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Primary Submaximal Ideals in Commutative Weak Idempotent Rings

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ABSTRACT: Let R be a commutative weak idempotent ring (cWIR, for short) with unity, N and R_B be the set of all nilpotent and idempotent elements of R respectively. In this paper, we study the structure of primary submaximal ideals in R and prove that, if P is a primary submaximal ideal of R and $n_1 \notin P$ for some $0 \neq n_1 \in N$, then $P + R_B n_1$ is a maximal ideal of R and $P + < n_1 > = P + R_B n_1$, where $R_B n_1 = \{b n_1 : b \in R_B\}$.

Key words/ phrases: cWIR, nilpotent and idempotent elements, Primary ideal, Primary submaximal ideal

INTRODUCTION

Foster (1946), introduced the notion of Boolean like rings as a generalization of Boolean rings. A Boolean like ring (BLR, for short) is a commutative ring R with unity of characteristic 2 and $ab(1 + a)(1 + b) = 0$ for all a, b in the ring R.

Let R be a Boolean like ring and $0 \neq n \in N$, where N is the set of all nilpotent elements of R. There exists a primary ideal P of R such that $n \notin P$ and for any $n_1 \in N$, either $n_1 \in P$ or $n + n_1 \in P$ (See Swaminathan V.(1982)).

Dereje Wasihun et al. (2022) introduced the notion of weak idempotent rings (WIR, for short) which is a ring with characteristic 2 and $a^4 = a^2$ for every element a in the ring. If a WIR with unity is local, then its idempotent elements are 0 and 1. A commutative WIR (cWIR, for short) with unity is a BLR but not conversely. For instance, a Quaternion ring over the field $\mathbb{Z}_2$ is a cWIR with unity which is not a Boolean like ring. Here
$R = \{0, 1, i, j, k, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j, 1 + j + k, 1 + i + k, i + j + k, 1 + i + j + k\}$ (See Dereje Wasihun et al. (2022), Example 10). From this ring, consider
$N = \{0, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j + k\}$, $Q = \{0\}$, $Q_1 = \{0, 1 + i + j + k\}$, $Q_2 = \{0, 1 + i, j + k, 1 + i + j + k\}$, $Q_3 = \{0, 1 + j, i + k, 1 + i + j + k\}$, $Q_4 = \{0, 1 + k, i + j, 1 + i + j + k\}$.

Q, Q_1, Q_2, Q_3 and Q_4 given above are all primary ideals of R and N is the maximal ideal of R. The following properties of a primary ideal of a commutative weak idempotent ring R with unity has been proved in Dereje Wasihun et al. (2022). Moreover, they have shown that in a commutative weak idempotent ring with unity, the intersection of all primary ideals is {0}. Furthermore, they have proved that every element of a weak idempotent ring can be written as a sum of an idempotent and nilpotent elements the ring.

Remark 1. Let $x = x_B + x_N \in R$ and I be an ideal of R such that $x \notin I$, where $x_N \in N$ (the set of nilpotent elements of R) and $x_B \in R_B$(the set of idempotent elements of R).
1. If $x_B \notin I$, then there exists a maximal ideal J of R such that $I \subset J$ and $x \notin J$.
2. If $x_N \notin I$, then there exists a primary ideal P of R such that $I \subset P$ and $x \notin P$.

Every proper ideal I of a cWIR R with unity is the intersection of all primary ideals of R which contains I (See Dereje Wasihun Mellese (2020)). An ideal I of a cWIR R with unity is contained in at least two maximal ideals of R if and only if I is not primary (See Venkateswarlu Kolluru et al. (2020)). That is, I is primary if and only if I is contained in only one maximal ideal of R.

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Note that an ideal of a ring is called semiprime if its radical is the ideal itself. If I is an ideal of a cWIR R with unity, then the following statements are equivalent (See Venkateswarlu Kolluru et al. (2020)).

1. I is semiprime.
2. The nilradical N of R is contained in I.
3. R/I is a Boolean ring.

An ideal I of a cWIR R is called submaximal if I is covered by a maximal ideal of R. That is, there exists a maximal ideal M of R such that $I \subsetneq M$ and for any ideal J of R such that $I \subseteq J \subseteq M$, then either J = I or J = M (See Venkateswarlu et al. (2020)). In Venkateswarlu Kolluru and Dereje Wasihun (2021), the structure of submaximal ideals of a cWIR R with unity has been studied. If an ideal I of R is submaximal, then R/I is either a four element Boolean ring or the Boolean like ring H_4, where $H_4 = \{0,1,p,q\}$. Every maximal ideal that contains a submaximal ideal I of R is a cover of I. Every submaximal ideal I of R is covered by at most two maximal ideals and it is primary if it is covered by a unique maximal ideal of R. That is, every submaximal ideal of R is either semiprime or primary.

In Venkateswarlu Kolluru et al. (2020), an ideal of a cWIR with unity is maximal if and only if it is prime.

In this work, we study some basic properties of a primary submaximal ideal of a cWIR with unity. In the second section we prove that the existence of a primary ideal of a cWIR R with unity which does not contain a nonzero nilpotent element of R. In the third section, we obtain some results of a primary submaximal ideal of a cWIR R with unity which does not contain a nonzero nilpotent element of R.

Primary Ideals

We begin with the following.

Recall that in a commutative ring R, an ideal P is primary if, for $a, b \in R, ab \in P$ implies either $a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$.

Note that throughout this paper, R denotes a commutative weak idempotent ring with unity and N represents the set of all nilpotent elements of R unless otherwise it is stated.

Remark 2.1. Let R be a cWIR with unity, P be an arbitrary primary ideal of R and $0 \neq n \in N$ such that $n \notin P$. Then

1. $P + < n >$ is an ideal of R which is neither maximal nor submaximal.
2. $P + R_B n$ may not be an ideal of R, where R_B denotes the set of all idempotent elements of R.

We clarify this remark by the following example.

Example2.1. $Q = \{0\}$ is a primary ideal of a cWIR R with unity (See Dereje Wasihun et al. (2022), Example 10) and $0 \neq n = 1 + i + j + k$ is a nilpotent element of R which does not belong to Q. Here, the ideal $Q + < n > = Q_1$, where $Q_1 = \{0, 1 + i, j + k, 1 + i + j + k\}$ which is neither a maximal nor a submaximal ideal of R. Again $n_1 = i + j$ is a non-zero nilpotent element of R which does not belong to Q and $Q + R_B n_1 = \{0, i + j\}$ which is not an ideal of R, where $R_B = \{0,1\}$.

Theorem 2.1. Let P be a non-zero primary ideal of a cWIR R with unity and n be a non-zero nilpotent element of R which does not belong to P. Then, $P + < n >$ is a primary ideal of R.

Proof. Let $Q = P + < n >$ and $z + Q$ be a non-zero idempotent element of R/Q. Then, $z^2 + Q = z + Q$ from which we obtained that $z(1 + z) \in Q = P + < n >$. So, $z(1 + z) = w + rn$ for some $w \in P$ and $r \in R$ as $< n > = \{rn : r \in R\}$. This implies $z^2(1 + z^2) \in P$ and hence $1 + z^2 \in P$. For: if $z^2 \in P$, then $z^2 \in Q$ which implies $z + Q = Q$ and hence $z \in Q$ which is a contradiction. Then, $z \in R_B$. For: if $z \in N$, then $1 \in P$ and this is a contradiction as P is a primary ideal. So, $1 + z \in P$ and hence $1 + z = y$ for some $y \in P$. Now, considering $z + Q = 1 + y + Q = 1 + Q$ as $y \in Q$. Thus, Q and $1 + Q$ are the only idempotent elements of R/Q. Therefore, R/Q is local and hence Q is a primary ideal of R.

Remark 2.4. Let n be a non-zero nilpotent element in a cWIR R with unity which does not belong to a primary ideal P of R.

If n_1 is a non-zero nilpotent element of R with $n_1 \neq n$, then both n_1 and $n_1 + n$ may not belong to P.

We justify this remark with the following example.

Example 2.2. Q_1 is a primary ideal of a cWIR R with unity with $n = 1 + i \notin Q_1$. Take $n_1 = 1 + j \notin Q1$ which gives $n_1 + n = i + j \notin Q_1$ (See Dereje Wasihun et al. (2022), Example10).

Theorem 2.2. Let R be a cWIR with unity and $0 \neq n \in N$. Then,

1. There exists a primary ideal P of R such that $n \notin P$.
2. For $0 \neq n_1 \in N$, either $n_1 \in P$ or $n + n_1 \in P$, provided that P is the primary ideal of R in (1) and $n \in P + R_B n_1$.

Proof. Suppose that R is a cWIR with unity and $0 \neq n \in N$.

1. Let $\Sigma_n = \{I : I$ is a proper ideal of R such that $n \notin I\}$. Then, Σ_n is a non-empty set as $\{0\}$ belongs to it. Considering Σ_n ordered by inclusion, it is a poset. Let $\mathcal{C}$ be a chain of ideals in Σ_n and $J = \cup_{I\in\mathcal{C}} I$. Then $I \subseteq J$ for all $I \in \mathcal{C}$, J is an ideal of R and $n \notin J$. Hence there exists a primary ideal P such that $J \subseteq P$ and $n \notin P$.

2. For $0 \neq n \in N$, there exists a primary ideal P of R such that $n \notin P$ and n belongs to all ideals of R those properly contain P by (1). Let $n_1 \in N$ such that $n_1 \notin P$. Hence $P \subset P + Rn_1$. We claim that $n + n_1 \in P$. If $n_1 = n$, clearly $n + n_1 \in P$. Suppose that $n_1 \neq n$. Then, $n \in P + Rn_1$ as $n_1 \notin P$ and $P + Rn_1 \notin \Sigma_n$ from (1). Let $n \in P + R_B n_1$. This implies that $n = x + bn_1$ for some $x \in P$ and $b \in R_B$. From this we have that $n + bn_1 = x \in P$. Then, $b \notin P$. For: if $b \in P$, then $bn_1 \in P$. Hence $n \in P$ and this is a contradiction. So, $b \notin P$. As P is a primary ideal of R and $b \notin P$, $1 + b \in P$ since $b(1 + b) = 0 \in P$ and $(1 + b)^m = 1 + b$ for all positive integer m. Thus, $n_1 + bn_1 = (1 + b)n_1 \in P$ and hence $n + n_1 = n + bn_1 + n_1 + bn_1 \in P$ as $n + bn_1 \in P$. Hence the theorem follows.

Primary Submaximal Ideals

Remark 3.1. In a cWIR R with unity, a non-zero nilpotent element of R may belong to every primary submaximal ideal of R.

We justify this remark with the following example.

Example 3.1. $n = 1 + i + j + k \in N$ which belongs to every primary submaximal ideals of R and $n = 1 + i + j + k = (1 + i)(1 + j)$, where both $1 + i$ and $1 + j$ are nilpotent elements of R (See Dereje Wasihun et al.(2022), Example 10).

Theorem 3.1. Let R be a cWIR with unity and n_1, n_2 are distinct non-zero nilpotent elements of R. Then, every primary submaximal ideal of R contains $n_1 n_2$.

Proof. Let n_1 and n_2 be two distinct non-zero nilpotent elements of a cWIR R with unity. If $n_1 n_2 = 0$, it is obvious. Let $n_1 n_2$ be a non-zero nilpotent element of R. Suppose there exists a primary submaximal ideal P of R such that $n_1 n_2 \notin P$. It follows that $P \subset P + \langle n_1 n_2 \rangle$. Moreover, $P \subset P + \langle n_1 n_2 \rangle \subset P + \langle n_1 \rangle$. Now we show that $P + \langle n_1 \rangle$ is a proper ideal of R. Assume that, $P + \langle n_1 \rangle$ is not a proper ideal of R. So, $1 \in P + \langle n_1 \rangle$ and $1 = x + bn_1$ for some $x \in P$ and $b \in R$. Thus, $1 + x$ is nilpotent as bn_1 is so. Hence $(1 + x)^2 = 0$ and $x^2 = 1$. So, x is a unit element in P and this contradicts the fact that P is a proper ideal of R. Therefore, $P + \langle n_1 \rangle$ is a proper ideal of R. Now, we claim that $P + \langle n_1 n_2 \rangle \neq P + \langle n_1 \rangle$. For: if $P + \langle n_1 n_2 \rangle = P + \langle n_1 \rangle$, then $n_1 = x + bn_1 n_2$ for some $x \in P$ and $b \in R$ as $n_1 \in P + \langle n_1 \rangle$. This implies $(1 + bn_2)n_1 = x \in P$ from which we get that $(1 + bn_2)^2 \in P$ as $n_1 \notin P$ and P is primary. Thus, $1 \in P$ and this is a contradiction as P is a proper ideal of R. Therefore, $P + \langle n_1 n_2 \rangle = P + \langle n_1 \rangle$. As both of them are proper ideals of R containing P, it is a contradiction. Therefore, our assumption $n_1 n_2 \notin P$ is false and hence every primary submaximal ideal of R contains $n_1 n_2$.

Based on theorem 3.1, we have the following remark.

Remark 3.2. Let R be a cWIR with unity. If n is a non-zero element of N and does not belong to some primary submaximal ideal of R, then n is not the product of two distinct elements of N.

Theorem 3.2. If P is a primary submaximal ideal of a cWIR R with unity which does not contain a non-zero nilpotent element n of R, then $P + R_B n$ is a maximal ideal of R.

Proof. Let n be a non-zero nilpotent element of R which does not belong to a primary submaximal ideal P of R. Then, $P \subset P + R_B n$ as $nN \subseteq P$ by theorem 3.1. Let $x, y \in P + R_B n$. Then, $x = m_1 + b_1 n$ and $y = m_2 + b_2 n$ for some $m_1, m_2 \in P$ and $b_1, b_2 \in R_B$. $x + y = (m_1 + m_2) + (b_1 + b_2)n \in P + R_B n$.

For $r \in R$, $rx = rm_1 + rb_1 n$. If $r \in N$, then $rx \in P \subset P + R_B n$ by theorem 3.1. If $r \in R_B$, then $rb_1 \in R_B$ and hence $rx \in P + R_B n$. Thus, in any case $rx \in P + R_B n$ and hence $P + R_B n$ is an ideal of R. And also $P + R_B n$ is a proper ideal of R. Assume, if possible that, it is not a proper ideal. Then, $1 \in P + R_B n$. This implies $1 = m + bn$ for some $m \in P$ and $b \in R_B$ which also gives us $1 + m^2 = 0 \in P$.Thus, $1 = m^2 \in P$ as $m \in P$. This implies m is a unit element in P and this is a contradiction as P is a proper ideal of R. Therefore, our assumption is false and hence $P + R_B n$ is a proper ideal of R. Therefore, $P + R_B n$ is a maximal ideal of R.

Theorem 3.3. Let P be a primary submaximal ideal of a cWIR R with unity and $0 \neq n_1 \notin P$ for some $n_1 \in N$. Then, $P + < n_1 > = P + R_B n_1$.

Proof. Let $0 \neq n_2 \in N$ and $n_2 \neq n1$. Then, $(1 + n_2)n_1 \in P + < n_1 >$ by theorem 3.1 as P is a primary submaximal ideal of R. So, $n_1 + n_2 n_1 = x + y n_1$ for some $x \in P$ and $y \in R$. This implies $n_1 + y n_1 = x + n_2 n_1 \in P$ by theorem 3.1 and hence $(1 + y)n_1 \in P$. As P is primary and $n_1 \notin P, (1 + y)^2 \in P$ which implies $1 + y^2 \in P$. Then, $y \in R_B$. For: if $y \in N$, then $y^2 = 0$ and hence $1 \in P$ which is a contradiction as P is a proper ideal of R. Hence, $y \in R_B$. Therefore, $P + < n_1 > = P + R_B n_1$.

Theorem 3.4. Let P be a primary submaximal ideal of a cWIR R with unity and n_1, n_2 are distinct non-zero nilpotent elements of R such that both do not belong to P. Then, $P + R_B n_1 = P + R_B n_2$.

Proof. Let P be a primary submaximal ideal of R and $n_1 \neq n_2$ be non-zero nilpotent elements of R such that $n_1, n_2 \notin P$. Assume that $P + R_B n_1 \neq P + R_B n_2$. Then, $P + < n_1 >$ and $P + < n_2 >$ are distinct proper ideals of R containing P as we have shown in the proof of theorem 3.1. But this contradicts the fact that a primary submaximal ideal of a commutative WIR is covered by a unique maximal ideal of the ring. Thus, our assumption is false and hence $P + R_B n_1 = P + R_B n_2$.

Here we have the following theorem

Theorem 3.5. Let P be a primary submaximal ideal of a cWIR R with unity and n1 be a non-zero nilpotent element of R such that $n_1 \notin P$. If there exists a non-zero nilpotent element n2 of R such that $n_2 \notin P$, then $n_1 + n_2 \in P$.

Proof. Suppose that P is a primary submaximal ideal of R and n_1, n_2 are non-zero nilpotent elements of R which do not belong to P. If $n_1 = n_2$, then clearly $n_1 + n_2 \in P$. Suppose that $n_1 \neq n_2$. Then, $P + R_B n_1 = P + R_B n_2$ by theorem 3.4. Hence, $n_1 = x + b n_2$ for some $x \in P$ and $b \in R_B$. Then, $b \notin P$. For: if $b \in P$, then $n_1 \in P$ and this is a contradiction. Thus, $b \notin P$ and hence $1 + b \in P$ as P is a primary ideal of R. As $n_2 \in P + R_B n_2 = P + R_B n_1$, $n_2 = y + b_2 n_1$ for some $y \in P$ and $b_2 \in R_B$. So, $n_1 + n_2 = x + b n_2 + y + b_2 n_1 = x + (1 + b)(y + b_2 n_1) \in P$ as $x, 1 + b \in P$. Therefore, $n_1 + n_2 \in P$.

Remark 3.3. The notion of primary submaximal and maximal ideals of a commutative weak idempotent ring with unity are different.

Consider the following example.

Example 3.2. For a commutative weak idempotent ring R with unity (See Dereje Wasihun et al. (2022), Example 10), Q_2, Q_3 and Q_4 are primary submaximal ideals of R which are not maximal. N is a maximal ideal of R which is primary but not submaximal. Here $0 \neq n = 1 + i + j + k \notin Q$ and Q is the only maximal primary ideal which is not submaximal.

Theorem 3.6. Let I be a submaximal ideal of a cWIR R with unity and $X(I) = \{P: P$ is a primary ideal of R such that $I \subseteq P\}$. Then X(I) has exactly two elements.

Proof. Suppose that I is a submaximal ideal of R. Then, I is either semiprime or primary by remark 2.2. If I is semiprime, then it is contained in exactly two maximal ideals of R both of them cover I. Both of them are also prime and hence primary. So, these are the only elements of X(I). If I is a primary ideal, then it is contained in a unique maximal ideal say M of R and both I and the unique maximal ideal M belongs to X(I). Hence, in either cases X(I) has exactly two elements.

Remark 3.4. Let R be a cWIR with unity and N be the nilradical of R. Let X be the set of all primary submaximal ideals of R. For $n \in N$, define $\overline{X_n} = \{P : P \in X$ and $n \notin P\}$. Then, $(N, +)$ is not isomorphic to $(\{\overline{X_n}\}_{n \in N}, \Delta)$, where Δ denotes the symmetric difference.

For this, look the following examples.

Example 3.3. $\{\overline{X_n}\}_{n \in N} = \{\emptyset, Q_2, Q_3, Q_4\}$, whereas N has eight elements. Here, $\overline{X_n}$ is an empty set for $n = 1 + i + j + k \in N$ (See Dereje Wasihun et al. (2022), Example 10).

Remark 3.5. Let I be an ideal of a cWIR R and N be the nilradical of R such that
$N \subset I$.

1. If I is a proper ideal which is not maximal, then I is contained in a submaximal semiprime ideal in R.
2. For $n_1, n_2 \in N$, if $n_1 \neq n_2$, then there exists a primary submaximal ideal P of R containing exactly one of them.
3. For $x \in R$, if $x \notin I$, then there exists an ideal P of R which is either maximal or primary submaximal such that $x \notin P$ and $I \subset P$.

CONCLUSIONS

In this work, we obtained some results in relation to nonzero nilpotent element and a primary ideal of a cWIR with unity. We also proved that a primary submaximal ideal of a cWIR R with unity which does not contain a nonzero nilpotent element of R is maximal. These may motivate to study further on the ideal structures of commutative weak idempotent rings with unity.

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