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Common fixed points of generalized F-contraction of multivalued mappings in bi-b-metric spaces

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ABSTRACT: In this article, we introduce a new concept of generalized F-contraction of nonlinear multivalued mappings and establish conditions for the existence of common fixed points of such mappings in the framework of bi- b-metric spaces. These findings combine, generalize, and expand on current and classic analogous findings in the literature.

Keywords/phrases: bi-b-metric space, common fixed point, generalized F-contraction, Maia fixed point theorem, set with two metrics

INTRODUCTION AND PRELIMINARIES

Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis. One of the fundamental pillars of the theory of metric fixed points is the Banach contraction principle. By examining the Banach contraction principle [9] in various directions, several writers have discovered generalizations, extensions, and applications of their discoveries. The study of new classes of spaces and their fundamental features is one of the most popular and interesting areas among them. For more details, we refer the readers to see ([2], [6], [8], [12], [19], [28]) and the references therein.

In contrast, Wardowski [28] developed a brand-new contraction in 2012 called F-contraction and demon- strated a fixed point result, which broadly generalizes Banach's contraction principle. Later, Wardoski's result generalized by many researchers, see Piri and Kumam [23], Suzuki [27], Wardowski et al.[29], and the references therein.

Nadler [21] developed the Banach contraction concept for multivalued mappings in complete metric spaces to generalize the well-known Banach contraction principle. Several scholars have extended and generalized Nadler's theorem in numerous ways, as evidenced by ([1], [2], [3], [18], [25]), and the references therein.

In 1993, Czerwik [12] introduced the concept of a b-metric space and also established the fixed point result in the setting of b-metric spaces which is a generalization of the Banach contraction principle. In 2015, Alsulami et al. [6] introduced the concepts of generalized F-Suzuki type contraction mappings and proved the fixed point theorems on complete b-metric spaces.

Suzuki [27] studied fixed point theorems for set-valued F-contractions in complete b-metric spaces and also investigated a fixed point theorem for single-valued F-contractions in complete b-metric spaces.

By using two metrics on a set X, Maia [20] extended the conclusions of the well-known Banach contraction principle. Maia's theorem has been generalized in the past few years, and fixed point theorems have been proved in a variety of approaches by Iseki [15], Iyer [16], Rus [24], Khan et al.[17], Berinde [10] and the references therein.

In 2015, Khan et al. [17] established fixed point results for continuous mappings satisfying a generalized contractive condition in the setting of two metrics endowed with a binary relation.

Inspired and motivated by the above-mentioned discussions, we establish some common fixed point results of generalized F-

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type contraction of nonlinear multivalued mappings in the framework of two b-metric spaces; called bi-b-metric space. Our results generalized and extended many existing common fixed point theorems, for generalized contractive and quasi-contractive mappings, in a b-metric space.

In this article, we present an improvement and generalization of the main results in the existing literature (see Maia [20], Iseki [15], Iyer [16], Rus [24] and Khan et al.[17], Acar [4], Balazs [7], Berinde [10], Petrusel [22], Sgroi [26], Suzuki [27] and the references therein).

Throughout this paper, $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ and $\mathbb{R}_+$denote the set of natural numbers, the set of nonnegative integers, the set of real numbers and the set of positive real numbers, respectively.

Consistent with [3], [12], [15], [16], [17] and [20], we start with some fundamental concepts, known definitions and results which will be needed in the sequel.

Definition 1.1 (See [12]) Let X be a nonempty set and $b_2 \geq 1$. A mapping $\varpi_2 : X \times X \to [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

$(b_1)\varpi_2(x, y) = 0$ if and only if $x = y$;

$(b_2)\varpi_2(x, y) = \varpi_2(y, x)$;

$(b_3)\varpi_2(x, y) \leq b_2[\varpi_2(x, z) + \varpi_2(z, y)]$.

Then, the pair (X, ϖ_2) is called a b-metric space with the b-metric constant b_2.

It is an obvious fact that a metric space is also a b-metric space with constant $b_2 = 1$, but the converse is not generally true. To support this fact, we have the following example.

Example 1.2 Consider the set $X = \mathbb{R}$ endowed with the function $\varpi_2 : X \times X \to [0, \infty)$ defined by $\varpi_2(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, (X, ϖ_2) is a b-metric space with $b_2 = 2$ but it is not a metric space.

Definition 1.3 (See [11]) Let (X, ϖ_2) be a b-metric space with constant b_2. The following notions are natural deductions from their metric counterparts.

(i) A sequence $\{x_n\}_{n=1}^{\infty}$ in X converges if and only if there exists $x \in X$ such that $\lim_{n\to\infty} \varpi_2(x_n, x) = 0$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called a Cauchy sequence if and only if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varpi_2(x_n, x_m) < \varepsilon$ for all $m, n \geq n_\varepsilon$. In this case, we write $\lim_{n,m\to\infty} \varpi_2(x_n, x_m) = 0$.

(iii) A b-metric space (X, ϖ_2) with constant b_2 is said to be complete if and only if each Cauchy sequence in X converges to some $x \in X$.

Remark 1.4 (See [11]) Notice that in a b-metric space (X, ϖ_2) the following statements hold:

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy;

(iii) in general, a b-metric is not continuous;

(iv) in general, a b-metric does not induce a topology on X.

Definition 1.5 (See [6]) Let (X, ϖ_X) and (Y, ϖ_Y) be b-metric spaces; a mapping $f : X \to Y$ is called:

(i) continuous at a point $x \in X$, if for every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} \varpi_X(x_n, x) = 0$, then $\lim_{n\to\infty} \varpi_Y(f(x_n), f(x)) = 0$.

(ii) continuous on X, if it is continuous at each point $x \in X$.

Since in general a b-metric is not continuous, we need the following Lemma about the b-convergent sequences in the proof of our main result.

Lemma 1.6 (See [5]) Let (X, ϖ_2) be a b-metric space with $b_2 \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have

$$\frac{1}{b_2^2}\varpi_2(x, y) \leq \liminf_{n\to\infty} \varpi_2(x_n, y_n) \leq \limsup_{n\to\infty} \varpi_2(x_n, y_n) \leq b_2^2 \varpi_2(x, y)$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} \varpi_2(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{b_2}\varpi_2(x, z) \leq \liminf_{n\to\infty} \varpi_2(x_n, z) \leq \limsup_{n\to\infty} \varpi_2(x_n, z) \leq b_2 \varpi_2(x, z)$$

In 1922, Banach [9] proved Banach fixed point theorem as follows:

Theorem 1.7 (See [9]) Let (X,ϖ_2) be a complete metric space and $T: X \to X$ be a contraction mapping, that is, there exists $k \in [0,1)$ such that $\varpi_2(Tx,Ty) \leq k\varpi_2(x,y)$ for all $x,y \in X$. Then, we have the following assertions hold: (i) T has a unique fixed point; (ii) for each $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for each $n \geq 0$ converges to the fixed point of T.

Maia [20] made a fascinating and enlightened generalization of Banach fixed Theorem in 1968 by distributing the assumptions over two metrics ϖ_1 and ϖ_2 defined on the set X.

Theorem 1.8 (See [20]) Let X be a set endowed with two metrics ϖ_1, ϖ_2 and a mapping $T: X \to X$ satisfy the following conditions:
(i) (X,ϖ_1) is a complete metric space;
(ii) $\varpi_1(x,y) \leq \varpi_2(x,y)$, for all $x,y \in X$;
(iii) T is continuous with respect ϖ_1;
(iv) T is a contraction with respect ϖ_2, that is, there exists $k \in [0,1)$ such that $\varpi_2(Tx,Ty) \leq k\varpi_2(x,y)$ for all $x,y \in X$.

Then T has a unique fixed point and for each $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for each $n \geq 0$ converges to the fixed point of T.

Now let us review definitions of F-contraction mappings introduced by Wardowski [28] and some results on F-contraction mappings, related to the existing literature.

Definition 1.9 (See [28]) Let (X,ϖ_2) be a metric space. A mapping $T: X \to X$ is said to be F-contraction on (X,ϖ_2) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x,y \in X$,

$$\varpi_2(Tx,Ty) > 0 \Rightarrow \tau + F(\varpi_2(Tx,Ty)) \leq F(\varpi_2(x,y))$$

where $\mathcal{F}$ be the family of all functions $F: \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:
(F1) F is strictly increasing, i.e. for all $a,b \in \mathbb{R}_+$such that $F(a) < F(b)$ whenever $a < b$;
(F2) for each sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+, \lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;
(F3) there exists $k \in (0,1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

Note that from (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous. Wardowski [28] gave generalization of Banach contraction principle as follows.

Theorem 1.10 (See [28]) Let (X,ϖ_2) be a complete metric space and $T: X \to X$ be F-contraction mapping. Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.

In 2014, Wardowski and Dung [29] introduced the notion of an F-weak contraction and proved a related fixed point theorem as follows.

Definition 1.11 (See [29]) Let (X,ϖ_2) be a metric space. A mapping $T: X \to X$ is said to be an F-weak contraction on (X,ϖ_2) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x,y \in X$,

$$\varpi_2(Tx,Ty) > 0 \Rightarrow \tau + F(\varpi_2(T(x),T(y))) \leq F(M(x,y))$$

where $M(x,y) = \max\left\{\varpi_2(x,y), \varpi_2(x,Tx), \varpi_2(y,Ty), \frac{\varpi_2(x,Ty)+\varpi_2(y,Tx)}{2}\right\}$.

Theorem 1.12 (See [29]) Let (X,ϖ_2) be a complete metric space and let $T: X \to X$ be an F-weak contraction mapping. If T or F is continuous, then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.

In 2014, Piri and Kumam [23] described a large class of functions by replacing the condition $(F3)$ in the definition of an F-contraction introduced by Wardowski [28] with the following one:

$(F3')F$ is continuous on $\mathbb{R}_+$.

They denote by $\mathcal{F}$ the family of all functions $F: \mathbb{R}_+ \to \mathbb{R}$ which satisfy conditions $(F1), (F2)$, and $(F3')$. Under this new set-up, Piri and Kumam proved some Wardowski and Suzuki type fixed point results in metric spaces as follows.

Theorem 1.13 (See [23]) Let (X,ϖ_2) be a complete metric space and let $T: X \to X$ be a mapping. Assume $F \in \mathcal{F}$ and there exists $\tau > 0$ such that, for all $x,y \in X$,

$$\varpi_2(Tx,Ty) > 0 \Rightarrow \tau + F(\varpi_2(Tx,Ty)) \leq F(\varpi_2(x,y))$$

Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.

Theorem 1.14 (See [23]) Let (X, ϖ_2) be a complete metric space and let $T: X \to X$ be a mapping. Suppose $F \in \mathcal{F}$ and there exists $\tau > 0$ such that, for all $x, y \in X$,

$$\frac{1}{2}\varpi_2(x, Tx) < \varpi_2(x, y) \Rightarrow \tau + F(\varpi_2(Tx, Ty)) \leq F(\varpi_2(x, y))$$

Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to u.

In the following: (X, ϖ_2) denotes a metric space, $P(X), CL(X), CB(X)$ and $K(X)$ denote the family of all nonempty subsets of X, the family of all nonempty closed subsets of X, the family of all nonempty closed and bounded subsets of X and the family of all nonempty compact subsets of X respectively. For $A, B \subseteq X$, we define $D(A) = \sup_{x,y\in A} \varpi_2(x, y)$ as the diameter of the set A and $\delta(A, B) = \sup_{x\in A} \varpi_2(x, B)$. Furthermore, the Hausdorff metric H on $CB(X)$ is defined as $H(A, B) := \max\{\delta(A, B), \delta(B, A)\}$, for all $A, B \in CB(X)$, where $\varpi(x, B) := \inf_{y\in B} \varpi_2(x, y)$ for all $x \in X$. Mapping H is said to be a Hausdorff metric induced by ϖ_2. If (X, ϖ_2) is complete, then $(CB(X), H)$ is also complete. A point $u \in X$ is a fixed point of a multivalued mapping $T: X \to P(X)$ if and only if $u \in Tu$. The set of all fixed points of multivalued mapping T is denoted by $F(T)$. A point $u \in X$ is a common fixed point of multivalued mappings $T, S: X \to P(X)$ if and only if $u \in Tu \cap Su$. The idea of common fixed point theorems for a family of multivalued generalized F-contraction mappings without using any commutativity condition in the setup of partially ordered metric spaces is due to Abbas et al. [3].

Lemma 1.15 [12, 13, 14] Let (X, ϖ_2) be a b-metric space. The following properties are satisfied.
(i) $\varpi(x, B) \leq \varpi_2(x, y)$ for all $x \in X, y \in B$ and $B \in CB(X)$.
(ii) $\varpi(x, B) \leq H(A, B)$ for all $x \in X$ and $A, B \in CB(X)$.
(iii) $\varpi(x, B) \leq b_2(\varpi_2(x, y) + \varpi(y, B))$ for all $x, y \in X$ and $B \in CB(X)$.

Lemma 1.16 [14] If $A, B \in CB(X)$ and $k > 1$, then for each $x \in A$, there exists $y \in B$ such that $\varpi_2(x, y) \leq kH(A, B)$.

Using Hausdorff metric, Nadler [21] introduced the concept of multivalued contraction and proved a multivalued version of the well-known Banach contraction principle.

Theorem 1.17 [21] Let (X, ϖ_2) be a complete metric space and let $T: X \to CB(X)$ be a mapping. Assume there exists $c \in [0,1)$ such that

$$H(Tx, Ty) \leq c\varpi_2(x, y), \forall x, y \in X$$

Then T has a fixed point $u \in X$.

In the past decades, various fixed point theorems concerning multivalued contractive mappings have been proved. Many researchers generalized Theorem 1.17 and proved a few fixed point theorems for multivalued contractive mappings.

Recent research by Acar et al. [4] produced a fixed point result and proposed the idea of generalized multivalued F-contraction mappings, which was a valid generalization of several multivalued fixed point theorems, including Nadler's.

Definition 1.18 [4] Let (X, ϖ_2) be a metric space and $T: X \to CL(X)$ be a multivalued mapping. Then T is said to be a generalized multivalued F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$x, y \in X,\ H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(M(x, y))$$

where

$$M(x, y) = \max\left\{\varpi_2(x, y), \varpi(x, Tx), \varpi(y, Ty), \frac{\varpi(x, Ty) + \varpi(y, Tx)}{2}\right\}$$

Theorem 1.19 [4] Let (X,ϖ_2) be a complete metric space and let $T: X \to K(X)$ be a generalized multivalued F-contraction. If F or T is continuous, then T has a fixed point X.

RESULTS

In this manuscript, we introduce generalized F-contraction for nonlinear multivalued mappings in the setting of bi-b-metric spaces and establish conditions for the existence of common fixed point of such mappings. In the sequel, we denote by $(X,\varpi_1,\varpi_2,b_1,b_2)$ a bi-b-metric space with (X,ϖ_1,b_1) and (X,ϖ_2,b_2) b-metric spaces. Throughout our work, we assume that the b-metrics ϖ_1 and ϖ_2 are continuous with respect to the first argument.

Definition 2.1 Let $(X,\varpi_1,\varpi_2,b_1,b_2)$ be a bi-b-metric space and $T_1,T_2: X \to CL(X)$ be multivalued mappings. A pair of mappings (T_1,T_2) is said to be generalized F-type contraction on (X,ϖ_2,b_2) if $F \in \mathcal{F}$, there exists $\tau > 0$ and $b_2 > 1$ such that for all $x,y \in X$ with $x \neq y$ satisfy the following condition:

$$H(T_1x,T_2y) > 0 \Rightarrow \tau + F\left(b_2^3 H(T_1x,T_2y)\right) \leq F\left(\frac{1}{b_2^3}G(x,y)\right)$$

where

$$G(x,y) = \max\left\{\varpi_2(x,y), \varpi(x,T_1x), \varpi(y,T_2y), \frac{\varpi(x,T_2y)+\varpi(y,T_1x)}{2b_2}, \frac{\varpi(x,T_1x)[1+\varpi(y,T_1x)]}{1+b_2\varpi_2(x,y)}, \frac{\varpi(y,T_2y)[1+\varpi(y,T_1x)]}{1+b_2\varpi_2(x,y)}\right\}$$

Theorem 2.2 Let $(X,\varpi_1,\varpi_2,b_1,b_2)$ be a bi-b-metric space. Let $T_1,T_2: X \to CB(X)$ be multivalued mappings and $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:
$(A_0)(X,\varpi_1,b_1)$ is a complete b-metric space;
(A_1) a pair (T_1,T_2) is generalized F-type contraction on (X,ϖ_2,b_2);
(A_2) there exists $c > 0$ such that $\varpi_1(x,y) \leq c.\varpi_2(x,y)$ for all $x,y \in X$;
(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{2n+1} \in T_1x_{2n}$ and $x_{2n+2} \in T_2x_{2n+1}$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X,ϖ_1,b_1); (A_4) both the mappings T_1 and T_2 are continuous in (X,ϖ_1,b_1).
Then T_1 and T_2 have a common fixed point in (X,ϖ_1,b_1).

Proof. By hypothesis, there exists $x_0 \in X$. Choose $x_1 \in T_1x_0$, if $x_0 = x_1$ or $x_1 \in T_2x_1$, then x_1 is a common fixed point of T_1 and T_2. Thus, the proof is complete. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin T_2x_1$. Thus, $\varpi(x_1,T_2x_1) > 0$, and hence

$$0 < \varpi(x_1,T_2x_1) \leq H(T_1x_0,T_2x_1)$$

According to (F_1), Lemma 1.15 (ii) and condition (A_1), we can write that

$$F\left(\varpi(x_1,T_2x_1)\right) \leq F\left(H(T_1x_0,T_2x_1)\right) \leq F\left(b_2^3 H(T_1x_0,T_2x_1)\right) \leq F\left(\frac{1}{b_2^3}G(x_0,x_1)\right) - \tau.$$

Using Lemma 1.16 for $k = b_2^3$, there exists $x_2 \in T_2x_1$ such that

$$F\left(\varpi_2(x_1,x_2)\right) \leq F\left(b_2^3 H(T_1x_0,T_2x_1)\right) \leq F\left(\frac{1}{b_2^3}G(x_0,x_1)\right) - \tau$$

where

$$G(x_0,x_1)=\max\Big\{\varpi_2(x_0,x_1),\varpi(x_0,T_1x_0),\varpi(x_1,T_2x_1),\frac{\varpi(x_0,T_2x_1)+\varpi(x_1,T_1x_0)}{2b_2},$$
$$\frac{\varpi(x_1,T_2x_1)[1+\varpi(x_1,T_1x_0)]}{1+b_2\varpi_2(x_0,x_1)},\frac{\varpi(x_0,T_1x_0)[1+\varpi(x_1,T_1x_0)]}{1+b_2\varpi_2(x_0,x_1)}\Big\}$$
$$\leq \max\Big\{\varpi_2(x_0,x_1),\varpi_2(x_1,x_2),\frac{\varpi_2(x_0,x_2)+\varpi_2(x_1,x_1)}{2b_2},$$
$$\frac{\varpi_2(x_1,x_2)[1+\varpi_2(x_1,x_1)]}{1+b_2\varpi_2(x_0,x_1)},\frac{\varpi_2(x_0,x_1)[1+\varpi_2(x_1,x_1)]}{1+b_2\varpi_2(x_0,x_1)}\Big\}$$
$$\leq \max\Big\{\varpi_2(x_0,x_1),\varpi_2(x_1,x_2),\frac{\varpi_2(x_0,x_1)+\varpi_2(x_1,x_2)}{2}\Big\}$$
$$=\max\{\varpi_2(x_0,x_1),\varpi_2(x_1,x_2)\}.$$

If $\max\{\varpi_2(x_0,x_1),\varpi_2(x_1,x_2)\}=\varpi_2(x_1,x_2)$, then from (2.4) and (F_1), we obtain

$$F\big(\varpi_2(x_1,x_2)\big)\leq F\left(\frac{1}{b_2^3}\varpi_2(x_1,x_2)\right)-\tau<F\big(\varpi_2(x_1,x_2)\big),$$

which is a contradiction. Thus, we have $\max\{\varpi_2(x_0,x_1),\varpi_2(x_1,x_2)\}=\varpi_2(x_0,x_1)$. According to the inequality (2.4), we get

$$F\big(\varpi_2(x_1,x_2)\big)\leq F\big(b_2^3H(T_1x_0,T_2x_1)\big)\leq F\left(\frac{1}{b_2^3}G(x_0,x_1)\right)-\tau$$
$$\leq F\left(\frac{1}{b_2^3}\varpi_2(x_0,x_1)\right)-\tau.$$

Clearly, if $x_1=x_2$ or $x_2\in T_1x_2$, then x_2 is a common fixed point of T_1 and T_2. Now, we assume that $x_1\neq x_2$ and $x_2\notin T_1x_2$. Thus, the proof is complete. Thus, $\varpi(x_2,T_1x_2)>0$, and hence

$$0<\varpi(x_2,T_1x_2)\leq H(T_2x_1,T_1x_2)$$

According to (F_1), Lemma 1.15 (ii) and condition (A_1), we can write that

$$F\big(\varpi(x_2,T_1x_2)\big)\leq F\big(H(T_2x_1,T_1x_2)\big)\leq F\big(b_2^3H(T_2x_1,T_1x_2)\big)$$
$$\leq F\left(\frac{1}{b_2^3}G(x_1,x_2)\right)-\tau.$$

Using Lemma 1.16 for $k=b_2^3$, there exists $x_3\in T_1x_2$ such that

$$F\big(\varpi_2(x_2,x_3)\big)\leq F\big(b_2^3H(T_2x_1,T_1x_2)\big)\leq F\left(\frac{1}{b_2^3}G(x_1,x_2)\right)-\tau$$

where

$$G(x_1,x_2)=\max\left\{\varpi_2(x_1,x_2),\varpi(x_1,T_2x_1),\varpi(x_2,T_1x_2),\frac{\varpi(x_{2j},T_1x_2)+\varpi(x_2,T_2x_1)}{2b_2},\right.$$
$$\left.\frac{\varpi(x_2,T_1x_2)[1+\varpi(x_2,T_2x_1)]}{1+b_2\varpi_2(x_1,x_2)},\frac{\varpi(x_1,T_2x_1)[1+\varpi(x_2,T_2x_1)]}{1+b_2\varpi_2(x_1,x_2)}\right\}$$
$$\le\max\left\{\varpi_2(x_1,x_2),\varpi_2(x_2,x_3),\frac{\varpi_2(x_1,x_3)+\varpi_2(x_2,x_2)}{2b_2},\right.$$
$$\left.\frac{\varpi_2(x_2,x_3)[1+\varpi_2(x_2,x_2)]}{1+b_2\varpi_2(x_1,x_2)},\frac{\varpi_2(x_1,x_2)[1+\varpi_2(x_2,x_2)]}{1+b_2\varpi_2(x_1,x_2)}\right\}$$
$$\le\max\left\{\varpi_2(x_1,x_2),\varpi_2(x_2,x_3),\frac{\varpi_2(x_1,x_2)+\varpi_2(x_2,x_3)}{2}\right\}$$
$$=\max\{\varpi_2(x_1,x_2),\varpi_2(x_2,x_3)\}.$$

If $\max\{\varpi_2(x_1,x_2),\varpi_2(x_2,x_3)\}=\varpi_2(x_2,x_3)$, then from (2.7) and (F_1), we obtain

$$F\big(\varpi_2(x_2,x_3)\big)\le F\left(\frac{1}{b_2^3}\varpi_2(x_2,x_3)\right)-\tau<F\big(\varpi_2(x_2,x_3)\big)$$

which is a contradiction. Thus, we have $\max\{\varpi_2(x_1,x_2),\varpi_2(x_2,x_3)\}=\varpi_2(x_1,x_2)$. According to the inequality (2.7), we get

$$F\big(\varpi_2(x_2,x_3)\big)\le F\big(b_2^3H(T_2x_1,T_1x_2)\big)\le F\left(\frac{1}{b_2^3}G(x_1,x_2)\right)-\tau$$
$$\le F\left(\frac{1}{b_2^3}\varpi_2(x_1,x_2)\right)-\tau.$$

By repeating this process, we construct a sequence $\{x_n\}$ in X such that $x_{2j+1}\in T_1x_{2j}$ for $j=0,1,2,\ldots$ and hence

$$0<\varpi\big(x_{2j+1},T_2x_{2j+1}\big)\le H\big(T_1x_{2j},T_2x_{2j+1}\big)$$

According to (F_1), Lemma 1.15 (ii) and condition (A_1), we can write that

$$F\left(\varpi\big(x_{2j+1},T_2x_{2j+1}\big)\right)\le F\left(H\big(T_1x_{2j},T_2x_{2j+1}\big)\right)\le F\left(b_2^3H\big(T_1x_{2j},T_2x_{2j+1}\big)\right)$$
$$\le F\left(\frac{1}{b_2^3}G\big(x_{2j},x_{2j+1}\big)\right)-\tau.$$

Using Lemma 1.16 for $k=b_2^3$, there exists $x_{2j+2}\in T_2x_{2j+1}$ such that

$$F\left(\varpi_2\big(x_{2j+1},x_{2j+2}\big)\right)\le F\left(b_2^3H\big(T_1x_{2j},T_2x_{2j+1}\big)\right)\le F\left(\frac{1}{b_2^3}G\big(x_{2j},x_{2j+1}\big)\right)-\tau$$

where

$$G\big(x_{2j},x_{2j+1}\big)=\max\left\{\varpi_2\big(x_{2j},x_{2j+1}\big),\varpi\big(x_{2j},T_1x_{2j}\big),\varpi\big(x_{2j+1},T_2x_{2j+1}\big),\frac{\varpi\big(x_{2j},T_2x_{2j+1}\big)+\varpi\big(x_{2j+1},T_1x_{2j}\big)}{2b_2},\right.$$
$$\left.\frac{\varpi\big(x_{2j+1},T_2x_{2j+1}\big)\big[1+\varpi\big(x_{2j+1},T_1x_{2j}\big)\big]}{1+b_2\varpi_2\big(x_{2j},x_{2j+1}\big)},\frac{\varpi\big(x_{2j},T_1x_{2j}\big)\big[1+\varpi\big(x_{2j+1},T_1x_{2j}\big)\big]}{1+b_2\varpi_2\big(x_{2j},x_{2j+1}\big)}\right\}$$
$$\le\max\left\{\varpi_2\big(x_{2j},x_{2j+1}\big),\varpi_2\big(x_{2j+1},x_{2j+2}\big),\frac{\varpi_2\big(x_{2j},x_{2j+2}\big)+\varpi_2\big(x_{2j+1},x_{2j+1}\big)}{2b_2},\right.$$
$$\left.\frac{\varpi_2\big(x_{2j+1},x_{2j+2}\big)\big[1+\varpi_2\big(x_{2j+1},x_{2j+1}\big)\big]}{1+b_2\varpi_2\big(x_{2j},x_{2j+1}\big)},\frac{\varpi_2\big(x_{2j},x_{2j+1}\big)\big[1+\varpi_2\big(x_{2j+1},x_{2j+1}\big)\big]}{1+b_2\varpi_2\big(x_{2j},x_{2j+1}\big)}\right\}$$
$$\le\max\left\{\varpi_2\big(x_{2j},x_{2j+1}\big),\varpi_2\big(x_{2j+1},x_{2j+2}\big),\frac{\varpi_2\big(x_{2j},x_{2j+1}\big)+\varpi_2\big(x_{2j+1},x_{2j+2}\big)}{2}\right\}$$
$$=\max\left\{\varpi_2\big(x_{2j},x_{2j+1}\big),\varpi_2\big(x_{2j+1},x_{2j+2}\big)\right\}.$$

If $\max\left\{\varpi_2\big(x_{2j},x_{2j+1}\big),\varpi_2\big(x_{2j+1},x_{2j+2}\big)\right\}=\varpi_2\big(x_{2j+1},x_{2j+2}\big)$, then from (2.10) and (F_1), we obtain

$$F\left(\varpi_2\left(x_{2j+1},x_{2j+2}\right)\right)\le F\left(\frac{1}{b_2^3}\varpi_2\left(x_{2j+1},x_{2j+2}\right)\right)-\tau<F\left(\varpi_2\left(x_{2j+1},x_{2j+2}\right)\right)$$

which is a contradiction. Thus, we have $\max\{\varpi_2(x_{2j},x_{2j+1}),\varpi_2(x_{2j+1},x_{2j+2})\}=\varpi_2(x_{2j},x_{2j+1})$. According to the inequality (2.10), we get

$$F\left(\varpi_2\left(x_{2j+1},x_{2j+2}\right)\right)\le F\left(b_2^3H\left(T_1x_{2j},T_2x_{2j+1}\right)\right)\le F\left(\frac{1}{b_2^3}G\left(x_{2j},x_{2j+1}\right)\right)-\tau$$
$$\le F\left(\frac{1}{b_2^3}\varpi_2\left(x_{2j},x_{2j+1}\right)\right)-\tau.$$

Thus for all $n\in\mathbb{N}_0$, we have

$$x_{2n+1}\in T_1x_{2n}\ \text{ and }\ x_{2n+2}\in T_2x_{2n+1}$$

Clearly, if $x_{2n+1}=x_{2n+2}$ or $x_{2n+1}\in T_2x_{2n+1}$, then x_{2n+1} is a common fixed point of T_1 and T_2. Now, we assume that $x_{2n+1}\neq x_{2n+2}$ and $x_{2n+1}\notin T_2x_{2n+1}$. Thus, the proof is complete. Thus, $\varpi(x_{2n+1},T_2x_{2n+1})>0$ which implies that $H(T_1x_{2n},T_2x_{2n+1})>0$ and

$$F\left(\varpi_2(x_{2n+1},x_{2n+2})\right)\le F\left(b_2^3H(T_1x_{2n},T_2x_{2n+1})\right)\le F\left(\frac{1}{b_2^3}G(x_{2n},x_{2n+1})\right)-\tau\le F\left(\frac{1}{b_2^3}\varpi_2(x_{2n},x_{2n+1})\right)-\tau.$$

Using (F_1) and repeating (2.13), we obtain

$$\begin{aligned}F\left(\varpi_2(x_{2n+1},x_{2n+2})\right)&\le F\left(\frac{1}{b_2^3}\varpi_2(x_{2n},x_{2n+1})\right)-\tau\\&\le F\left(\varpi_2(x_{2n},x_{2n+1})\right)-\tau\\&\le F\left(\frac{1}{b_2^3}\varpi_2(x_{2n-1},x_{2n})\right)-2\tau\\&\vdots\\&\le F\left(\frac{1}{b_2^3}\varpi_2(x_0,x_1)\right)-(2n+1)\tau.\end{aligned}$$

Repeating the above steps, we get

$$F\left(\varpi_2(x_{2n},x_{2n+1})\right)\le F\left(\frac{1}{b_2^3}\varpi_2(x_0,x_1)\right)-(2n)\tau$$

Combining (2.14) and (2.15), we obtain a sequence $\{x_n\}$ in X such that

$$F\left(\varpi_2(x_n,x_{n+1})\right)\le F\left(\frac{1}{b_2^3}\varpi_2(x_0,x_1)\right)-n\tau$$

Taking the limit as $n\to\infty$ in (2.16), we obtain

$$\lim_{n\to\infty}F\left(\varpi_2(x_n,x_{n+1})\right)=-\infty$$

From property (F_2), we get

$$\lim_{n\to\infty}\varpi_2(x_{n+},x_{n+1})=0$$

Next, we claim that the sequence $\{x_n\}$ is a b-Cauchy sequence in (X,ϖ_1). Suppose on the contrary that $\{x_{2n}\}$ is not a b-Cauchy sequence. Then there exists an $\varepsilon>0$ for which we can find two sequences of positive integers $\{2n(k)\}$ and $\{2m(k)\}$ such that for all positive integer k, and assuming that $\{2n(k)\}$ is the smallest number, we obtain

$$2n(k)>2m(k)>k,\ \varpi_2\left(x_{2n(k)},x_{2m(k)}\right)\ge\varepsilon,\ \varpi_2\left(x_{2n(k)-1},x_{2m(k)}\right)<\varepsilon$$

By using triangle inequality and from (2.18), we have

$$\varepsilon \leq \varpi_2(x_{2n(k)}, x_{2m(k)}) \leq b_2[\varpi_2(x_{2n(k)}, x_{2n(k)-1}) + \varpi_2(x_{2n(k)-1}, x_{2m(k)})]$$
$$< b_2[\varepsilon + \varpi_2(x_{2n(k)}, x_{2n(k)-1})].$$

By taking the upper limit as $k \to \infty$ in (2.19) and using (2.17), we get

$$\varepsilon \leq \limsup_{k\to\infty} \varpi_2(x_{2n(k)}, x_{2m(k)}) < b_2\varepsilon$$

Applying the triangle inequality and from (2.18), we have

$$\varepsilon \leq \varpi_2(x_{2n(k)}, x_{2m(k)}) \leq b_2[\varpi_2(x_{2n(k)}, x_{2m(k)+1}) + \varpi_2(x_{2m(k)+1}, x_{2m(k)})]$$
$$\leq b_2^2[\varpi_2(x_{2n(k)}, x_{2m(k)}) + \varpi_2(x_{2m(k)}, x_{2m(k)+1})] + b_2\varpi_2(x_{2m(k)+1}, x_{2m(k)})$$

By taking the upper limit as $k \to \infty$ in (2.21) and using (2.17), (2.20), we have

$$\frac{\varepsilon}{b_2} \leq \limsup_{k\to\infty} \varpi_2(x_{2n(k)}, x_{2m(k)+1}) \leq b_2^2\varepsilon.$$

Similarly,

$$\frac{\varepsilon}{b_2} \leq \limsup_{k\to\infty} \varpi_2(x_{2n(k)+1}, x_{2m(k)}) \leq b_2^2\varepsilon.$$

By triangle inequality, we have

$$\varpi_2(x_{2n(k)}, x_{2m(k)+1}) \leq b_2[\varpi_2(x_{2n(k)}, x_{2n(k)+1}) + \varpi_2(x_{2n(k)+1}, x_{2m(k)+1})],$$

Taking the upper limit as $k \to \infty$ in (2.24) and using (2.17), (2.22), we have

$$\frac{\varepsilon}{b_2^2} \leq \limsup_{k\to\infty} \varpi_2(x_{2n(k)+1}, x_{2m(k)+1}).$$

From (2.20) and the inequality

$$\varpi_2(x_{2n(k)+1}, x_{2m(k)+1}) \leq b_2[\varpi_2(x_{2n(k)+1}, x_{2m(k)}) + \varpi_2(x_{2m(k)}, x_{2m(k)+1})]$$
$$\leq b_2^2[\varpi_2(x_{2n(k)+1}, x_{2n(k)}) + \varpi_2(x_{2n(k)}, x_{2m(k)})] + b_2\varpi_2(x_{2m(k)}, x_{2m(k)+1})$$

we have

$$\limsup_{k\to\infty} \varpi_2(x_{2n(k)+1}, x_{2m(k)+1}) \leq b_2^3\varepsilon.$$

It follows from (2.25) and (2.26) that

$$\frac{\varepsilon}{b_2^2} \leq \limsup_{k\to\infty} \varpi_2(x_{2n(k)+1}, x_{2m(k)+1}) \leq b_2^3\varepsilon$$

Assume $x_{2m(k)+1} \neq x_{2n(k)}$. Thus $\varpi(x_{2m(k)+1}, T_2x_{2n(k)}) > 0$ and hence

$$0 < \varpi(x_{2m(k)+1}, T_2x_{2n(k)}) \leq H(T_1x_{2m(k)}, T_2x_{2n(k)})$$

By (F_1), Lemma 1.16 and condition (A_1), we obtain

$$F\left(\varpi_2(x_{2m(k)+1}, x_{2n(k)+1})\right) \leq F\left(b_2^3 H(T_1x_{2m(k)}, T_2x_{2n(k)})\right)$$
$$\leq F\left(\frac{1}{b_2^3} G(x_{2m(k)}, x_{2n(k)})\right) - \tau$$

where

$$\varpi_2(x_{2m(k)}, x_{2n(k)})$$
$$\leq G(x_{2m(k)}, x_{2n(k)})$$
$$= \max\{\varpi_2(x_{2m(k)}, x_{2n(k)}), \varpi(x_{2m(k)}, T_1x_{2m(k)}), \varpi(x_{2n(k)}, T_2x_{2n(k)}),$$
$$\frac{\varpi(x_{2m(k)}, T_2x_{2n(k)}) + \varpi(x_{2n(k)}, T_1x_{2m(k)})}{2b_2}, \frac{\varpi(x_{2n(k)}, T_2x_{2n(k)})[1 + \varpi(x_{2n(k)}, T_1x_{2m(k)})]}{1 + b_2\varpi_2(x_{2m(k)}, x_{2n(k)})},$$
$$\left.\frac{\varpi(x_{2m(k)}, T_1x_{2m(k)})[1 + \varpi(x_{2n(k)}, T_1x_{2m(k)})]}{1 + b_2\varpi_2(x_{2m(k)}, x_{2n(k)})}\right\}$$
$$\leq \max\{\varpi_2(x_{2m(k)}, x_{2n(k)}), \varpi_2(x_{2m(k)}, x_{2m(k)+1}), \varpi_2(x_{2n(k)}, x_{2n(k)+1})$$
$$\frac{\varpi_2(x_{2m(k)}, x_{2n(k)+1}) + \varpi_2(x_{2n(k)}, x_{2m(k)+1})}{2b_2}$$
$$\frac{\varpi_2(x_{2n(k)}, x_{2n(k)+1})[1 + \varpi_2(x_{2n(k)}, x_{2m(k)+1})]}{1 + b_2\varpi_2(x_{2m(k)}, x_{2n(k)})}$$
$$\left.\frac{\varpi_2(x_{2m(k)}, x_{2m(k)+1})[1 + \varpi_2(x_{2n(k)}, x_{2m(k)+1})]}{1 + b_2\varpi_2(x_{2m(k)}, x_{2n(k)})}\right\}$$

Taking the upper limit as $k \to \infty$ in the above inequality and using (2.17), (2.20), (2.22), and (2.23), we obtain

$$\varepsilon = \max\left\{\varepsilon, \frac{\frac{\varepsilon}{b_2} + \frac{\varepsilon}{b_2}}{2b_2}\right\} \leq \limsup_{k\to\infty} G(x_{2m(k)}, x_{2n(k)})$$
$$\leq \max\left\{b_2\varepsilon, \frac{b_2^2\varepsilon + b_2^2\varepsilon}{2b_2}\right\} = b_2\varepsilon$$

Using (2.25), (2.28) and (2.29), we get

$$F(b_2\varepsilon) = F\left(b_2^3\left(\frac{\varepsilon}{b_2^2}\right)\right) \leq F\left(b_2^3 \limsup_{k\to\infty} \varpi_2(x_{2m(k)+1}, x_{2n(k)+1})\right)$$
$$\leq F\left(b_2^3 \cdot \frac{1}{b_2^3} \limsup_{k\to\infty} G(x_{2m(k)}, x_{2n(k)})\right) - \tau$$
$$= F\left(\limsup_{k\to\infty} G(x_{2m(k)}, x_{2n(k)})\right) - \tau$$
$$\leq F(b_2\varepsilon) - \tau,$$

which contradicts as $\tau > 0$. Therefore, $\{x_n\}$ is a b-Cauchy sequence in (X, ϖ_2). By condition (A_2), $\{x_n\}$ is a b-Cauchy sequence in (X, ϖ_1). By condition (A_3), the b-Cauchy sequence $\{x_n\}$ defined by (2.12) has convergent subsequence $\{x_{n_k}\}$ in (X, ϖ_1) that converges to x^* in (X, ϖ_1). Thus, $\{x_n\}$ also converges to x^* in (X, ϖ_1) since (X, ϖ_1) is a complete b-metric space by condition (A_0). Therefore,

$$\lim_{n\to\infty} \varpi_1(x_n, x^*) = \lim_{n\to\infty} \varpi_1(x_{2n}, x^*) = \lim_{n\to\infty} \varpi_1(x_{2n+1}, x^*) = 0$$

Finally, we show that x^* is a common fixed point of T_1 and T_2. By condition (A_4), we obtain

$$\varpi(x^*, T_1x^*) = \varpi\left(x^*, \lim_{n\to\infty} T_1(x_{2n})\right) = \lim_{n\to\infty} \varpi(x^*, T_1(x_{2n})) \leq \lim_{n\to\infty} \varpi_1(x^*, x_{2n+1}) = 0,$$

which implies that $x^* \in T_1x^*$. Analogously,

$$\varpi(x^*, T_2x^*) = \varpi\left(x^*, \lim_{n\to\infty} T_2(x_{2n} + 1)\right) = \lim_{n\to\infty} \varpi(x^*, T_2(x_{2n} + 1)) \leq \lim_{n\to\infty} \varpi_1(x^*, x_{2n+2}) = 0,$$

which implies that $x^* \in T_2x^*$.

Therefore, T_1 and T_2 have a common fixed point $x^* \in X$. This completes the proof.

Corollary 2.3 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $T_1, T_2: X \to CB(X)$ be

multivalued mappings and $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:
$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;
(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$H(T_1x, T_2y) > 0 \Rightarrow \tau + F\left(b_2^3 H(T_1x, T_2y)\right) \leq F\left(\frac{1}{b_2^3} M(x, y)\right)$$

where

$$M(x, y) = \max\left\{\varpi_2(x, y), \varpi(x, T_1x), \varpi(y, T_2y), \frac{\varpi(x, T_2y) + \varpi(y, T_1x)}{2b_2}\right\}$$

(A_2) there exists $c > 0$ such that $\varpi_1(x, y) \leq c.\varpi_2(x, y)$ for all $x, y \in X$;
(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{2n+1} \in T_1x_{2n}$ and $x_{2n+2} \in T_2x_{2n+1}$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);
(A_4) both the mappings T_1 and T_2 are continuous in (X, ϖ_1, b_1).
Then T_1 and T_2 have a common fixed point in (X, ϖ_1, b_1).

Proof. Since $M(x, y) \leq G(x, y)$, thus the proof follows immediately from Theorem 2.2.

Corollary 2.4 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $T: X \to CB(X)$ be a multivalued mapping and $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:
$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;
(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$H(Tx, Ty) > 0 \Rightarrow \tau + F(b_2^3 H(Tx, Ty)) \leq F\left(\frac{1}{b_2^3} G(x, y)\right)$$

where

$$G(x, y) = \max\left\{\varpi_2(x, y), \varpi(x, Tx), \varpi(y, Ty), \frac{\varpi(x, Ty) + \varpi(y, Tx)}{2b_2}, \frac{\varpi(x, Tx)[1 + \varpi(y, Tx)]}{1 + b_2\varpi_2(x, y)}, \frac{\varpi(y, Ty)[1 + \varpi(y, Tx)]}{1 + b_2\varpi_2(x, y)}\right\}$$

(A_2) There exists $c > 0$ such that $\varpi_1(x, y) \leq c.\varpi_2(x, y)$ for all $x, y \in X$;
(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{n+1} \in Tx_n$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);
(A_4) the mapping T is continuous in (X, ϖ_1, b_1).
Then T has a fixed point in (X, ϖ_1, b_1).

Proof. Taking $T_1 = T_2$ in Theorem 2.2, thus the result follows.

Corollary 2.5 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $T: X \to CB(X)$ be a multivalued mapping and $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:
$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;
(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$H(Tx, Ty) > 0 \Rightarrow \tau + F(b_2^3 H(Tx, Ty)) \leq F\left(\frac{1}{b_2^3} M(x, y)\right)$$

where

$$M(x, y) = \max\left\{\varpi_2(x, y), \varpi(x, Tx), \varpi(y, Ty), \frac{\varpi(x, Ty) + \varpi(y, Tx)}{2b_2}\right\}$$

(A_2) there exists $c > 0$ such that $\varpi_1(x, y) \leq c.\varpi_2(x, y)$ for all $x, y \in X$;
(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{n+1} \in Tx_n$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);
(A_4) the mapping T is continuous in (X, ϖ_1, b_1).
Then T has a fixed point in (X, ϖ_1, b_1).

Proof. Taking $T_1 = T_2$ in Corollary 2.3, thus the result follows.

APPLICATION TO SINGLE VALUED MAPPINGS

In this part, we use bi-b-metric spaces to derive various common fixed point results of single-valued mappings. These findings add to, unite, and generalize what has already been discovered in the literature.

Theorem 3.1 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $f, g: X \to X$ be single valued mappings, $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:

$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;

(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$\varpi_2(fx, gy) > 0 \Rightarrow \tau + F(b_2^3 \varpi_2(fx, gy)) \leq F\left(\frac{1}{b_2^3} G(x, y)\right)$$

where

$$G(x, y) = \max\left\{\varpi_2(x, y), \varpi_2(x, fx), \varpi_2(y, gy), \frac{\varpi_2(x, gy) + \varpi_2(y, fx)}{2b_2},\right.$$
$$\left.\frac{\varpi_2(x, fx)[1 + \varpi_2(y, fx)]}{1 + b_2\varpi_2(x, y)}, \frac{\varpi_2(y, gy)[1 + \varpi_2(y, fx)]}{1 + b_2\varpi_2(x, y)}\right\}$$

(A_2) there exists $c > 0$ such that $\varpi_1(x, y) \leq c.\varpi_2(x, y)$ for all $x, y \in X$;

(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);

(A_4) both the mappings f and g are continuous in (X, ϖ_1, b_1).

Then f and g have a common fixed point in (X, ϖ_1, b_1).

Proof. Define $T_1, T_2: X \to CB(X)$ as $T_1x = \{fx\}$ and $T_2x = \{gx\}$ for $x \in X$. Hence T_1 and T_2 satisfy all conditions of Theorem 2.2. Thus, f and g have a common fixed point in (X, ϖ_1, b_1). That is, there exists $x^* \in X$ such that $x^* = fx^* = gx^*$.

Corollary 3.2 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $f, g: X \to X$ be single valued mappings, $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:

$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;

(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$\varpi_2(fx, gy) > 0 \Rightarrow \tau + F(b_2^3 \varpi_2(fx, gy)) \leq F\left(\frac{1}{b_2^3} M(x, y)\right)$$

where

$$M(x, y) = \max\left\{\varpi_2(x, y), \varpi_2(x, fx), \varpi_2(y, gy), \frac{\varpi_2(x, gy) + \varpi_2(y, fx)}{2b_2}\right\}$$

(A_2) there exists $c > 0$ such that $\varpi_1(x, y) \leq c.\varpi_2(x, y)$ for all $x, y \in X$;

(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);

(A_4) both the mappings f and g are continuous in (X, ϖ_1, b_1).

Then f and g have a common fixed point in (X, ϖ_1, b_1).

Proof. Since $M(x, y) \leq G(x, y)$, thus the proof follows immediately from Theorem 3.1.

Corollary 3.3 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $f: X \to X$ be a single valued mapping, $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:

$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;

(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$\tau + F(b_2^3\varpi_2(fx,fy)) \leq F\left(\frac{1}{b_2^3}G(x,y)\right)$$

where

$$G(x,y) = \max\left\{\varpi_2(x,y), \varpi_2(x,fx), \varpi_2(y,fy), \frac{\varpi_2(x,fy)+\varpi_2(y,fx)}{2b_2}, \frac{\varpi_2(x,fx)[1+\varpi_2(y,fx)]}{1+b_2\varpi_2(x,y)}, \frac{\varpi_2(y,fy)[1+\varpi_2(y,fx)]}{1+b_2\varpi_2(x,y)}\right\};$$

(A_2) there exists $c > 0$ such that $\varpi_1(x,y) \leq c.\varpi_2(x,y)$ for all $x,y \in X$;
(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{n+1} = fx_n$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);
(A_4) the mapping f is continuous in (X, ϖ_1, b_1).
Then f has a fixed point in (X, ϖ_1, b_1).

Proof. Taking $f = g$ in Theorem 3.1, thus the result follows.

Corollary 3.4 Let $(X, \varpi_1, \varpi_2, b_1, b_2)$ be a bi-b-metric space. Let $f: X \to X$ be a single valued mapping and $b_1 \geq 1$ and $b_2 > 1$. Suppose the following conditions hold:
$(A_0)(X, \varpi_1, b_1)$ is a complete b-metric space;
(A_1) if there exist $F \in \mathcal{F}, \tau > 0$ and $b_2 > 1$ such that for all $x, y \in X$ with $x \neq y$ satisfy the following condition:

$$\tau + F(b_2^3\varpi_2(fx,fy)) \leq F\left(\frac{1}{b_2^3}M(x,y)\right)$$

where

$$M(x,y) = \max\left\{\varpi_2(x,y), \varpi_2(x,fx), \varpi_2(y,fy), \frac{\varpi_2(x,fy)+\varpi_2(y,fx)}{2b_2}\right\}$$

(A_2) there exists $c > 0$ such that $\varpi_1(x,y) \leq c.\varpi_2(x,y)$ for all $x,y \in X$;
(A_3) there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ of iterates defined as $x_{n+1} = fx_n$ for any $n \in \mathbb{N}_0$ has a convergent subsequence x_{n_k} converging to x^* in (X, ϖ_1, b_1);
(A_4) the mapping f is continuous in (X, ϖ_1, b_1).
Then f has a fixed point in (X, ϖ_1, b_1).
Proof. Taking $f = g$ in Corollary 3.2, thus the result follows.

Remark 3.5

1. In all our results, if $\varpi_1 = \varpi_2$, then the obtained results will extend and generalize numerous corresponding results in the literature.
2. Theorem 2.2, corollary 2.3, corollary 2.4, corollary 2.5, Theorem 3.1, corollary 3.2, corollary 3.3, and corollary 3.4, are new results in the existing literature.
3. Corollary 3.3 extends and generalizes the main results: Theorem 1.10 and Theorem 1.12 of Wardowski [28] and [29].
4. Corollary 3.4 extends and generalizes main result of Theorem 1.13 and Theorem 1.13 of Piri [23] and Theorem 1.8 of Maia [20].
5. If we take $T_1 = T_2$ in generalized F-type contraction of multivalued mappings, then we obtain the fixed point results for generalized F-type contraction of multivalued mapping.

CONCLUSION

In this manuscript, an interesting generalization of the fixed point theorems such as Banach's fixed point theorem, Nadler's fixed point theorem [9], Wardowski's fixed point theorems [28] and [29], Suzuki's fixed point results [27], Sgroi [26], Acar [4], Alsulami [6] was shown by introducing the notion of generalized F-type contraction, which as a new type of contraction, have been applied to obtain common fixed point results for single-valued mappings and multivalued mappings in bi-b-metric spaces.

Existence of fixed point and common fixed point results of such type of F-contraction in complete bi-bmetric space are established. The new concepts lead to further investigations and applications. It will be also interesting to apply these concepts in a different bi-metric spaces.

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