \wedge e\}^n] \leq a * e \leq e$. Thus $a * e = e$ and hence $a = e$. Therefore L is subdirectly reducible.

Corollary 4.9. Any subdirectly irreducible Boolean algebra is a chain.

Corollary 4.10. Any subdirectly irreducible commutative l-group is a chain.

Proof. Let L be a BH-lattice and (L, o) is a group. Then by 1 of Theorem 2.5, 8 of Theorem 2.7 and Theorem 3.3, for any $x \in L$, $e = (x \vee x^{-1}) o (x \vee x^{-1})^{-1} = (x \vee x^{-1}) o (x^{-1} \wedge x) \leq (x \vee x^{-1})^2$. Hence $\{(x \vee x^{-1})^{-1}\}^2 \leq e$. Thus $(x \vee x^{-1})^{-1} \leq e$ and hence $e \leq x \vee x^{-1}$. Therefore for any commutative l-group L, and $x, y \in L$, $e \leq (x o y^{-1}) \vee (x o y^{-1})^{-1} = (x o y^{-1}) \vee (x^{-1} o y) = (x \rightarrow y) \vee (y \rightarrow x)$

Theorem 4.17. Let L be a BH-lattice, $x, y, z \in L$ such that $x, y, z \leq e$, Then

1. $(x \wedge y) o (x \wedge z) \leq x \wedge (y o z)$
2. $(x \vee y) o (x \vee z) \leq x \vee (y o z)$.

Proof. By 1 of Theorem 2.5, $x \leq e$, $y \leq e$ implies that $x o y \leq x \wedge y$. Hence by distributive property of o over $\wedge$, $(x \wedge y) o (x \wedge z) = (x o x) \wedge (y o x) \wedge (x o z) \wedge (y o z) \leq (x \wedge x) \wedge (y \wedge x) \wedge (x \wedge z) \wedge (y o z) \leq x \wedge x \wedge x \wedge (y o z) = x \wedge (y o z)$. Further, $(x \vee y) o (x \vee z) = (x o x) \vee (y o x) \vee (x o z) \vee (y o z) \leq (x \wedge x) \vee (y \wedge x) \vee (x \wedge z) \vee (y o z) \leq x \vee x \vee x \vee (y o z) = x \vee (y o z)$.

Theorem4.18. Let L be a BH-lattice, T be a filter of L and $a, b \in L$. Then

1. $F(T \cup \{a\}) = \{x \in L : t o (a * e)^n \leq x * e,\ t \in T,\ n \geq 0\}$, where for $x \in L$, $x^0 = e$.
2. $F(a) \wedge F(b) \subseteq F(a * e \wedge b * e) = F((a * e) o (b * e)) \subseteq F(a) \vee F(b)$
3. $F(a) \vee F(b) \supseteq F((a * e) \vee (b * e))$

Proof. The first one is trivial. For the rest two consider the following. Since $a * e = (a * e) * e$, it is obvious that $F(a) = F(a * e)$. Further, by Theorem 3.2, $(a * e) o (b * e) \leq (a * e) \wedge (b * e) \leq a * e,\ b * e$. Hence by Theorem 3.1 and induction we have $((a * e) o (b * e))^n \leq (a * e \wedge b * e)^n \leq (a * e)^n,\ (b * e)^n,\ n \in N$. Observe that, $F(a), F(b) \subseteq F((a * e) \wedge (b * e))$. Hence $F(a) \wedge F(b) \subseteq F(a), F(b) \subseteq F((a * e) \wedge (b * e)) \subseteq F((a * e) o (b * e))$. Let G be any filter of L such that $F(a), F(b) \subseteq G$ and let $x \in F((a * e) o (b * e))$. Then $((a * e) o (b * e))^n \leq x * e$, for some $n \in N$. Nevertheless, $a * e,\ b * e \in G$, thus $(a * e) o (b * e) \in G$. Therefore $x \in G$. Hence $F((a * e) o (b * e)) \subseteq F((a * e) \wedge (b * e))$, $F(a) \vee F(b)$. Hence (2) holds. Furthermore, as $a * e,\ b * e \leq (a * e) \vee (b * e)$, it follows that $(a * e)^n, (b * e)^n \leq ((a * e) \vee (b * e))^n$, $n \in N$. Hence it follows that $F((a * e) \vee (b * e))$

$\subseteq F(a*e)$, $F(b*e) \subseteq F(a) \vee F(b)$.

In 2 and 3 of Theorem 4.18 equality does not hold. To show this consider F_1 and F_2 in example 4.2. Observe that $F(2) = \{x \in B : (2*e)^n \leq x*e\} = \{x \in B : 2 \leq x\} = F_1 = \{2, 6, 10, 30\}$. Similarly $F(3) = F_2 = \{3, 6, 15, 30\}$. Then clearly $F(2) \vee F(3) = B = \{1, 2, 3, 5, 6, 10, 15, 30\}$ while $F((2*e) \vee (3*e)) = F(2 \vee 3) = F(6) = \{6, 30\}$. Additionally, $F(2) \wedge F(3) = \{6, 30\}$, while $F((2*e) \wedge (3*e)) = F(1) = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

Theorem 4.19. Let L be any BH-lattice, F and T be filters of L. Then $F \vee T = F(F \cup T) = \{x \in L : fot \leq x*e, f \in F, t \in T, f, t \leq e\}$ is the smallest filter containing $F \cup T$.

Proof. Trivially follows from Theorem 4.2 and Theorem 4.15.

Theorem 4.20. Let L be any BH-lattice. Then the set of filters ***F*** of L forms a complete bounded distributive lattice (under $\subseteq$).

Proof. Let L be any BH-lattice and F and T be filters of L. Take $F \wedge T = F \cap T$; $F \vee T = F(F \cup T)$.

Then clearly it is a bounded lattice with $0 = \bigcap F$ and $1 = L$. To show it is distributive it suffices to show $(F \vee G) \wedge (F \vee T) \subseteq F \vee (G \wedge T)$ or $F \wedge (G \vee T) \subseteq (F \wedge G) \vee (F \wedge T)$ for $F, G, T \in \boldsymbol{F}$. Let $x \in F \wedge (G \vee T)$. This implies that $x \in F$ and $x \in G \vee T$. Hence by Theorem 4.19 there exists $t \in T$, $g \in G$ such that $tog \leq x*e$, $t, g \leq e$ and $x \in F$. Then $((x*e) \vee t)o((x*e) \vee g) \leq (x*e) \vee (tog) = x*e$. Since $(x*e) \vee t \in F \cap T$, $(x*e) \vee g \in F \cap G$, $((x*e) \vee t)o((x*e) \vee g) \in (F \cap G) \vee (F \cap T)$. Hence $x \in (F \cap G) \vee (F \cap T)$. Therefore the lattice in theorem is distributive.

Let $\emptyset \neq X \subseteq L$. Any $x \in L$ is in $F(X)$ iff there exists elements $a_1, a_2, \ldots, a_n \in X$ such that $(a_1 * e)o(a_2 * e)o\ldots o(a_n * e) \leq x * e$. Therefore $x \in F(a_1, a_2, \ldots, a_n)$ and thus $F(X) = \bigcup\{F(Y) : Y \subseteq X, |Y| \leq n, n \in N\}$. Further, clearly for $X, Y \subseteq L$, $X \subseteq F(X)$, $X \subseteq Y$ implies that $F(X) \subseteq F(Y)$ and $F(F(X)) = F(X)$. Hence the mapping $X \rightarrow F(X)$ is an algebraic closure operator whose closed subsets are filters. Since the set of all closed subsets (with set inclusion as the partial ordering) is a complete lattice, the lattice in the theorem is complete. Hence $(F, \subseteq)$ is a complete bounded distributive lattice.

In any lattice $(P, \leq)$, for $a, b \in P$, the relative Pseudocomplement of a with respect to b is a maximal element c such that $a \wedge c \leq b$ and it is denoted by $b \rightarrow a$.

Theorem 4.21. Let L be a BH-lattice and T and K be filters of L. The relative pseudocomplement of T with respect to K is given by $K \rightarrow T = \{x \in L : (x*e) \vee (a*e) \in K, \forall a \in T\}$.

Proof. Let $H = \{x \in L : (x*e) \vee (a*e) \in K, \forall a \in T\}$. Since for any $a \in L$, $a*e \leq e$, $(e*e) \vee (a*e) = e \in K$, $e \in H$. So that H is non empty set. Let $x, y \in H$. Then for each $a \in T$, $(x*e) \vee (a*e)$, $(y*e) \vee (a*e) \in K$. Since $(x*e)o(y*e) \leq (xoy)*e$, by Theorem 4.17, $\{(x*e) \vee (a*e)\}o\{(y*e) \vee (a*e)\} \leq \{(x*e)o(y*e)\} \vee (a*e) \leq ((xoy)*e) \vee (a*e)$. Thus $((xoy)*e) \vee (a*e) \in K$ and hence $xoy \in H$. Now let $x \in H$ and $x*e \leq y*e$. Then $(x*e) \vee (a*e) \leq (y*e) \vee (a*e) \in K$. Hence $y \in H$ and consequently H is a filter.

If $x \in T \cap H$, then $x*e = (x*e) \vee (x*e) \in K$ and hence $x \in K$. Hence $T \cap H \subseteq K$. Furthermore, let C be any filter such that $T \cap C \subseteq K$. For $x \in C$ and for each $a \in T$, $x*e, a*e \leq \{(x*e) \vee (a*e)\}*e = (x*e) \vee (a*e) \in T \cap C \subseteq K$. Hence $x \in C$ implies that $x \in H$. Therefore H is the maximal element C such that $T \cap C \subseteq K$. Hence $H = K \rightarrow T$.

Corollary 4.11. For any BH-lattice L and the set of filters F, the system $(F, \subseteq)$ forms a Heyting algebra with $K \rightarrow T = \{x \in L : (x*e) \vee (a*e) \in K, \forall a \in T\}$, $K, T \in \boldsymbol{F}$.

Observe that for $K, T \in \boldsymbol{F}$, $K \rightarrow T = \vee\{C \in F : T \cap C \subseteq K\}$.

Theorem 4.22. If any BH-lattice L is totally order, then so does $(\boldsymbol{F}, \subseteq)$.

Proof. Let $F, T \in \boldsymbol{F}$ such that $F \neq T$. Then there exists an element $x \in F - T$ or $x \in T - F$. Suppose that $x \in F - T$. Then by Theorem 4.9, $x*e \in F - T$. Let $y \in T$. Then $y*e \in T$. If $y*e \leq x*e$, then $x \in T$ which is a contradiction. Thus $x*e \leq y*e$ and hence $y \in F$. Hence $T \subseteq F$. Therefore $(F, \subseteq)$ is totally ordered.

CONGRUENCERELATIONS

In this section we introduce the notion of congruence relations in BH-lattices and prove certain properties of filters.

Definition 5.1. An equivalence relation θ on a BH-lattice $(L, \circ, e, \leq, \to)$ is called a congruence relation on L if for (x, y), (c, d) $\in \theta$, $(x \wedge c, y \wedge d)$, $(x \vee c, y \vee d)$, (xoc, yod), $(x \to c, y \to d) \in \theta$. That is, a lattice congruence θ on L is a congruence on L if for (x, y), (c, d) $\in \theta$, (xoc, yod), $(x \to c, y \to d) \in \theta$.

For a congruence relation θ on BH-lattice L, the congruence class containing x is denoted by [x] or $[x]_\theta$ and the set of all congruence classes is denoted by L/θ.

Theorem 5.1. Let F be a filter and define $\theta^F = \{(x, y) \in L X L : xoh \leq y, yoh \leq x$ for some $h \in F\}$. Then θ^F is a congruence relation on L.

Proof. For any $x \in L$, $xoe \leq x$ and by Theorem 4.1, $e \in F$. so $(x, x) \in \theta^F$. Clearly θ^F is symmetric. Let (x, y), (y, z) $\in \theta^F$. Then there exist elements $h, h' \in F$ such that $xoh \leq y$, $yoh \leq x$, $yoh' \leq z$, $zoh' \leq y$. By 1 of Theorem 2.5, $(xoh)oh' \leq yoh' \leq z$ and $(zoh')oh \leq yoh \leq x$ and $hoh' = h'oh \in F$. So $(x, z) \in \theta^F$. Thus θ^F is an equivalence relation on L.

To establish the substitution properties, let (a, b), (x, y) $\in \theta^F$. Then there exist elements $h, h' \in F$ such that $aoh \leq b$ and $boh \leq a$, $xoh' \leq y$ and $yoh' \leq x$. Let $k = hoh' \in F$, then by Theorem 3.1, $(aoh)o(xoh') \leq boy$ and $(boh)o(yoh') \leq aox$. This implies that $(aox)o(hoh') \leq boy$ and $(boy)o(hoh') \leq aox$. Thus, $(aox, boy) \in \theta^F$. Further, let $j = h \wedge h'$. By 1 and 14 of Theorem 2.5, $aoj \leq aoh \leq b$, $boj \leq boh \leq a$ and $xoj \leq xoh' \leq y$, $yoj \leq yoh' \leq x$. This implies that $a \leq b \to j$, $b \leq a \to j$, $x \leq y \to j$, $y \leq x \to j$. Hence $a \wedge x \leq (b \to j) \wedge (y \to j) = (b \wedge y) \to j$ and $b \wedge y \leq (a \to j) \wedge (x \to j) = (a \wedge x) \to j$. Therefore, $(a \wedge x)oj \leq b \wedge y$ and $(b \wedge y)oj \leq a \wedge x$ and hence $(a \wedge x, b \wedge y) \in \theta^F$. Moreover, $(a \vee x)oj = (aoj) \vee (xoj)$ and $j \leq h, h'$ and this implies that $boj \leq boh \leq a$, $yoj \leq yoh' \leq x$ and $aoj \leq aoh \leq b$, $xoj \leq xoh' \leq y$. Hence, $(aoj) \vee (xoj) = (a \vee x)oj \leq b \vee y$ and $(boj) \vee (yoj) = (b \vee y)oj \leq a \vee x$. Thus, $(a \vee x, b \vee y) \in \theta^F$. Therefore, the relation θ^F is a congruence relation.

Theorem 5.2. Let F be a filter of a BH-lattice L. The relation defined by $x \equiv y(\theta_F)$ iff $x * y = (x \to y) \wedge (y \to x) \in F$ is a congruence relation.

Proof. Let F be a filter of BH-lattice L. Define $x \equiv y(\theta_F)$ iff $x * y = (x \to y) \wedge (y \to x) \in F$. Since e belongs to F, it follows that for any element x in L, $x \equiv x(\theta_F)$. So it is reflexive. Let for x, y in L, $x \equiv y(\theta_F)$. Since $x * y = y * x$, it follows that $y \equiv x(\theta_F)$. Hence it is symmetric.

Now let $x \equiv y(\theta_F)$, $y \equiv z(\theta_F)$. Then $x * y$ and $y * z$ both belong to F. Hence $(x * y)o(y * z)$ belong to F. Using Theorem 3.2, $(x * y)o(y * z) \leq x * z \leq e$. Hence using 9 of Theorem 2.5, $[(x * y)o(y * z)] * e = (x * y)o(y * z) \leq x * z = (x * z) * e$. So it follows that $x * z \in F$ and hence $x \equiv z(\theta_F)$. Thus it is transitive Hence $x \equiv y(\theta_F)$ is an equivalence relation.

Let $x \equiv y(\theta_F)$ and $t \equiv s(\theta_F)$.

Then using 9 of Theorem 2.5, Theorem 3.1 and the distributive property of o over $\wedge$, $[(x * y)o(t * s)] * e = (x * y)o(t * s) = [(x \to y) \wedge (y \to x)]o(t * s) = \{(x \to y)o(t * s)\} \wedge \{(y \to x)o(t * s)\}$
$= \{(x \to y)o(t \to s)\} \wedge \{(y \to x)o(t \to s)\} \wedge \{(y \to x)o(s \to t)\} \wedge \{(x \to y)o(s \to t)\}$
$\leq \{(x \to y)o(t \to s)\} \wedge \{(y \to x)o(s \to t)\}$. Moreover, by 2, 4 and 11 of Theorem 2.5, $x \leq xos \to s \Rightarrow x \to y \leq (xos \to s) \to y = xos \to yos$. Similarly, $t \to s \leq tox \to xos$. Hence, by 11 of Theorem 2.5 and Theorem 3.1, $(x \to y)o(t \to s) \leq xot \to yos$. Analogously, $(y \to x)o(s \to t) \leq yos \to xot$. Thus, $\{(x \to y)o(t \to s)\} \wedge \{(y \to x)o(s \to t)\} \leq \{(xot) \to (yos)\} \wedge \{(yos) \to (xot)\} = (xot) * (yos) = [(xot) * (yos)] * e$. Hence by the definition of a filter $(xot) * (yos) \in F$. So $(xot) \equiv (yos)(\theta_F)$.

Further by 9 and 14 of Theorem 2.5, $[(x \wedge t) * (y \wedge s)] * e = (x \wedge t) * (y \wedge s) = (x \wedge t \to y \wedge s) \wedge (y \wedge s \to x \wedge t) = (x \to y \wedge s) \wedge (t \to y \wedge s) \wedge (y \to x \wedge t) \wedge (s \to x \wedge t) \geq (x \to y) \wedge (t \to s) \wedge (y \to x) \wedge (s \to t) = (x * y) \wedge (t * s) = [(x * y) \wedge (t * s)] * e$. Since by Theorem 4.2, $(x * y) \wedge (t * s)$ belongs to F, it follows that $(x \wedge t) * (y \wedge s)$ belongs to F. Hence

$x \wedge t \equiv y \wedge s(\theta_F)$.

By 8 and 13 of Theorem 2.5, $[(x\vee t)*(y\vee s)]*e = (x\vee t)*(y\vee s) = (x\vee t\to y\vee s) \wedge (y\vee s\to x\vee t) = (x\vee t\to y) \wedge (x\vee t\to s)\wedge(y\vee s\to x)\wedge(y\vee s\to t) \geq (x\to y) \wedge (t\to s) \wedge (y\to x) \wedge (s\to t) = (x*y)\wedge(t*s) = [(x*y)\wedge(t*s)]*e$. Since $(x*y)\wedge(t*s)$ belongs to F, it follows that $x\vee t \equiv y\vee s(\theta_F)$.

Finally, Since $s*t$, $x*y \in F$ and using Theorem 3.2, $[x*y]*e = x*y = (x\to y)\wedge(y\to x)\leq (x\to y)*e, (y\to x)*e$ and $[s*t]*e = s*t = (s\to t)\wedge(t\to s) \leq (s\to t)*e, (t\to s)*e$, it follows that $x\to y$, $y\to x$, $t\to s$, $s\to t$ all belong to F. And since F is a filter it follows that$(s\to t)o(x\to y)$ and $(y\to x)o(t\to s)$ belong to F. Now, by 3 and 11 of Theorem 2.5, $to(y\to s)o(s\to t)o(x\to y) \leq to(y\to t)o(x\to y) \leq yo(x\to y) \leq x$. Hence it follows that $(x\to t)\to(y\to s) \geq (s\to t)o(x\to y)$. Similarly, as so $(x\to t)o(t\to s)o(y\to x) \leq$ so $(x\to s)o(y\to x) \leq xo(y\to x) \leq y$, it follows that $(y\to s)\to(x\to t) \geq (t\to s)o(y\to x)$.

Hence by 9 of Theorem 2.5 and Theorem 3.2, it follows that $[(x\to t)*(y\to s)]*e = (x\to t)*(y\to s) \geq \{(s\to t)o(x\to y)\}\wedge\{(t\to s)o(y\to x)\} = [\{(s\to t)o(x\to y)\}\wedge\{(t\to s)o(y\to x)\}]*e$. So that $(x\to t)*(y\to s)$ belongs to F, as $\{(s\to t)o(x\to y)\}\wedge\{(t\to s)o(y\to x)\} \in F$. Hence $(x\to t) \equiv (y\to s)(\theta_F)$. Thus if F is a filter of BH-lattice L, then $x\equiv y(\theta_F)$ iff $x*y \in F$ is a congruence relation on L.

Definition 5.2. Let L be a BH-lattice and F be a filter of L. The congruence relation in Theorem 5.2 is called congruence relation related to F denoted by θ_F and for simplicity the se t of all congruence classes L/θ_F is simply denoted by L/F.

Theorem 5.3. Let θ be a congruence relation on BH-lattice L. If e is comparable for each $x \in L$, then $N=\{x\in L : x\equiv e(\theta)\}$ is a filter of L.

Proof. Let θ be a congruence relation on L and $N = \{x\in L : x\equiv e(\theta)\}$. Obviously e belongs to N, so N is non empty subset of L. For x, y belong to N, by the definition of a congruence relation, it follows that xoy belongs to N. Let x belongs to N and $x*e \leq y*e$. Then $x\equiv e(\theta)$, $x \equiv x(\theta) \Rightarrow x*e = x\wedge(e\to x) \equiv e(\theta)$. $\Rightarrow (x*e)\wedge(y*e) \equiv e(\theta) \Rightarrow [(x*e)\wedge(y*e)]\vee(y*e) = y*e \equiv e(\theta)$. If $y \leq e$, then by 9 of Theorem 2.5, $y*e = y$. Hence $y \equiv e(\theta)$. If $e \leq y$,then $e \to y \leq e \leq y$. Thus $y*e = e \to y \equiv e(\theta)$. This implies that $yo(e\to y) \equiv y(\theta)$. Hence $\{yo(e\to y)\}\vee e \equiv y \vee e(\theta)$.Thus $e \equiv y(\theta)$, so that $y \equiv e(\theta)$. Hence in both cases we have that $y \equiv e(\theta)$. Thus y belongs to N and hence N is a filter.

Theorem 5.4. Let L be a BH-lattice such that e is comparable for each $x \in L$. Then there is one to one correspondence between the set of all filters F of L and the set of all congruence relations of L, ConL.

Proof. From Theorem 5.2 and Theorem 5.3, it suffices to show the following. Define a function $f:F\longrightarrow ConL$ by $f(F)=\theta_F$. For F, $G \in F$, let $f(F)=f(G)$. Then this implies that $\theta_F = \{(x, y): x*y \in F\} = \{(x, y); x*y \in G\} = \theta_G$. Hence, $x \in F$ implies that $x*e \in F$, so $x*e \in G$. Thus by Theorem 4.9 $x \in G$. Therefore $F\subseteq G$. By a similar argument, it follows that $G\subseteq F$. Thus $F=G$ and hence f is one to one function. Let $\theta \in$ conL. Then by Theorem 5.3, $N=\{x\in L : x\equiv e(\theta)\}$ is a filter. Moreover, if $x\to y \in N$, then $x\to y \equiv e(\theta)$. Hence by the reflexive property of θ, the definition ofcongruence in L and Theorem 2.6, $(x\wedge y)=\{(x\to y)\wedge e\}oy\equiv y(\theta)$. Similarly, if $y\to x \in N$, then $y\to x\equiv e(\theta)$ and hence $(x\wedge y)=\{(y\to x)\wedge e\}ox \equiv x(\theta))$. Thus, if $x\to y$, $y\to x \in N$, then $x\equiv y(\theta)$. Therefore, $f(N)=\{(x, y): x*y \in N\} = \{(x, y): x\to y, y\to x \in N\} = \{(x, y): x\equiv y(\theta)\} = \theta$. i.e; N is the pre image of θ. Hence f is on to. Hence, there is a one to one correspondence between congruence relations on L and its filters.

Corollary 5.1. There is one to one correspondence between the set of all filters F of Heyting algebra H and the set of all congruence relations of H, ConH.

Theorem 5.5. In a BH-lattice L, for a filter F in L and [x], $[y] \in L/F$, $[x] \leq [y] \Leftrightarrow (y\to x)\wedge e \in F$.

Proof. Let $[x] \leq [y]$. This implies $[x]\wedge[y] = [x] \Rightarrow [x \wedge y] = [x] \Rightarrow (x \wedge y)*x \in F$. As $x\wedge y \leq x$, $x \to x = e \leq x\to x\wedge y$. Hence $(x\wedge y)*x = (x\to(x \wedge y)) \wedge ((x\wedge y)\to x) = (x\to(x\wedge y)) \wedge (x\to x) \wedge (y\to x) = (y\to x)\wedge e$. This implies that $(y\to x)\wedge e \in F$.

Conversely, let $(y\to x)\wedge e \in F$. Since $x\wedge y \leq x$, $x\to x = e \leq x\to x\wedge y$. Hence $x*(x\wedge y)= (y\to x)\wedge e \in F$. Thus it follows that $[x\wedge y]= [x]$ and hence $[x]\leq [y]$.

Theorem 5.6. Let L be a BH-lattice and F be a filter of L. The system $(L/F, o, [e], \vee, \wedge, \to)$ is a BH-lattice (called quotient BH-lattice corresponding to F), where the operations in the quotient are defined by [x]o[y] = [xoy], [x]∨[y] = [x∨y], [x]∧[y] = [x∧y], [x]→[y] = [x→y]. Further, if L is bounded above, so is (L/F, o, [e], ∨, ∧, →).

Proof. Obviously all the operations in the quotient system are well defined [5], [3]. Also it is clear that (L/F, o, [e]) is a commutative monoid with identity element [e] and (L/F, ∨, ∧) is a lattice. Further, by Theorem 5.5 we have the following. Let [x], [a], [b] ∈ L / F such that [x]o[b] ≤ [a] ⇔ [xob] ≤ [a] ⇔ (a→xob)∧e ∈ F ⇔{(a→b)→x } ∧e ∈ F ⇔ [x]≤ [a→b] ⇔ [x]≤ [a]→[b]. Hence the system (L/F, o, [e], ∨, ∧, →) is a BH-lattice. Trivially all the other axioms of BH-lattices hold. Finally if L is bounded above, then for x ∈ L, x≤ e ⇒ e ≤ e→x ≤ e ⇒ e= e→x ∈ F . Hence for any [x] ∈ L/F, [x] ≤ [e]. Therefore it is bounded above.

ACKNOWLEDGMENTS

The authors are thankful to the reviewers for their valuable suggestions which enhance the readability of the paper. Moreover, the second author would like to thank the authorities of International Science Program (ISP) of Upsal University, Sweden for their partial financial support.

REFERENCES

1. Birkhoff, G. (1973). Lattice Theory, Amer. Math. Soc. Colloq Publications VolXXV.
2. Blyth, T. S. (2005). Lattices and Ordered Algebraic Structures. Springer-Verlag, London Limited.
3. Burris, S. and Sankappanavar, H.P. (2000). A Course in Universal Algebra. Springer-Verlag New York Inc.
4. Esakia, L. (2019). Heyting algebras: Duality theory. Volume 50, Springer Nature Switzerland AG.
5. Gabor Szasz, Introduction to Lattice Theory, 3rd revised and enlarged edn.
6. George Gratzer, Lattice Theory- First Concepts and Distributive Lattices, University of Manitoba, 1999
7. Hohle, U. (1995).Commutative Residuated l-monoids: Non-classical logics and their applica-tions to Fuzzy subsets. Springer Science Busiuen Media, B.V.
8. Kopytov, V.M. and Medvedev, N.Y. (1994). The Theory of Lattice-Ordered Groups. Springer-Science+Business Media, B.V.
9. K.Venkateswarlu, M.Mamo, and Y.Yitayew, Decomposition theorems on BH-lattices, SINET: Ethiopian Journal of Science, 46(2023), pp. 99-108.
10. Nordhaus, E.A and Leolapidus, L. (1954). Brouwerian geometry. Canad. J. Math. Soc. 6, pp. 217-229.
11. Rutherford, D.E. (1965). Introduction to lattice theory. Oliver and Boyd Ltd, Tweeddalecourt, Edinburgh
12. Sankappanavar, H. P. (2007). Semi-Heyting algebras: An abstraction from Heyting algebras. State University of New York at New Paltz.
13. Swamy, K.L.N. (2020). Brouwer-Heyting Monoids. International Journal of Engineering Sciences and Research Technology, 9(3), pp.28-33.
14. Swamy, K.L.N. (1965). Dually Residuated Lattice Ordered Semigroups. Math. Ann. 159, pp. 105–114.
15. Swamy, K.L.N. (1965). Dually Residuated Lattice Ordered Semigroups II. Math.Ann. 160, pp. 64–71.
16. Swamy, K.L.N. (1966). Dually Residuated Lattice Ordered Semigroups III. Math. Ann. 167pp. 71–74.
17. Ward, M. and Dilworth, R.P. (1939). Residuated Lattices. Trans. Amer. Math. Soc. 45, pp.335-354.