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3-GDDs with 4 Groups and Block Size 5

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ABSTRACT: This paper studies a special case of group divisible designs (GDDs) called 3-GDDs, which were defined by extending the definitions of a group divisible designs and a t-design. In particular, the paper looks at a 3-GDD(n, 4, 5; μ_1, μ_2) with 4 groups and block size 5. Necessary conditions for the existence of such GDDs are developed, the non-existence of a 3-GDD(n, 4, 5; μ_1, 0) is proved and several specific instances of non-existence are given.

Key words/phrases: Balanced Incomplete Block Designs, Group Divisible Designs, Partially Balanced Incomplete Block Designs, t-designs, 3-GDDs

INTRODUCTION

A balanced incomplete block design, BIBD(v, b, r, k, λ), is a set V of v elements together with a collection $\mathbb{B}$ of k-subsets(called blocks) of V, where each element occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks. The number $|V|=v$ is called the order of the BIBD. The parameters of BIBD(v, b, r, k, λ) must satisfy the necessary conditions $\lambda(v-1) = r(k-1)$ and $vr = bk$ for a BIBD to exist, and a BIBD(v, b, r, k, λ) is commonly denoted by BIBD(v, k, λ) (Street and Street, 1996; Fu and Roger, 1998).

A group divisible design, GDD(n, m, k; λ_1, λ_2), is a collection of k-element subsets (called blocks) of an mn-set V, which satisfies the following properties: The elements of V are partitioned into m subsets (called groups) of size n each. Points within the same group are called the first associates of each other and appear together in λ_1 blocks; any two points not in the same group are called the second associates and appear together in λ_2 blocks (Brouwer et al., 1977; Hurd and Sarvate, 2008; Sarvate and Zhang, 2016; Sarvate et al., 2018).

The evolution of combinatorial design theory has been remarkable because of its deep connections with fundamental mathematics and the desire to produce order from apparent chaos (Stinson, 2008). Since group divisible designs have been studied for their usefulness in statistics and for their universal application to constructions of new designs (Street and Street, 1987; Mullin and Gronau, 1996; Street and Street, 1996), the existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto (1952), who began classifying such designs. Recently, a 3-GDD was defined by extending the definitions of a group divisible design and a t-design, and some necessary conditions for its existence were given (Sarvate and Bezire, 2018; Sarvate and Cowden, 2018). This new definition has the potential to raise many more generalizations and challenging existence problems. In Sarvate and Bezire (2018), the authors proved that the necessary conditions are sufficient for the existence of a 3-GDD(n, 2, 4; λ_1, λ_2) except possibly when $n \equiv 1, 3 \pmod 6$, $n \neq 3, 7, 13$ and $\lambda_1 > \lambda_2$ and in Sarvate and Cowden (2018), the authors settled that the necessary conditions are sufficient for the existence of a 3-GDD(n, 2, 4; λ_1, λ_2) for $n \equiv 1, 7, 9 \pmod{12}$.

In this paper, we continue to focus on the definition of 3-GDDs and explicitly consider the case when the required designs have 4 groups of size n each and block size 5. Throughout this paper, such GDD is denoted by 3-GDD (n, 4, 5; μ_1, μ_2). In this work, some necessary conditions for the existence of such GDDs are determined, and several specific instances of their non-existence are proved.

The rest of the paper is organized as follows: Section 2 presents some well-known definitions and concepts that will be used in the succeeding section. Section 3 is the result section, and it presents and

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discusses the findings of this paper.

PRELIMINARIES

In this section, we review some important definitions and concepts that we will be used later to study the main results.

Definition 1. [Sarvate and Cowden (2018)] A t-(v, k, λ) design, or a t-design is a pair ($\mathbb{X}$, $\mathbb{B}$), where $\mathbb{X}$ is a v-set of points and $\mathbb{B}$ is a collection of k-subsets (blocks) of $\mathbb{X}$ with the property that every t-subset of $\mathbb{X}$ is contained in exactly λ blocks. The parameter λ is called the index of the design.

It is possible to generalize the concepts of GDDs and t-designs in many ways. Here is a generalization for GDDs with two groups and block size k:

Definition 2. [Sarvate and Bezire (2018)] A 3-GDD(n, 2, k; λ_1, λ_2) is a set $\mathbb{X}$ of 2n elements partitioned into two parts of size n, called groups, together with a collection of k-subsets of $\mathbb{X}$ called blocks, such that

(i) every 3-subset of each group occurs in λ_1 blocks and

(ii) every 3-subset where two elements are from one group and one element from the other group occursin λ_2 blocks.

Example 1. Let $\mathbb{X}$ = {1, 2, 3, a, b, c}, G_1={1, 2, 3} and G_2={ a, b, c}. $\mathbb{B}$={{1, 2, 3, a}, {1, 2, 3, b}, {1, 2, 3, c}, {a, b, c, 1}, {a, b, c, 2}, {a, b, c, 3}} gives the required blocks of the 3-GDD(3, 2, 4;3, 1).

The above definition from Sarvate and Bezire (2018), has been extended by Sarvate to include more than two groups as follows:

Definition 3. A 3-GDD(n, m, k; μ_1, μ_2) is a pair ($\mathbb{X}$, $\mathbb{B}$), where $\mathbb{X}$ is a set of mn elements partitioned into m n-subsets (groups) and $\mathbb{B}$ is a collection of k-subsets (blocks) of $\mathbb{X}$ such that

(i) every triple occurs in exactly μ_1 blocks if it contains elements from at most 2 groups,

(ii) it occurs in exactly μ_2 blocks if it contains all three elements from different groups.

We extrapolate the above to a type of a 3-PBIBD (Partially Balanced Incomplete Block Design) and denote it simply by 3-PBIBD(n, m, k; Λ_1, Λ_2, Λ_3), where

(i) Every triple formed from elements of only a single group occurs in Λ_1 blocks,

(ii) Every triple formed from elements of only two groups occurs in Λ_2 blocks,

(iii) Every triple formed from elements of all three groups occurs in Λ_3 blocks.

A 3- GDD(n, 4, 5; μ_1, μ_2) is a 3-PBIBD(n, m, k; Λ_1, Λ_2, Λ_3) , where Λ_1= Λ_2, denoted by μ_1 while Λ_3 is denoted by μ_2.

In this paper, we explicitly consider the case in which m = 4 and k = 5 in Definition 3. Throughout this paper, such GDD is denoted by 3-GDD(n, 4, 5; μ_1, μ_2) .

Remark 1.
A 3-GDD(n, 4, 5; μ_1, μ_2), where μ_1=μ_2 is the same as a 3-(4n, 5, μ_1) and a 3-GDD(n, 4, 5; μ_1, μ_2) exists if and only if a 3-(4n, 5, μ_1) exists.

RESULTS

In this section, the main results of this paper will be discussed. We obtain some necessary conditions for the existence of a 3-PBIBD(n, m, k; Λ_1, Λ_2, Λ_3) and a 3-GDD(n, 4, 5, μ_1, μ_2), and assuming such designs exist, we count the number of blocks containing any given element (called the replication number r), the number of blocks, say λ_1, containing a given first associate pair, the number of blocks, say λ_2 containing a given second associate pair, and the required number of blocks, say b, for the design. In addition, several specific instances of non-existence are presented.

NECESSARY CONDITIONS

Theorem 1. In 3-PBIBD(n, m, k; Λ_1, Λ_2, Λ_3) with m=4 and k=5, we have

$$r = \frac{1}{12}((n-1)(n-2)\Lambda_1 + 9n(n-1)\Lambda_2 + 6n^2\Lambda_3) \quad (1)$$

$$b = \frac{n}{15}((n-1)(n-2)\Lambda_1 + 9n(n-1)\Lambda_2 + 6n^2\Lambda_3) \quad (2)$$

$$\lambda_1 = \frac{1}{3}((n-2)\Lambda_1 + 3n\Lambda_2) \quad (3)$$

$$\lambda_2 = \frac{1}{3}(2(n-2)\Lambda_2 + 2n\Lambda_3) \quad (4)$$

Proof:

(1). We count the number of triples containing a fixed element x in the design in two ways.

First, given an element x, it appears in $\binom{n-1}{2}$triples of the type (3, 0), 3n(n-1)+3$\binom{n}{2}$ triples of the type (2, 1) and $3n^2$ triples of the type (1, 1, 1) containing x, and these triples are repeated Λ_1, Λ_2 and Λ_3 times, respectively.

In sum, there are $\binom{n-1}{2}\Lambda_1+(3n(n-1)+3\binom{n}{2})\Lambda_2+3n^2\Lambda_3$ triples containing x.

Second, in every block containing x, there are 6 triples containing x and x occurs in r blocks, which meansthere are $6r$ triples containing x.

Equating the two, the result follows.

(2). In a design with 4 groups, block size 5 and b blocks, counting the number of triples in a design in two ways, we get: $\binom{5}{3} b = 4\binom{n}{3} \Lambda_1 + 6n^2(n-1)\Lambda_2 + 4n^3\Lambda_3$, which leads to

$$b=\frac{n}{15}((n-1)(n-2)\Lambda_1+9n(n-1)\Lambda_2+6n^2\Lambda_3).$$

(3). To count the number of blocks containing a given pair of points from the same group, say (x_1, x_2), consider the following two cases:

(a) There are $(n-2)$ triples of the type (3, 0) and $3n$ triples of the type (2, 1) containing (x_1, x_2). In sum, there are $(n-2)\Lambda_1+3n\Lambda_2$ triples containing (x_1, x_2).

(b) In a design with block size 5, the block containing a first associate pair, contains three triples containing (x_1, x_2). Hence there are $3\lambda_1$ such triples.

Since (a) and (b) counts the same,

$3\lambda_1=(n-2)\Lambda_1+3n\Lambda_2$ and $\lambda_1=\frac{1}{3}((n-2)\Lambda_1+3n\Lambda_2)$.

(4). Again counting the number of blocks containing a given pair of points from different groups, say (x, y) in two ways:

First, the pair occurs in triples of the type (2, 1) and (1, 1, 1) only. It occurs in $2(n-1)\Lambda_2$ triples of type (2, 1) or $2n\Lambda_3$ triples of type (1, 1, 1).

Hence, there are $2(n-1)\Lambda_2+2n\Lambda_3$ triples containing the pair (x, y).

Second, each of the λ_2 blocks containing the pair (x, y) has three triples containing (x, y). Hence, there are $3\lambda_2$ triples containing (x, y).

Since both cases count the same, the result follows.

Example 2. 3-PBIBD(2, 4, 5; 0, 4, 7) exists and is constructed on the groups {1, 2}, {a, b}, {x, y} and {w, z}.Then the blocks are constructed by combining both elements of a group with every-three subsets (taken from distinct groups) of the union of the remaining three groups.

1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a
2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b
a	a	a	a	b	b	b	b	1	1	1	1	2	2	2	2
x	x	y	y	x	x	y	y	x	x	y	y	x	x	y	y
w	z	w	z	w	z	w	z	w	z	w	z	w	z	w	z
x	x	x	x	x	x	x	x	w	w	w	w	w	w	w	w
y	y	y	y	y	y	y	y	z	z	z	z	z	z	z	z
1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2
a	a	b	b	a	a	b	b	a	a	b	b	a	a	b	b
w	z	w	z	w	z	w	z	x	y	x	y	x	y	x	y

Example 3. 3-PBIBD(3, 4, 5; 36, 9, 3) exists.

If G_1={1, 2, 3}, G_2={a, b, c}, G_3={x, y, z} and G_4={w, s, t} , then the blocks of the design are constructed by joining G_i for i = 1, 2, 3, 4 with every two subsets of

$H=\cup_{j=1.j\neq i}^{4} G_j$

Example 4. 3-PBIBD(4, 4, 5; 12, 1, 0) exists.

With G_1={1, 2, 3, 4}, G_2={a, b, c, d}, G_3={x, y, z, w} and G_4={r, s, t, u}, the blocks of this design are obtained by joining all the four elements of a group with every element of the union of the remaining three groups.

In 3-PBIBD(n, m, k; Λ_1, Λ_2, Λ_3) when $\Lambda_1=\Lambda_2$ is denoted is by μ_1 and Λ_3 is denoted by μ_2, the following Corollary directly follows from Theorem 1. So we omit their proofs.

Corollary 1. Given 3-GDD(n, 4, 5; μ_1, μ_2),

$$r=\frac{(n-1)(5n-1)\mu_1+3n^2\mu_2}{6} \tag{5}$$

$$b=\frac{2n\left((n-1)(5n-1)\mu_1+3n^2\mu_2\right)}{15} \tag{6}$$

$$\lambda_1 = \frac{(4n-2)\mu_1}{3} \tag{7}$$

$$\lambda_2 = \frac{2(n-1)\mu_1 + 2n\mu_2}{3} \tag{8}$$

Corollary 2. In a 3-GDD(n, 4, 5; μ_1, μ_2), $\mu_1 \neq 0$ and $\lambda_1 \neq 0$.

From (5) and (6), the necessary conditions are satisfied under the following conditions:

Lemma 1. For

i. $n \equiv 0 \pmod 5$, μ_1 and μ_2 are free in regards to the number of blocks b.
ii. $n \equiv 0 \pmod 6$ and $\mu_1 \equiv 0 \pmod 6$, r is an integer for any chosen μ_2 .
iii. $n \equiv 1 \pmod 6$ and even μ_2, r is an integer for any chosen μ_1.
iv. $n \equiv 4 \pmod 6$ and even μ_1, r is an integer for any chosen μ_2.

From (7) and (8), a more compact necessary condition is given as follows:

Lemma 2. In regards to:

i. λ_1, for $n \equiv 2 \pmod 3$, μ_1 and μ_2 are free and for $n \not\equiv 2 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ and μ_2 is free.
ii. λ_2, for $n \equiv 0 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ and μ_2 is free, when $n \equiv 1 \pmod 3$, μ_1 is free and $\mu_2 \equiv 0 \pmod 3$ and for $n \equiv 2 \pmod 3$, $\mu_1 + 2\mu_2 \equiv 0 \pmod 3$.

From Corollary 1, as the values of b and r must be integers, we have the following table.

Table 1: Table of congruence restrictions (all values are considered to be in terms of (mod 30) unlessotherwise stated):

n	*r*		*b*	
	μ_1	μ_2	μ_1	μ_2
0	0(mod6)	All	All	All
1, 11	All	0(mod2)	All	0(mod5)
2, 7, 17	0(mod2)	All	*1	*1
3, 9	*2	*2	*3	*3
4, 8, 14, 28	0(mod2)	All	*3	*3
5, 25	All	0(mod2)	All	All
6	0(mod6)	All	All	0(mod5)
10, 20	0(mod2)	All	All	All
12	0(mod6)	All	*1	*1
13, 19, 23, 29	All	0(mod2)	*3	*3
15	*2	*2	All	All
16, 26	0(mod2)	All	All	0(mod5)
18, 24	0(mod6)	All	*3	*3
21	*2	*2	All	0(mod5)
22	0(mod2)	All	*1	*1
27	*2	*2	*1	*1

Where $*_1 = \mu_1 + 3\mu_2 \equiv 0 \pmod 5$, $*_2 = 4\mu_1 + 3\mu_2 \equiv 0 \pmod 6$ *and* $*_3 = \mu_1 + 4\mu_2 \equiv 0 \pmod 5$.

Remark 2. From Table 1 and Lemma 2, we have more compact necessary conditions for different valuesof n as follows:

- $n \equiv 0 \pmod{30}$, for all μ_2 and $\mu_1 \equiv 0 \pmod 6$
- $n \equiv 1 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod{30}$
- $n \equiv 4, 28 \pmod{30}$, $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{15}$ or, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{15}$ or, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 12 \pmod{15}$ or, $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{15}$ or, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{15}$.
- $n \equiv 5 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod 6$ or, $\mu_1 \equiv 2 \pmod 3$ and $\mu_2 \equiv 2 \pmod 6$ or, $\mu_1 \equiv 1 \pmod 3$ and $\mu_2 \equiv 4 \pmod 6$.
- $n \equiv 6 \pmod{30}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod 5$.
- $n \equiv 7 \pmod{30}$, $\mu_1 \equiv 0 \pmod{15}$ and $\mu_2 \equiv 0 \pmod{30}$

or, $\mu_1\equiv 3 \pmod{15}$ and $\mu_2\equiv 24 \pmod{30}$ or, $\mu_1\equiv 6 \pmod{15}$ and $\mu_2\equiv 18 \pmod{30}$ or, $\mu_1\equiv 9 \pmod{15}$ and $\mu_2\equiv 12 \pmod{30}$ or, $\mu_1\equiv 12 \pmod{15}$ and $\mu_2\equiv 6 \pmod{30}$.

- $n\equiv 10 \pmod{30}$, $\mu_1\equiv 0 \pmod 6$ and $\mu_2\equiv 0 \pmod 3$.
- $n\equiv 11 \pmod{30}$, $\mu_1\equiv 0 \pmod 3$, and $\mu_2\equiv 0 \pmod{30}$ or, $\mu_1\equiv 1 \pmod 3$ and $\mu_2\equiv 10 \pmod{30}$ or, $\mu_1\equiv 2 \pmod 3$ and $\mu_2\equiv 20 \pmod{30}$.
- $n\equiv 12 \pmod{30}$, $\mu_1\equiv 0 \pmod{30}$ and $\mu_2\equiv 0 \pmod 5$ or, $\mu_1\equiv 6 \pmod{30}$ and $\mu_2\equiv 3 \pmod 5$ or, $\mu_1\equiv 18 \pmod{30}$ and $\mu_2\equiv 4 \pmod 5$ or, $\mu_1\equiv 24 \pmod{30}$ and $\mu_2\equiv 2 \pmod 5$.
- $n\equiv 13, 19 \pmod{30}$, $\mu_1\equiv 0 \pmod{15}$ and $\mu_2\equiv 0 \pmod{30}$ or, $\mu_1\equiv 3 \pmod{15}$ and $\mu_2\equiv 18 \pmod{30}$ or, $\mu_1\equiv 6 \pmod{15}$ and $\mu_2\equiv 6 \pmod{30}$ or, $\mu_1\equiv 9 \pmod{15}$ and $\mu_2\equiv 24 \pmod{30}$ or, $\mu_1\equiv 12 \pmod{15}$ & $\mu_2\equiv 12 \pmod{30}$.
- $n\equiv 15 \pmod{30}$, $\mu_1\equiv 0 \pmod 3$ and $\mu_2\equiv 0 \pmod 2$.
- $n\equiv 16 \pmod{30}$, $\mu_1\equiv 0 \pmod 6$ and $\mu_2\equiv 0 \pmod{15}$.
- $n\equiv 18, 24 \pmod{30}$, $\mu_1\equiv 0 \pmod{30}$ and $\mu_2\equiv 0 \pmod 5$ or, $\mu_1\equiv 6 \pmod{30}$ and $\mu_2\equiv 1 \pmod 5$ or, $\mu_1\equiv 12 \pmod{30}$ and $\mu_2\equiv 2 \pmod 5$ or, $\mu_1\equiv 18 \pmod{30}$ and $\mu_2\equiv 3 \pmod 5$ or, $\mu_1\equiv 24 \pmod{30}$ and $\mu_2\equiv 4 \pmod 5$.
- $n\equiv 20 \pmod{30}$, $\mu_1\equiv 0 \pmod 6$ and $\mu_2\equiv 0 \pmod 3$ or, $\mu_1\equiv 2 \pmod 6$ and $\mu_2\equiv 2 \pmod 3$ or, $\mu_1\equiv 4 \pmod 6$ and $\mu_2\equiv 1 \pmod 3$.
- $n\equiv 21 \pmod{30}$, $\mu_1\equiv 0 \pmod 3$ and $\mu_2\equiv 0 \pmod{10}$.
- $n\equiv 22 \pmod{30}$, $\mu_1\equiv 0 \pmod{30}$ and $\mu_2\equiv 0 \pmod{15}$ or, $\mu_1\equiv 6 \pmod{30}$ and $\mu_2\equiv 3 \pmod{15}$ or, $\mu_1\equiv 12 \pmod{30}$ and $\mu_2\equiv 6 \pmod{15}$ or, $\mu_1\equiv 18 \pmod{30}$ and $\mu_2\equiv 9 \pmod{15}$ or, $\mu_1\equiv 24 \pmod{30}$ and $\mu_2\equiv 12 \pmod{15}$.
- $n\equiv 25 \pmod{30}$, $\mu_1\equiv 0 \pmod 3$ and $\mu_2\equiv 0 \pmod 6$.
- $n\equiv 26 \pmod{30}$, $\mu_1\equiv 0 \pmod 6$ and $\mu_2\equiv 0 \pmod{15}$ or, $\mu_1\equiv 2 \pmod 6$ and $\mu_2\equiv 5 \pmod{15}$ or, $\mu_1\equiv 4 \pmod 6$ and $\mu_2\equiv 10 \pmod{15}$.

When $*_1$, $*_2$ and $*_3$ are as given in Table 1, $*= \mu_1+2\mu_2\equiv 0 \pmod 3$ and $\cap$ denotes intersection, the original necessary conditions in Corollary 1 are satisfied for the remaining n as follows:

When $n\equiv 2 \pmod{30}$, $\mu_1\equiv 0 \pmod 2 \cap *_1 \cap *$ and $\mu_2\equiv *_1 \cap *$, taking all the possible combinations, the original necessary conditions are satisfied for:

$\mu_1\equiv 0 \pmod{30}$ and $\mu_2\equiv 0 \pmod{15}$ or, $\mu_1\equiv 2 \pmod{30}$ and $\mu_2\equiv 11 \pmod{15}$ or, $\mu_1\equiv 4 \pmod{30}$ and $\mu_2\equiv 7 \pmod{15}$ or, $\mu_1\equiv 6 \pmod{30}$ and $\mu_2\equiv 3 \pmod{15}$ or, $\mu_1\equiv 8 \pmod{30}$ & $\mu_2\equiv 14 \pmod{15}$ or, $\mu_1\equiv 10 \pmod{30}$ and $\mu_2\equiv 10 \pmod{15}$ or, $\mu_1\equiv 12 \pmod{30}$ & $\mu_2\equiv 6 \pmod{15}$ or, $\mu_1\equiv 14 \pmod{30}$ and $\mu_2\equiv 2 \pmod{15}$ or, $\mu_1\equiv 16 \pmod{30}$ and $\mu_2\equiv 13 \pmod{15}$ or, $\mu_1\equiv 18 \pmod{30}$ and $\mu_2\equiv 9 \pmod{15}$ or, $\mu_1\equiv 20 \pmod{30}$ & $\mu_2\equiv 5 \pmod{15}$ or, $\mu_1\equiv 22 \pmod{30}$ and $\mu_2\equiv 1 \pmod{15}$ or, $\mu_1\equiv 24 \pmod{30}$ and $\mu_2\equiv 12 \pmod{15}$ or, $\mu_1\equiv 26 \pmod{30}$ and $\mu_2\equiv 8 \pmod{15}$ or, $\mu_1\equiv 28 \pmod{30}$ and $\mu_2\equiv 4 \pmod{15}$.

- When $n\equiv 3, 9 \pmod{30}$, $\mu_1\equiv 0 \pmod 3 \cap *_2 \cap *_3$ and $\mu_2\equiv *_2 \cap *_3$
- When $n\equiv 8, 14 \pmod{30}$, $\mu_1\equiv 0 \pmod 2 \cap *_3 \cap *$ and $\mu_2\equiv *_3 \cap *$
- When $n\equiv 17 \pmod{30}$, $\mu_1\equiv *_1 \cap *$ and $\mu_2\equiv 0 \pmod 2 \cap *_1 \cap *$
- When $n\equiv 23, 29 \pmod{30}$, $\mu_1\equiv *_3 \cap *$ and $\mu_2\equiv 0 \pmod 2 \cap *_3 \cap *$
- When $n\equiv 27 \pmod{30}$, $\mu_1\equiv 0 \pmod 3 \cap *_2 \cap *_1$ and $\mu_2\equiv *_2 \cap *_1$.

Theorem 2. A 3-GDD$(n, 4, 5; \mu_1, 0)$ doesn't exist.

Proof. If a 3-GDD$(n, 4, 5; \mu_1, 0)$ exists, then blocks of the design are of type (5, 0), (4, 1) or (3, 2) only. Let x, y and z denote the number of blocks of type (5, 0), (4, 1) and (3, 2) respectively.

i. Note that, for n = 3, both x and y are zeros and for n = 4, x is zero.

- When $n = 3$, only allowed blocks are of type (3, 2) and such blocks give 9 (2, 1) triples and 1(3, 0) triple.

So in a design with z blocks, there are z triples of type (3, 0) and $9z$ triples of type (2, 1), but $z = 4\mu_1$ (number of triples of type (3, 0)) and $9z=108\mu_1$ (number of triples of type (2, 1)). From the two equations, we get $4\mu_1=12\mu_1$, which is impossible.

- When $n = 4$, number of triples of type (3, 0) is

$$4y+z = 16\mu_1 \quad (9)$$

and number of triples of type (2, 1) is

$$6y+9z = 288\mu_1 \quad (10)$$

From (9) and (10), simultaneously solving for z gives

$$5z = 176\,\mu_1 \quad (11)$$

From (11), $z > 35\,\mu_1$ and hence equation (9) is impossible.

ii. For $n \geq 5$, the number of triples of type (3, 0) and (2, 1) are respectively given by:

$$10x+4y+z=4\binom{n}{3}\mu_1=\frac{2n(n-1)(n-2)\mu_1}{3} \quad (12)$$

and

$$6y+9z=4\binom{n}{2}*3\binom{n}{1}\mu_1= 6n^2(n-1)\mu_1 \quad (13)$$

Solving for z from equation (13) and substituting this value in to equation (12) gives:

$$30x + 10y + 2n^2(n-1)\mu_1=2n(n-1)(n-2)\mu_1 \quad (14)$$

Since $2n^2(n-1)\mu_1 > 2n(n-1)(n-2)\mu_1$, equation (14) is impossible.

Theorem 3. Given 3-GDD(n, 4, 5; μ_1, μ_2)

(i) $\mu_1 \geq \frac{2n\mu_2}{7(n-1)}$

(ii) $b \geq \frac{3n^2(n-1)\,\mu_1+2n^3\mu_2}{5}$

Proof.

(i) In 3-GDD(n, 4, 5; μ_1, μ_2), in each of the blocks which can have a (1, 1, 1) triples, the number of (2, 1) triples are more except in the blocks of type (2, 1, 1, 1). Here clearly, $\frac{7}{3}$ times the number of (2, 1) triples in the design must be greater than or equal to the number of (1, 1, 1) triples occurring in the design.

Hence, we have $\frac{7}{3} * 6n^2(n-1)\mu_1 > 4n^3\mu_2$, which implies $\mu_1 \geq \frac{2n\mu_2}{7(n-1)}$.

(ii) The inequality for b is derived from b without triples of the form (3, 0).

NON-EXISTENCE

Remark 3. When $\mu_1, \mu_2 \neq 0$, blocks of a 3-GDD(n, 4, 5; μ_1, μ_2) are of type (5, 0), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1) or (2, 1, 1, 1).

If u, v, x, y, z and w denotes the number of blocks of type (5, 0), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1) and (2, 1, 1, 1) respectively, then

$$10u + 4v + x + y = 4\binom{n}{3}\mu_1 \quad (15)$$

$$6v + 9x + 6y + 6z + 3w = 6n^2(n-1)\mu_1 \quad (16)$$

$$3y + 4z + 7w = 4n^3\mu_2 \quad (17)$$

where (15), (16) and (17) denote the number of triples of type (3, 0), (2, 1) and (1, 1, 1) respectively.

Theorem 4. A 3-GDD(n, 4, 5; μ_1, μ_2) does not exist if

i). $n = 3$ and $\mu_2 < \frac{4}{9}\mu_1$.

ii). $n = 4$ and $\mu_2 < \frac{3}{8}\mu_1$

iii). $n \geq 5$ and $\mu_2 < \frac{2n}{7(n-1)}\mu_2$

(14)

Proof. Let u, v, x, y, z and w denotes the number of blocks of type (5, 0), (4, 1), (3, 2), (3, 1, 1) , (2, 2, 1) and (2, 1, 1, 1) respectively.

i). When n = 3, we have $u = v = 0$.

Multiplying (16) by $\frac{2}{3}$ and subtracting (15) gives

$$5x+ 3y + 4z + 2w = 68\,\mu_1 \quad (18)$$

Subtracting (18) from (17) yields

$$5w = 108\,\mu_2 - 68\,\mu_1+ 5x \quad (19)$$

Since $x \leq 4\mu_1$ (from (15)), $5w \leq 108\,\mu_2 - 68\,\mu_1+ 20\mu_1$ and $108\,\mu_2 - 48\,\mu_1 < 0$ implies $\mu_2 < \frac{4}{9}\,\mu_1$.

But w cannot be negative, and when $\mu_2 < \frac{4}{9}\mu_1$ a 3-GDD(3, 4, 5; μ_1, μ_2) does not exist.

ii). When $n = 4$, we have $u = 0$.
Similarly, multiplying (16) by $\frac{2}{3}$ and subtracting (15) gives

$$5x + 3y + 4z + 2w = 176\,\mu_1 \quad (20)$$

Subtracting (20) from (17), we have

$$5w = 256\,\mu_2 - 176\,\mu_1 + 5x \quad (21)$$

When n=4, $x \le 16\,\mu_1$ and $5w \le 256\,\mu_2 - 176\,\mu_1 + 80\,\mu_1$. Then $256\,\mu_2 - 96\,\mu_1 < 0$ implies $\mu_2 < \frac{3}{8}\mu_1$.
Therefore, 3-GDD(4, 4, 5; μ_1, μ_2) does not exist when $\mu_2 < \frac{3}{8}\mu_1$.

iii). From (15), (16) and (17), solving for x in terms of the remaining, we have

$$5x = 4n^2(n-1)\,\mu_1 - \frac{2n(n-1)(n-2)}{3}\mu_1 - 4n^3\,\mu_2 + 10u + 5w \quad (22)$$

But
$5x \le 4n^2(n-1)\mu_1 - 4\binom{n}{3}\mu_1 - 4n^3\,\mu_2 + 4\binom{n}{3}\mu_1 + 10n^2(n-1)\mu_1$
[From (15) and (16), $10u \le 4\binom{n}{3}\mu_1$ and $w \le 2n^2(n-1)\mu_1$].
Thus
$5x \le 14n^2(n-1)\mu_1 - 4n^3\mu_2$ and $14n^2(n-1)\mu_1 - 4n^3\mu_2 < 0$ implies $\mu_1 < \frac{2n}{7(n-1)}\mu_2$.
For $n \ge 5$, if $\mu_1 < \frac{2n}{7(n-1)}\mu_2$, then $x < 0$ (which is impossible).

Remark 4. For n=3, the original necessary conditions for 3-GDD(n, 4, 5; μ_1, μ_2) are satisfied only when:

- $\mu_1 \equiv 0 \pmod{15}$ and $\mu_2 \equiv 0 \pmod{10}$ or,
- $\mu_1 \equiv 3 \pmod{15}$ and $\mu_2 \equiv 8 \pmod{10}$ or,
- $\mu_1 \equiv 6 \pmod{15}$ and $\mu_2 \equiv 6 \pmod{10}$ or,
- $\mu_1 \equiv 9 \pmod{15}$ and $\mu_2 \equiv 4 \pmod{10}$ or,
- $\mu_1 \equiv 12 \pmod{15}$ and $\mu_2 \equiv 2 \pmod{10}$.

There are cases when the original necessary conditions are satisfied, but the designs do not exist. From theorem 4(*i*) and Remark 4, the following result follows:

Corollary 3. For non-negative integers s and t:

(i) If $t > \frac{3s}{2}$, then 3-GDD(3, 4, 5; 15t, 10s) and 3-GDD(3, 4, 5; 15t+9, 10s+4), do not exist.

(ii) 3-GDD(3, 4, 5; 15t+3, 10s+8) does not exist if $t > \frac{3s+2}{2}$

(iii) 3-GDD(3, 4, 5; 15t+6, 10s+6) does not exist if $t > \frac{3s+1}{2}$

(iv) 3-GDD(3, 4, 5; 15t+12, 10s+2) does not exist if $t > \frac{3s-1}{2}$

Even when the necessary conditions are satisfied, here below are lists of some of these designs, which do not exist for n = 3:

3-GDD(3, 4, 5; 15t, 10) for t ≥ 2, 3-GDD(3, 4, 5; 15t, 20) for t ≥ 4, 3-GDD(3, 4, 5; 15t, 30) for t ≥ 5, 3-GDD(3, 4, 5; 15t + 9, 4) for t ≥ 1,
3-GDD(3, 4, 5; 15t + 9, 14) for t ≥ 2, 3-GDD(3, 4, 5; 15t + 3, 8) for t ≥ 2, 3-GDD(3, 4, 5; 15t + 3, 18) for t ≥ 3, 3-GDD(3, 4, 5; 15t + 12, 2) for t ≥ 0,
3-GDD(3, 4, 5; 15t + 12, 12) for t ≥ 2, 3-GDD(3, 4, 5; 15t + 6, 6) for t ≥ 1, 3-GDD(3, 4, 5; 15t + 6, 16) for t ≥ 3, etc.

Remark 5. For n = 4, the original necessary conditions for 3-GDD(n, 4, 5; μ_1, μ_2) are satisfied only when:
$\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{15}$ or, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{15}$ or, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 12 \pmod{15}$ or, $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{15}$ or, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{15}$.

When n = 4, from Theorem 4 (*ii*) and Remark 5, the following result follows:

Corollary 4. For non-negative integers s and t:

i. If $t > \frac{4s}{3}$, then 3-GDD(4, 4, 5; 30t, 15s) and 3-GDD(n, 4, 5; 30t+24, 15s+9), do not exist.

ii. If $t > \frac{4s+1}{3}$, then 3-GDD(4, 4, 5; 30t+6, 15s+6) does not exist

iii. 3-GDD(4, 4, 5; 30+12t, 15s+12) does not exist if $t > \frac{4s+2}{3}$.

iv. 3-GDD(4, 4, 5; 30t+18, 15s+3) does not exist if $t > \frac{4s-1}{3}$.

CONCLUSIONS

In this paper, definition 3 is used to study a special type of 3-GDDs with 4 groups and block size 5, which is denoted by 3-GDD(n, 4, 5; μ_1, μ_2). Some of the main results of this work are; some necessary conditions for the existence of such designs are developed, when $\mu_2 = 0$, the non-existence of a 3-GDD(n, 4, 5; μ_1, μ_2) is proved, and even when the original necessary conditions are satisfied, several specific instances of non- existence are given.

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